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ON THE WEAKLY SEMI-FREDHOLM STRUCTURE OF A CLASS OF SINGULAR HILBERT TYPE OPERATOR EQUATIONS

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We studied the solution space of a linear singular Hilbert type operator equation

$$T\left(a(z)f(z)\right) + b(z)f(z) = 0$$

in the Banach space $L_q(\mathbb{C}, d\sigma), q > p > 1$, specified by coefficients $a, b \in L_r(\mathbb{C}, d\sigma)$ and the operator action

$$Tf(z) = \frac{1}{\pi} v.p. \int_{\mathbb{C}} \frac{f(\xi) - f(z)}{(z-\xi)^2} d\sigma(\xi, \bar{\xi}),$$

$f \in L_p(\mathbb{C}, d\sigma)$, $r = pq/(q-p)$. Based on classical functional analysis A.N. Kolmogorov and F. Riesz compactness theorems, we formulated conditions under which the the singular operator equation under regard possesses the weakly semi-Fredholm structure.

1. Introduction. Singular operator equations on complex domains are widely used [5, 17, 18, 21] in many investigations of important problems of modern mechanics, radiophysics, solid state physics and other applied sciences. Their theory is closely related [6, 17] with partial differential equations, geometry of complex-analytic and quasi-conformal mappings. In particular, the following [5, 10] class of expressions

$$Af(z) \colon = T\left(a(z)f(z)\right) + b(z)f(z), \tag{1}$$

characterized by its coeficients $a, b \in L_r(\mathbb{C}, d\sigma), q > r > 1$, and acting on complex function $f \in L_q(\mathbb{C}, d\sigma), q > 1$, endowed with the norm

$$||f||_q = \left(\int_{\mathbb{C}} |f(\xi)|^q d\sigma(\bar{\xi}, \xi) \right)^{1/q} \tag{2}$$

with respect to the Lebesgue measure $d\sigma(\xi, \bar{\xi}) = d\bar{\xi} \wedge d\xi/(2i)$ on the complex plane $\mathbb{C}$, define singular integral operator equations

$$Af(z) = 0, \tag{3}$$

specified by the singular Beurling-Ahlfors [1, 8, 9, 20, 22] transform

$$Tf(z) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|z-\xi| > \varepsilon} \frac{f(z) - f(\xi)}{(z-\xi)^2} d\sigma(\xi, \bar{\xi}), \tag{4}$$

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which is equivalent to the Kalderon-Zygmund [8] mapping

$$T = \partial_z \partial_{\bar{z}}^{-1} \colon L_q(\mathbb{C}, d\sigma) \to L_q(\mathbb{C}, d\sigma), q > 1, \text{ for all } z \in \mathbb{C}.$$

The linear singular operator equation (3) makes it possible to pose a related problem of studying the functional analytic and geometric properties of the corresponding reduced operator

$$A \colon L_q(\mathrm{K}, d\sigma) \to L_p(\mathrm{K}, d\sigma), \tag{5}$$

defined on the closed space of functions $f \in L_q(\mathrm{K}, d\sigma)$ with a compact support $\mathrm{K} \subset \mathbb{C}$, where $K \colon P = \overline{\Omega}$ for some bounded domain $\Omega \subset \mathbb{C}$, and, respectively, its solution space $\operatorname{Ker} A \subset L_q(\mathrm{K}, d\sigma)$, whose properties are of the most interest for applications.

Let us introduce the following definition, which reflects [10, 17, 20] some weaker versions of the classical Fredholm operators.

