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# ON THE SHARPNESS OF THE HAYMAN-TYPE ANALOGUE OF THE WIMAN INEQUALITY FOR ENTIRE DIRICHLET SERIES

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We prove the sharpness of Sheremeta's analog of the Wiman-type inequality for entire Dirichlet series. Consider $F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}$, $0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(1 \le n \to +\infty)$ and $|n(t) - \Delta t^\rho| \le \mathcal{D}$ $(t \ge t_0)$, $\Delta \in \mathbb{R}_+$, $\mathcal{D} \in \mathbb{R}_+$, $\rho \ge \frac{1}{2}$, $n(t) = \sum_{n\colon \lambda_n \le t} 1$. There exists (Sheremeta, 1978) a set $E$ of finite measure and for each $k \in \mathbb{N}$ there exists a number $\sigma_k$ such that for all $\sigma \in [\sigma_k; +\infty)\backslash E$ we have

$$M(\sigma, F) \le \mu(\sigma, F) \cdot \ln^{\rho-\frac{1}{2}} \mu(\sigma, F) \cdot u_{k-1}(\ln_2^\rho \mu(\sigma, F)) \ln_{k+1}^\delta \mu(\sigma, F),$$

where $M(\sigma, F) = \sup\{|F(\sigma + iy)|\colon y \in \mathbb{R}\}$, $\mu(\sigma, F) = \max\{|a_n|e^{\sigma\lambda_n}\colon n \ge 0\}$, $u_k(x) = u_{k-1}(x) \ln_k x$, $u_0(x) = x$, $u_{-1}(x) = 1$, $\ln_k x = \ln(\ln_{k-1} x)$, $\ln_1 x = \ln x$. All powers of each iteration of the logarithm of maximal term in this inequality are sharp. We obtain the following statement (Theorem 4): If a sequence $(\lambda_n)$ satisfies condition (4), then for every $\delta > 0$ there exist an entire Dirichlet series $F$ of the form (3) and $\sigma_0 > 0$ such that for each $\sigma \ge \sigma_0$

$$M(\sigma, F) \ge \mu(\sigma, F) \ln^{\rho-\frac{1}{2}} \mu(\sigma, F) \cdot u_{k-1}(\ln_2^\rho \mu(\sigma, F)).$$

Also are obtained analogues of this result for entire gap power series.

**1. Introduction.** Denote by $\mathcal{E}_0$ the class of transcendental entire functions of the form $f(z) = \sum_{n=0}^{+\infty} a_n z^n$. For any $r > 0$ and $f \in \mathcal{E}$ we denote

$$M_f(r) = \max\{|f(z)|\colon |z| \le r\}, \quad \mu_f(r) = \max\{|a_n|r^n\colon n \ge 0\},$$
$$\ln_1 x := \ln x, \quad \ln_{k+1} x := \ln \ln_k x, \quad \exp_1 x = e^x, \quad \exp_{k+1} x = \exp(\exp_k x),$$
$$u_{-1}(x) = 1, \quad u_0(x) = x, \quad u_k(x) = x \prod_{j=1}^{k} \ln_j x, \quad k \in \mathbb{N}.$$

In [1] W. K. Hayman proved such Wiman's type inequality.

**Theorem 1** ([1]). *Let $k \in \mathbb{N}$. For every $f \in \mathcal{E}$ and $\delta > 0$ inequality*

$$M_f(r) \le \mu_f(r) \ln^{1/2} \mu_f(r) u_{k-2}(\ln_2 \mu_f(r)) \cdot \ln_k^\delta \mu_f(r) \tag{1}$$

*holds for all $r \in (1; +\infty)\backslash E$, where $E$ is a set of finite logarithmic measure ($\int_E d\ln r < +\infty$).*

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At $k = 1$, inequality (1) coinsides with the classical Wiman inequality. The question of the sharpness of inequality (1) was considered in [2, 3]. From the Theorem 4 formulated in [2] we can obtain the following statement.

**Theorem 2.** *For any $k \in \mathbb{N}$, there exists an entire function $f \in \mathcal{E}$ such that for all $r \geq r_0$ we have*

$$M_f(r) \geq \mu_f(r) \ln^{1/2} \mu_f(r) u_{k-2}(\ln_2 \mu_f(r)). \tag{2}$$

So, inequality (1) in the general class $\mathcal{E}$ are not improveable.