A bounded operator $A \colon L_q(\mathrm{K}, d\sigma) \to L_p(\mathrm{K}, d\sigma)$ with closed Range $A \subset L_p(\mathrm{K}, d\sigma)$ is called *semi-Fredholm*, if its kernel $\operatorname{Ker} A \subset L_q(\mathrm{K}, d\sigma)$, or its cokernel

$$\operatorname{coKer} A = L_p(\mathrm{K}, d\sigma)/\overline{\text{Range } A}$$

is finite dimensional. If the image Range $A \subset L_p(\mathrm{K}, d\sigma)$ is neither closed nor its cokernel $L_p(\mathrm{K}, d\sigma)/\text{Range } A$ is finite dimensional, the bounded operator $A \colon L_q(\mathrm{K}, d\sigma) \to L_p(\mathrm{K}, d\sigma)$ is called *weakly semi-Fredholm.*

In particular, our special interest lies in proving the weakly semi-Fredholm structure of the operator (5), which consists in stating the finite dimension of its kernel $\operatorname{Ker} A \subset L_q(\mathrm{K}, d\sigma)$ and the infinite dimension of its coimage $L_p(\mathrm{K}, d\sigma)/\overline{\text{Range } A}$. Based on the functional analytic properties of the Beurling-Ahlfors singular transform (4) as well as on the structural properties of the singular operator mapping (4) on the Banach space $L_p(\mathrm{K}, d\sigma)$, we stated our main theorem.

Theorem 1. *The bounded operator (5) on the Banach space* $L_q(\mathrm{K}, d\sigma), q > p > 1$, *is weakly semi-Fredholm, that is* $\dim \operatorname{Ker} A < \infty$ *and* Range $A \subset L_p(\mathrm{K}, d\sigma)$ *is not closed, if the coefficients* $0 \neq a,\ b \in L_r(\mathrm{K}, d\sigma)$ *and the chosen factorization* $b = b_-^{-1} b_+$ *is such, that for some measurable mapping* $\omega \colon \mathrm{K} \to \mathbb{C}$ *there hold conditions:* $\omega(z) b_+(z)^{-1} \in L_s(\mathrm{K}, d\sigma)$ *and* $\omega(z) b_-(z)^{-1} \sim (1 + |z|^2)^{\frac{q - m\mu}{2qm} + \nu} \in L_{\frac{qm\mu}{q-m} + \nu, loc}(\mathrm{K}, d\sigma)$, *where* $1/m = 1/r + 1/q + 1/s - \mu/2$ *and* $\delta < |\omega(z) b_-(z)^{-1}|$ *for some* $\delta > 0$ *almost everywhere.*

The above statement in its main part depends on the following classical functional analysis A.N. Kolmogorov and F. Riesz theorems [7, 12, 13, 15, 16, 19], describing linear mappings between Banach spaces.

Theorem 2 (A.N. Kolmogorov)**.** *Let* $1 \leq p < \infty$ *and* $\mathcal{F}$ *be a subset of* $L_p(\mathrm{K})$, $K \subseteq \mathbb{C}^n$. *Then the set* $\mathcal{F}$ *is totally bounded if and only if (i)* $\mathcal{F}$ *is bounded; (ii) for every* $\varepsilon > 0$ *there is such* $r > 0$, *that for every* $f \in \mathcal{F}$ *the estimation* $\int_{\mathrm{K} \cap \{|z|| > r\}} |f(z)|^p d\sigma(z, \bar{z}) < \varepsilon^p$ *holds; (iii) for every* $\varepsilon > 0$ *there is such* $\rho > 0$ *that for every* $f \in \mathcal{F}$ *and* $y \in \mathrm{K}$ *with* $|y| < \rho$ *the estimation* $\int_{\mathrm{K}} |f(y) - f(\xi)|^p d\sigma(\xi, \bar{\xi}) < \varepsilon^p$ *holds.*

Theorem 3 (F. Riesz)**.** *If a bounded operator* $A \colon X \to Y$ *from a Banach space* X *to a Banach space* Y *is compact and its image* $\operatorname{Im} A \subset Y$ *is closed, then* $\dim(AX) < \infty$. *Moreover, a linear subspace of a normed space* X *is locally compact if and only if it is finite dimensional.*

Below we proceed to studying the functional analytic properties of the singular Hilbert type transform (4) and next proving the weakly semi-Fredholm property of the linear singular operator (1).

2. Singular Beurling-Ahlfors operator T and its properties. We consider the Cauchy type integral

$$Pf(z):=\frac{1}{2\pi i}\int_{\mathrm{K}} f(\xi)\left(\frac{1}{\xi-z}-\frac{1}{\xi}\right)d\sigma(\xi,\bar{\xi}) \tag{6}$$

for a complex variable $z\in \mathrm{K}\subset\mathbb{C}$ on the dense subspace $C_0^2(\mathrm{K},d\sigma)\subset L_q(\mathrm{K},d\sigma)$ for $q>1$, naturally defining [1,6,21] the following two functions:

$$\partial Pf(z)/\partial\bar{z}=f(z),\ \partial Pf(z)/\partial z=Tf(z) \tag{7}$$

for any point $z\in\mathrm{K}$. It is easy to check that the operator $T=\partial_z\partial_{\bar{z}}^{-1}\colon L_q(\mathrm{K},d\sigma)\to L_q(\mathrm{K},d\sigma)$ for $z\in\mathrm{K}$ and $1<q<\infty$. Moreover, applying the Holder inequality to the expression (6), we obtain

$$|Pf(z)|\ \le K_q||f||_q|z|^{\frac{-2}{q}+1} \tag{8}$$

for arbitrary function $f\in L_q(\mathrm{K},d\sigma)$ and any $z\in\mathrm{K}$, where

$$K_q=\left(\int_{\mathrm{K}}|\xi|^{\frac{-q}{q-1}}|(\xi-1)|^{\frac{q}{1-q}}d\sigma(\xi)\right)^{\frac{q-1}{q}}<\infty.$$

In addition, from the following identity

$$Pf_{z_1}(z_2-z_1)=\frac{1}{2\pi i}\int_{\mathrm{K}} f(\xi)\left(\frac{1}{\xi-z_2}-\frac{1}{\xi-z_1}\right)d\sigma(\xi)\ =Pf(z_2)-Pf(z_1) \tag{9}$$

for the shifted function $f_{z_1}(\xi):=f(\xi+z_1),\xi\in\mathrm{K}$, and the estimation (8) one derives the inequality

$$|Pf(z_1)-Pf(z_2)|\ \le K_q\ ||f||_q\ |z_2-z_1|^{\frac{-2}{q}+1} \tag{10}$$

for arbitrary $z_1,z_2\in\mathrm{K}$, meaning the Hölder continuity of the mapping (6). To formulate our main statement about the solution set of the equation (3), we first state that the operator $T\colon L_q(\mathrm{K},d\sigma)\to L_q(\mathrm{K},d\sigma)$ for $1<q<\infty$ is bounded. To do this, we first observe this property for the particular case $q=2$, having calculated the integral

$$\begin{aligned}\int_{\mathrm{K}}|Tf(z)|^2\ d\sigma(z,\bar{z})\ &=\int_{\mathrm{K}}|Tf(z)|^2d\sigma(z,\bar{z})=\int_{\mathrm{K}}\partial^2\overline{Pf(z)}/\partial\bar{z}\partial z\ Pf(z)\ d\sigma(z,\bar{z})=\\ &=\int_{\mathrm{K}}\partial\overline{f(z)}/\partial\bar{z}\ Pf(z)\ d\sigma(z,\bar{z})=\int_{\mathrm{K}}\overline{f(z)}\ \partial Pf(z)/\partial\bar{z}\ d\sigma(z,\bar{z})=\\ &=\int_{\mathrm{K}}\overline{f(z)}\ f(z)\ d\sigma(z,\bar{z})=\int_{\mathrm{K}}|f(z)|^2d\sigma(z,\bar{z}),\end{aligned} \tag{11}$$