But this inequality seems to be significantly improved for entire functions given by lacunar power series ([6]) or by random power series ([14]).

**2. Dirichlet series and Wiman's inequality.** We consider entire Dirichlet series of the form

$$F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n} \tag{3}$$

$0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(1 \leq n \to +\infty)$.

Denote

$$M(\sigma, F) = \sup\{|F(\sigma + iy)| \colon y \in \mathbb{R}\}, \quad \mu(\sigma, F) = \max\{|a_n| e^{\sigma\lambda_n} \colon n \geq 0\}.$$

For Dirichlet series was proved analogues of inequality (1) in [4, 5]. So, by condition $\lambda_{n+1} - \lambda_n \geq h > 0$ $(n \geq 0)$ in [4] was proved that for all $\delta > 0$

$$M(\sigma, F) \leq \mu(\sigma, F) \ln^{\frac{1}{2}+\delta} \mu(\sigma, F)$$

holds for all $\sigma \in [0; +\infty)\backslash E$, where $E$ is a set of finite measure ($\int_E dr < +\infty$).

For this series was proved ([7, 8]) such a statement. It can be shown that Sheremeta's result is true in the following formulation.

**Theorem 3** ([7, 8]). *Let $F$ be Dirichlet series of the form* (3) *such that*

$$|n(t) - \Delta t^\rho| \leq \mathcal{D}\ (t \geq t_0),\ \Delta \in \mathbb{R}_+,\ \mathcal{D} \in \mathbb{R}_+,\ \rho \geq \frac{1}{2},\ n(t) = \sum_{n \colon \lambda_n \leq t} 1. \tag{4}$$

*Then there exists a set $E$ of finite measure and for each $k \in \mathbb{N}$ there exists a number $\sigma_k$ such that for all $\sigma \in [\sigma_k; +\infty)\backslash E$ we have*

$$M(\sigma, F) \leq \mu(\sigma, F) \cdot \ln^{\rho-\frac{1}{2}} \mu(\sigma, F) \cdot u_{k-1}(\ln_2^\rho \mu(\sigma, F)) \ln_{k+1}^\delta \mu(\sigma, F). \tag{5}$$

The analogues of Theorems 1 and 2 for Dirichlet series are unknown. In this regard, a *problem naturally arises*: to obtain analogues of (2) for entire Dirichlet series with sharp powers of $\ln_k \mu(F, \sigma)$, $k \in \mathbb{N}$.

The following theorem contains a statement about the sharpness of inequality (5).

**Theorem 4.** *If a sequence $(\lambda_n)$ satisfies condition (4), then for every $\delta > 0$ there exist an entire Dirichlet series $F$ of the form* (3) *and $\sigma_0 > 0$ such that for each $\sigma \geq \sigma_0$*

$$M(\sigma, F) \geq \mu(\sigma, F) \ln^{\rho-\frac{1}{2}} \mu(\sigma, F) \cdot u_{k-1}(\ln_2^\rho \mu(\sigma, F)).$$

*Proof.* Without loss of generality, we assume that $\ln n = o(\lambda_n)$ $(n \to +\infty)$. Denote $n_0 = \min\{n\colon \lambda_n \geq \exp_{k+4}(1)\}$. For $\beta > 0$ we choose $a_n = \exp\{-\beta\lambda_n \ln_{k+4} \lambda_n\}$ for $n \geq n_0$ and $a_n = 0$, for $n < 0$, and consider Dirichlet series $F$ of the form (3). We denote

$$h(x) = -\beta x \ln_{k+4} x + \sigma x, \quad x > \exp_{k+4}(1).$$

Since $h(x)$ is a convex function on the interval $[\exp_{k+4}(1), +\infty)$, there is a unique maximum point $x(\sigma)$ of $h(x)$ on $[\exp_{k+4}(1), +\infty)$. Hence, for sufficiently large $\sigma$, we have $x(\sigma) \in (\exp_{k+4}(1), +\infty)$, $x(\sigma) \to +\infty$ $(\sigma \to +\infty)$ and

$$h'(x(\sigma)) = -\beta\Big(\ln_{k+4} x + \frac{1}{u_{k+2}(\ln x)}\Big)\Big|_{x(\sigma)} + \sigma = 0.$$