where we took into account that the space $C_0^2(\mathrm{K},d\sigma)\subset L_2(\mathrm{K},d\sigma)$ is dense in $L_2(\mathrm{K},d\sigma)$. The result (11) evidently means that the mapping $T\colon L_2(\mathrm{K},d\sigma)\to L_2(\mathrm{K},d\sigma)$ is isometry of the Hilbert space $L_2(\mathrm{K},d\sigma)$ with the norm $||T||=1$. Moreover, for any $f\in L_q(\mathrm{K},d\sigma)$ there holds [1,21,22], the following general Zygmund-Kalderon inequality

$$||Tf||_q\le\alpha_q\ ||f||_q, \tag{12}$$

where $\alpha_q>0, 1<q<\infty$, are some bounded constants. The estimation (12) in its main part significantly depends on the operator representation $T=-S^2\colon L_q(\mathrm{K},d\sigma)\to L_q(\mathrm{K},d\sigma)$, where, by definition,

$$Sf(z)=\frac{1}{2\pi i}\int_{\mathrm{K}}\frac{f(\xi)-f(z)}{(\xi-z)\ |\xi-z|}d\sigma(\xi,\bar{\xi}). \tag{13}$$

The latter is considered on the dense subspace $C_0^2(\mathrm{K}, d\sigma) \subset L_q(\mathrm{K}, d\sigma)$ for $q \geq 2$ and satisfies the following inequality

$$||Sf||_q \ \leq \sqrt{\alpha_q}||f||_q \tag{14}$$

for $f \in C_0^2(\mathrm{K}, d\sigma) \subset L_2(\mathrm{K}, d\sigma)$ and some constant $\alpha_q > 0$, $q \geq 2$, which makes it possible to extend the operator $T = -S^2 \colon C_0^2(\mathrm{K}, d\sigma) \to L_q(\mathrm{K}, d\sigma)$ by continuity on the whole space $L_q(\mathrm{K}, d\sigma)$, $q \geq 2$, and to obtain the classical Zygmund-Calderón inequality (12) for all $f \in L_q(\mathrm{K}, d\sigma)$, $q \geq 2$. Based on the properties of the adjoint operator

$$T^* = T \colon L_{q'}(\mathrm{K}, d\sigma) \to L_{q'}(\mathrm{K}, d\sigma),\ 1 < q' \leq 2,$$

one finally obtains that the operator $T \colon L_q(\mathrm{K}, d\sigma) \to L_q(\mathrm{K}, d\sigma)$ is well defined and bounded on the space $L_q(\mathrm{K}, d\sigma)$ for all $1 < q < \infty$. Moreover, the singular operator (13), as well as that of (4), can be interpolated [3,4,14,20] as operators from $L_{q_1}(\mathrm{K}, d\sigma)$ to $L_{q_2}(\mathrm{K}, d\sigma)$, where $1 < q_1 < q < q_2$, where they become compact, thus presenting a useful tool for studying functional properties of the operator (5). So, proceed now to studying them in detail and proving Theorem 1.

3. Operator A and its weakly semi-Fredholm structure. We consider the operator mapping (1) and observe that, owing to the inequality (12), the following estimation

$$||Af||_p \leq ||T\,(af)\,||_p + ||b||_r\ ||f||_q \leq (\alpha_q\ ||a||_r + ||b||_r)||f||_q, \tag{15}$$

holds for $1 < p < q < \infty$, where $r = pq/(q-p)$, and $f \in L_q(\mathrm{K}, d\sigma)$, thus meaning that the corresponding operator norm $||A|| \leq (\alpha_q\ ||a||_r + ||b||_r)$, that is bounded.

Now we formulate our main theorem about the weakly semi-Fredholm structure of the bounded operator (1) from the Banach space $L_q(\mathrm{K}, d\sigma), 1 < q < \infty$, into the Banach space $L_p(\mathrm{K}, d\sigma), q > p \geq 1$.