Therefore,

$$\ln_{k+4} x(\sigma) = \frac{\sigma}{\beta} - \frac{1}{u_{k+2}(\ln x(\sigma))}, \quad \ln \mu(\sigma, F) \leq \frac{\beta x(\sigma)}{u_{k+2}(\ln x(\sigma))} \quad (\sigma \to +\infty). \tag{6}$$

Denote $t_\pm = x(\sigma) \pm \big(u_{k+3}(x(\sigma)) \ln^\delta_{k+3} x(\sigma)\big)^{1/2} \pm 1$, $I(\sigma) = (t_-, t_+)$. Remark also that

$$\big(u_{k+3}(x) \ln^\delta_{k+3} x\big)^{1/2} = o(x), \quad h''(x) \sim -\beta/u_{k+3}(x) \quad (x \to +\infty).$$

For every $x \in I(\sigma)$ there exists $\eta \in I(\sigma)$ such that

$$h(x) = h(x(\sigma)) + \frac{h''(\eta)}{2}(x - x(\sigma))^2 \geq h(x(\sigma)) - \Big(\frac{\beta}{2} + o(1)\Big) \ln^\delta_{k+3} x(\sigma) \quad (\sigma \to +\infty).$$

Let $n_1(I)$ denote the number of the points of the sequence $(\lambda_n)$ on the interval $I$. Then, as $\sigma \to +\infty$ we have

$$\mathfrak{M}(\sigma, F) \geq \sum_{n \in I(\sigma)} |a_n| e^{\sigma\lambda_n} \geq \mu(\sigma, F) n_1(I(\sigma)) \exp\Big\{\Big(-\frac{\beta}{2} + o(1)\Big) \ln^\delta_{k+3} x(\sigma)\Big\}. \tag{7}$$

Since, $t_+ - t_- = 2\big(u_{k+3}(x(\sigma)) \ln^\delta_{k+3} x(\sigma)\big)^{1/2} + 2$,

$$\begin{aligned} n_1(I(\sigma)) &\geq \Delta(t_+^\varrho - t_-^\varrho) - 2\mathcal{D} \geq \Delta\varrho(t_+ - t_-)\big(x(\sigma) + o(x(\sigma))\big)^{\varrho-1} - 2\mathcal{D} = \\ &= 2\Delta\varrho(1 + o(1))\big(u_{k+3}(x(\sigma)) \ln^\delta_{k+3} x(\sigma)\big)^{1/2} x(\sigma)^{\varrho-1} \geq \\ &\geq \Delta\rho x^\rho(\sigma) \frac{\sqrt{u_{k+3}(x(\sigma)) \ln^\delta_{k+3} x(\sigma)}}{x(\sigma)} = \Delta\rho x^{\rho-\frac{1}{2}}(\sigma) u_{k+2}(\ln^{1/2} x(\sigma)) \ln^{\delta/2}_{k+3} x(\sigma) \end{aligned}$$

as $\sigma \to +\infty$. Using (6), we get

$$\begin{gathered} \beta x(\sigma) \geq \ln \mu(\sigma, F) u_{k+2}(\ln x(\sigma)), \quad \ln \beta + \ln x(\sigma) \geq \ln_2 \mu(\sigma, F) + 2\ln_2 x(\sigma), \\ \ln x(\sigma) - 2\ln_2 x(\sigma) + \ln \beta \geq \ln_2 \mu(\sigma, F), \quad \ln x(\sigma) \geq (1 + o(1)) \ln_2 \mu(\sigma, F, \quad \sigma \to +\infty. \end{gathered}$$