Theorem 4. *The bounded operator (5) on the Banach space* $L_q(\mathrm{K}, d\sigma), q > p > 1$, *is weakly semi-Fredholm, that is* $\dim \mathrm{Ker}\ A < \infty$ *and* $\mathrm{Range}\ A \subset L_p(\mathrm{K}, d\sigma)$ *is not closed, if the coefficients* $0 \neq a,\ b \in L_r(\mathrm{K}, d\sigma)$ *and the chosen factorization* $b = b_-^{-1}b_+$ *is such, that for some measurable mapping* $\omega \colon \mathrm{K} \to \mathbb{C}$ *there hold conditions:* $\omega(z)b_+(z)^{-1} \in L_s(\mathrm{K}, d\sigma)$ *and* $\omega(z)b_-(z)^{-1} \ \sim (1+|z|^2)^{\frac{q-m\mu}{2qm}+\nu} \in L_{\frac{qm\mu}{q-m}+\nu,loc}(\mathrm{K}, d\sigma)$, *where* $1/m = 1/r + 1/q + 1/s - \mu/2$ *and* $\delta < |\omega(z)b_-(z)^{-1}|$ *for some* $\delta > 0$ *almost everywhere.*

First of all, take into account that for $q > p \geq 1$ the image $\mathrm{Range}\ A \subset L_p(\mathrm{K}, d\sigma)$ can not be closed, as $\dim \mathrm{Range}\ A = \infty$. Moreover, the image $\mathrm{Range}\ A^* \subset L_q(\mathrm{K}, d\sigma)'$ of the adjoint singular transformation $A^* = A - \{T, b\} \colon L_p(\mathrm{K}, d\sigma)' \to L_q(\mathrm{K}, d\sigma)'$, where $\{T, b\} := T \circ b + b \circ T$, is not also closed, as owing to the Banach closed image theorem one obtains both the equality $\mathrm{codim\ Range}\ A^* = \dim \mathrm{Ker}\, A < \infty$ and $\dim \mathrm{Ker}\, A^* = \infty$, contradicting the condition $\mathrm{codim\ Range}\ A < \infty$. Thus, to state that $\dim \mathrm{Ker}\ A < \infty$, we will construct a new suitably reduced operator on the kernel subspace $\mathrm{Ker}\ A \subset L_q(\mathrm{K}, d\sigma)$, next extend it on the whole space $L_q(\mathrm{K}, d\sigma)$ and state its compactness by means of both the interpolating properties of the singular Beurling-Ahlfors operator (4) and the classical A.N. Kolmogorov Theorem 2, formulated in the case of the Banach space $L_q(\mathrm{K}, d\sigma)$. As this compact operator on $L_q(\mathrm{K}, d\sigma)$, if reduced on its closed subspace $\mathrm{Ker}\, A \subset L_q(\mathrm{K}, d\sigma)$, persists to be compact and its image is closed, we right away derive, as a consequence from the F. Riesz Theorem 3, that $\dim \mathrm{Ker}\, A < \infty$.

Namely, consider the closed kernel subspace $\operatorname{Ker} A \subset L_q(\mathrm{K}, d\sigma)$ of the operator (5), that is the condition $A \operatorname{Ker} A = 0$, which is equivalently rewritten as the equation

$$\mathcal{R}\omega(z) b_+^{-1} T a f = -\mathcal{R}\omega(z) b_-^{-1} f \tag{16}$$

for some nonzero mapping $\omega \colon \mathrm{K} \to \mathbb{C}$ and

$$\mathcal{R}(\cdot) := -\frac{w_\nu(z)}{\pi} \int_{\mathrm{K}} \frac{(\cdot) d\sigma(\xi, \bar{\xi})}{|\xi - z|^{2-\mu}} \tag{17}$$

is the so called fractional Cauchy-Vekua-Riesz [21] operator, where $1 < \mu < 2$, $\nu > \mu/2 + 1/m$ and $w_\nu(z) := (1 + |z|^2)^{-\nu}$, $z \in \mathrm{K}$. The lefthand side expression of (16) defines a singular operator $A_T \colon L_q(\mathrm{K}, d\sigma) \to W_m^\mu(\mathrm{K}, w_\nu d\sigma)$, $1/m = 1/r + 1/q + 1/s - \mu/2$, where

$$A_T f := \mathcal{R}\omega(z) b_+(z)^{-1} T a f \tag{18}$$

for any $f \in L_q(\mathrm{K}, d\sigma)$ and Range $A_T \subset W_m^\mu(\mathrm{K}, w_\nu d\sigma)$, if the function $\omega(z) b_+^{-1}(z) \in L_s(\mathrm{K}, d\sigma)$, owing to the classical [11] Hardy-Littlwood theorem. Similarly, the righthand side of (16) defines a singular operator $B_T \colon L_q(\mathrm{K}, d\sigma) \to W_m^\mu(\mathrm{K}, w_\nu d\sigma)$, where

$$B_T f := -\mathcal{R}\omega(z) b_-(z)^{-1} f \tag{19}$$

for any $f \in L_q(\mathrm{K}, d\sigma)$, where $\omega(z) b_-(z)^{-1} \sim (1 + |z|^2)^{\frac{q-m}{2qm} + \nu} \in L_{\frac{qm\mu}{q-m} + \nu, loc}(\mathrm{K}, d\sigma)$ and $\delta < |\omega(z) b_-(z)^{-1}|$ for some $\delta > 0$ almost everywhere. The operators (18) and (19) are characterized by the following lemma.

Lemma 1. *Let* $a(z) \in L_r(\mathrm{K}, d\sigma)$ *and the factoriazation* $b(z) := b_-(z)^{-1} b_+(z) \in L_r(\mathrm{K}, d\sigma)$, $1/r = 1/p - 1/q$, *satisfy for almost all* $z \in \mathrm{K}$ *the following conditions: there exists a measurable mapping* $\omega \colon \mathrm{K} \to \mathbb{C}$, *such that* $\omega(z) b_+(z)^{-1} \in L_s(\mathrm{K}, d\sigma)$ *and* $\omega(z) b_-(z)^{-1} \sim (1 + |z|^2)^{\frac{q-m\mu}{2qm} + \nu} \in L_{\frac{qm\mu}{q-m} + \nu, loc}(\mathrm{K}, d\sigma)$, *where* $1/m = 1/r + 1/q + 1/s - \mu/2$ *and* $\delta < |\omega(z) b_-(z)^{-1}|$ *for some* $\delta > 0$ *almost everywhere. Then 1) the linear operator (18) is bounded and compact; 2) the linear operator (19) maps the closed subspaces into the closed ones.*

Proof. To state the lemma above we first observe that, owing to the classical Kolmogorov 2 and Riesz theorem 3, the operator (18) is bounded and compact, taking into account the condition $\omega(z) b_+(z)^{-1} \in L_s(\mathrm{K}, d\sigma)$, emposed above, and the smoothing property of the Cauchy-Vekua-Riesz operator (17), giving rise, owing to the limiting condition

$$\lim_{\varepsilon \to 0} \Big[\int_{|z| > 1/\varepsilon} |A_T f(z)|^m w_\nu(z) d\sigma(z, \bar{z}) +$$
$$+ \Big(\int_{|\xi| > 1/\varepsilon} \int_{|z| > 1/\varepsilon} \frac{|A_T f(z) - A_t f(\xi)|^m}{|z - \xi|^{2+m\mu}} w_\nu(z) w_\nu(\xi) d\sigma(z, \bar{z}) d\sigma(\xi, \bar{\xi}) \Big) \Big]^{1/m} = 0$$

for any $f \in L_s(\mathrm{K}, d\sigma)$, to the compact embedding of the image Range $A_T = W_m^\mu(\mathrm{K}, w_\nu d\sigma)$, if $1/m = 1/r + 1/q + 1/s - \mu/2$. Moreover, since the Vecua type multiplicator $\omega(z) b_-(z)^{-1} \sim (1 + |z|^2)^{\frac{q-m}{2qm} + \nu} \in L_{\frac{qm}{q-m} + \nu, loc}(\mathrm{K}, d\sigma)$, as well as $\delta < |\omega(z) b_-(z)^{-1}|$ for some $\delta > 0$ almost everywhere, the smoothing property of the Cauchy-Vekua-Riesz operator (17)

$$\mathcal{R} \colon L_m(\mathrm{K}, w_\nu d\sigma) \to W_m^\mu(\mathrm{K}, w_\nu d\sigma)$$

gives rise to the surjective condition Range $B_T = W_m^\mu(\mathrm{K}, w_\nu d\sigma)$. $\square$