Then, for some $C_1 > 0$ we obtain

$$n_1(I(\sigma)) \geq C_1 \ln^{\rho-1/2} \mu(\sigma, F) \cdot u_{k+2}(\ln^{\rho-1/2} x(\sigma)) \cdot u_{k+2}(\ln^{1/2} x(\sigma)) \cdot \ln^{\delta/2}_{k+3} x(\sigma) =$$

$$= C_1 \ln^{\rho-1/2} \mu(\sigma, F) \cdot u_{k+2}(\ln^\rho x(\sigma)) \cdot \ln_{k+3}^{\delta/2} x(\sigma) \geq$$
$$\geq \ln^{\rho-\frac{1}{2}} \mu(\sigma, F) \cdot u_{k+2}(\ln_2^\rho \mu(\sigma, F)) \cdot \ln_{k+4}^\delta \mu(\sigma, F) \quad (\sigma \to +\infty).$$

Using $\exp\{(-\frac{\delta}{2}+o(1)) \ln_{k+3}^\delta y\} \geq \ln_{k+2}^{-\delta} y,\ y \to +\infty$, and choosing in inequality (7) $\beta \leq \delta$, we get

$$\mathfrak{M}(\sigma, F) \geq \mu(\sigma, F) \ln^{\rho-\frac{1}{2}} \mu(\sigma, F) \cdot u_{k+2}(\ln_2^\rho \mu(\sigma, F)) \cdot \ln_{k+4}^\delta \mu(\sigma, F) \times$$
$$\times \exp\Big\{\Big(-\frac{\delta}{2} + o(1)\Big) \ln_{k+3}^\delta \mu(\sigma, F)\Big\} \geq$$
$$\geq \mu(\sigma, F) \ln^{\rho-\frac{1}{2}} \mu(\sigma, F) u_k(\ln_2^\rho \mu(\sigma, F)) \ln_{k+2}^{-\delta} \mu(\sigma, F) \geq$$
$$\geq \mu(\sigma, F) \ln^{\rho-\frac{1}{2}} \mu(\sigma, F) \cdot u_{k-1}(\ln_2^\rho \mu(\sigma, F)), \quad \sigma \to +\infty.$$

□

Theorems 1 and 2 are consequences of theorems 3 and 4, respectively.

**Question. 1.** Similar problems concerning various asymptotic relations have also been considered for Laplace–Stieltjes type integrals (see, e.g., [9, 10, 11, 12]). A natural question arises as to what the corresponding analogues of Theorems 3 and 4 are in this case.
**2.** The same applies to entire functions of several complex variables represented by multiple power series ([15]) or multiple Dirichlet series, as well as those represented by power series with gaps in terms of homogeneous polynomials (see [13]) or those that are random analytic functions (see [14]).

**3. Corollary for gap power series.** Let us consider the class $\mathcal{E}(\mathbf{m})$ an entire functions of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^{m_n}, \quad m_n \in \mathbb{Z}_+ \quad (n \geq 0).$$

Let us denote by $m(t) = \sum_{n\colon m_n \leq t} 1$ the counting function of the sequence $\{m_n\}$.

From Theorems 3 and 4 for entire Dirichlet series we can obtain such a corollary.

**Theorem 5.** *If $f \in \mathcal{E}(\mathbf{m})$ and a sequence $\mathbf{m} = \{m_n\}$ satisfies condition where sequence $\mathbf{m} = \{m_n\}$ satisfies the following condition*

$$(\exists \Delta \in (0; +\infty))(\exists \rho \in [1/2; 1])(\exists \mathcal{D} > 0)\colon |m(t) - \Delta t^\rho| \leq \mathcal{D} \quad (t \geq t_0), \tag{8}$$

*then for each $k \in \mathbb{N}$ $(\forall \delta > 0)(\exists E_1 \subset (1; +\infty)\colon \int_{E_1} r^{-1} dr < +\infty)$ $(\forall r \in [1; +\infty)\backslash E_1)$ we have*

$$M_f(r) \leq \mu_f(r) \ln^{\rho-\frac{1}{2}} \mu_f(r) \cdot u_{k-1}(\ln_2^\rho \mu_f(r)) \ln_{k+1}^\delta \mu_f(r).$$

*On the other hand, for any sequence $\mathbf{m} = \{m_n\}$ such that satisfies condition (8) there exist a function $F \in \mathcal{E}(\mathbf{m})$ such that*

$$M_F(r) \geq \mu_F(r) \ln^{\rho-\frac{1}{2}} \mu_F(r) u_{k-1}(\ln_2^\rho \mu_F(r)).$$

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