Proof of the main Theorem. We first observe that the reduced operator

$$A_T\colon \text{ Ker } A \to W_m^{\mu}(\mathrm{K}, w_\nu d\sigma)$$

is compact too, and whose image

$$\text{Range } A_T|_{\text{Ker } A} = -\mathcal{R}\omega b_-^{-1} \text{ Ker } A \subset W_m^{\mu}(\mathrm{K}, w_\nu d\sigma)$$

is *a priori* closed, owing its surjectivity of the operator $B_T\colon L_q(\mathrm{K}, d\sigma) \to W_m^{\mu}(\mathrm{K}, w_\nu d\sigma)$. Then, based on the F. Riesz compactness Theorem 3, we derive the boundedness of the $\dim \text{Range } A_T$, and as a consequence, the boundedness $\dim \text{Ker} A < \infty$, since the kernel $\text{Ker } \mathcal{R} = \{0\}$ of the Cauchy-Vekua-Riesz operator (17) is trivial. The latter means that the operator (1) possesses the weakly semi-Fredholm structure, thus proving main theorem. □

Remark 1. Taking into account the properties (7) of the P - operator (6), the above equation (16) can be reformulated as the following partial differential equation

$$b_+ \frac{\partial Pf(z)}{\partial \bar{z}} + b_- \frac{\partial P\,(af)\,(z)}{\partial z} = 0, \tag{20}$$

generating quasi-conformal mappings of the complex plane K, and whose solutions naturally generate the kernel $\text{Ker } A \subset L_q(\mathrm{K}, d\sigma)$ of the operator (1). This observation provides an alternative viewpoint on equation (16) and a more detailed analysis of the subspace $\text{Ker } A \subset L_q(\mathrm{K}, d\sigma)$ makes it possible to establish Theorem 1 in a somewhat more general form. It is also worth to mention that similar to (20) equations were extensively studied in [2], and analytical techniques there devised can be useful for studying problems, treated in the present work.

4. Conclusion. Based on the functional analytic properties of the singular Kalderon-Zygmund type integral operator

$$Tf(z) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\mathrm{K} \cap |z-\xi| > \varepsilon} \frac{f(z) - f(\xi)}{(z-\xi)^2} d\sigma(\xi, \bar{\xi}),$$

on the Banach space $L_q(\mathrm{K}, d\sigma)$, we have studied the kernel $\text{Ker} A \subset L_q(\mathrm{K}, d\sigma)$ of the linear singular operator expression

$$Af(z) := T\,(a(z)f(z)) + b(z)f(z) \tag{21}$$

with coeficients $a, b \in L_r(\mathrm{K}, d\sigma)$, $b = b_+^{-1} b_-$, where for some measurable mapping $\omega\colon \mathrm{K} \to \mathbb{C}$ there hold the embeddings $\omega(z)b_+(z)^{-1} \in L_s(\mathrm{K}, d\sigma)$ and $\omega(z)b_-(z)^{-1} \sim (1+|z|^2)^{\frac{q-m\mu}{2qm}+\nu} \in L_{\frac{qm\mu}{q-m}+\nu, loc}(\mathrm{K}, d\sigma)$, where $1/m = 1/r + 1/q + 1/s - \mu/2$ and $\delta < |\omega(z)b_-(z)^{-1}|$ for some $\delta > 0$ almost everywhere. Making use of the reduction operator approach and classical functional analysis results Theorem 3 and Lemma 1, we succeeded in stating the weakly semi-Fredholm property of the above operator expression (21).

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