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# BLOCK-SYMMETRIC AND WEAKLY SYMMETRIC POLYNOMIALS ON BANACH SPACES OF ABSOLUTELY SUMMABLE IN A POWER $p \in [1, +\infty)$ SEQUENCES

A. I. Bandura[1], S. I. Nykorovych[2], T. V. Vasylyshyn[3]. *Block-symmetric and weakly symmetric polynomials on Banach spaces of absolutely summable in a power $p \in [1, +\infty)$ sequences*, Mat. Stud. **66** (2026), 71–79.

The work is devoted to the study of block-symmetric and weakly symmetric continuous polynomials on the Banach space $\ell_p^{(\mathbb{K})}$ of all sequences $x = (x_1, x_2, \ldots)$ with $x_j \in \mathbb{K}$ for $j \in \mathbb{N}$ such that the series $\sum_{j=1}^{\infty} |x_j|^p$ is convergent, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $p \in [1, +\infty)$. We construct an algebraic basis of the algebra of $n$-block-symmetric continuous polynomials on $\ell_p^{(\mathbb{R})}$. Also we construct a generating system of the algebra of weakly symmetric continuous polynomials on $\ell_p^{(\mathbb{K})}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and we show that elements of this system are linearly dependent. We show that the weakly symmetric continuous linear functionals, in contrast to the block-symmetric functions, separate points of the space $\ell_1^{(\mathbb{K})}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

**1. Introduction.** Algebras of polynomials (see [5]) and analytic functions (see [1, 3, 12]) on Banach spaces with various symmetry properties have many applications, especially in statistical quantum physics (see [4, 6]). One of the important types of symmetry is the so-called block-symmetry (see, e.g., [7, 8, 13]). In [13], it is constructed an algebraic basis of the algebra of continuous $n$-block symmetric complex-valued polynomials on the Banach space $\ell_p^{(\mathbb{C})}$ of all absolutely summable in the power $p \in [1, +\infty)$ sequences of complex numbers. In this work, we establish the analogical result for the real version, $\ell_p^{(\mathbb{R})}$, of this space. These two results gave us the opportunity to construct a generating system of the algebra of weakly symmetric continuous polynomials on $\ell_p^{(\mathbb{K})}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We show that elements of this system are linearly dependent. Consequently, the system is not an algebraic basis. Also, we prove that weakly symmetric continuous linear functionals, in contrast to block-symmetric functions, separate points of the space $\ell_1^{(\mathbb{K})}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

**2. Preliminaries. Polynomials.** We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_+$ the set of all nonnegative integers. Let $X$ and $Y$ be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ resp. A mapping $A\colon X^m \to Y$, where $m \in \mathbb{N}$, is said to be *$m$-linear* if it is linear in each argument separately, when all other $m-1$ arguments are fixed. A mapping $P\colon X \to Y$ is called an $m$-homogeneous polynomial with $m \in \mathbb{N}$, if there exists an $m$-linear mapping $A_P\colon X^m \to Y$ such that $P(x) = A_P\big(\underbrace{x, \ldots, x}_{m}\big)$ for every $x \in X$. A mapping

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$P\colon X \to Y$ is called a 0-homogeneous polynomial, if it is a constant mapping. A mapping $P\colon X \to Y$ is called a polynomial of degree at most $N$, if it can be represented in the form $P = P_0 + P_1 + \ldots + P_N$, where $P_j\colon X \to Y$ is a $j$-homogeneous polynomial for every $j \in \{0, 1, \ldots, N\}$.

**Algebraic basis.** Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $A$ be a unital commutative algebra over the field $\mathbb{K}$. For every polynomial $Q\colon \mathbb{K}^n \to \mathbb{K}$ of the form

$$Q(z_1, \ldots, z_n) = \sum_{(k_1,\ldots,k_n)\in\Omega} \alpha_{(k_1,\ldots,k_n)} z_1^{k_1} \cdots z_n^{k_n},$$

where $\alpha_{(k_1,\ldots,k_n)} \in \mathbb{K}$ and $\Omega$ is some nonempty finite subset of $\mathbb{Z}_+^n$, let us define the mapping $Q_A\colon A^n \to A$ by

$$Q_A(a_1, \ldots, a_n) = \sum_{(k_1,\ldots,k_n)\in\Omega} \alpha_{(k_1,\ldots,k_n)} a_1^{k_1} \cdots a_n^{k_n},$$

where $a_1, \ldots, a_n \in A$ (we consider the zero power $a_j^0$ of an element $a_j$ to be the unit element of $A$).

Let $a, a_1, \ldots, a_n \in A$. The element $a$ is called an *algebraic combination* of $a_1, \ldots, a_n$, if there exists a polynomial $Q\colon \mathbb{K}^n \to \mathbb{K}$ such that $a = Q_A(a_1, \ldots, a_n)$. A nonempty set $B \subset A$ is called a *generating system* of $A$, if every element of $A$ can be represented as an algebraic combination of some elements of $B$. A nonempty set $B \subset A$ is called an *algebraic basis* of $A$ if every element of $A$ can be uniquely represented as an algebraic combination of some elements of $B$. A finite nonempty set $\{a_1, \ldots, a_n\} \subset A$ is called algebraically independent, if the equality $Q_A(a_1, \ldots, a_n) = 0$ is possible only if the polynomial $Q$ is identically equal to 0. An infinite set $A_0 \subset A$ is called algebraically independent, if its each finite nonempty subset is algebraically independent. Evidently, every algebraic basis is algebraically independent. Furthermore, every algebraically independent generating system is an algebraic basis.

**The space $\ell_p^{(\mathbb{K})}$.** Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $p \in [1, +\infty)$. Let us denote $\ell_p^{(\mathbb{K})}$ the Banach space of all sequences $x = (x_1, x_2, \ldots)$, where $x_j \in \mathbb{K}$ for $j \in \mathbb{N}$, such that the series $\sum_{j=1}^{\infty} |x_j|^p$ is convergent, with norm

$$\|x\|_p = \Big(\sum_{j=1}^{\infty} |x_j|^p\Big)^{1/p}.$$

**Symmetric and weakly symmetric mappings.** Let $X$ and $Y$ be nonempty sets. Let $G$ be a set of mappings which act from $X$ to itself. A function $f\colon X \to Y$ is called $G$-symmetric if $f(a(x)) = f(x)$ for every $a \in G$ and $x \in X$. Let

$$\mathcal{G} = \{G_\alpha \colon \alpha \in \Lambda\}$$

be a family of sets $G_\alpha$ of mappings which act from $X$ to itself, indexed by elements of some index set $\Lambda$, such that for every $\alpha, \beta \in \Lambda$ there exists $\gamma \in \Lambda$ such that $G_\gamma \subset G_\alpha \cap G_\beta$. A function $f\colon X \to Y$ is called $\mathcal{G}$-weakly symmetric if there exists $\alpha \in \Lambda$ such that $f$ is $G_\alpha$-symmetric.

**Groups of coordinate permutation operators.** Let $B$ be some set of bijections, acting from $\mathbb{N}$ to itself. Let $X$ be an arbitrary vector space of sequences such that $(x_{b(1)}, x_{b(2)}, \ldots)$ belongs to $X$ for every $b \in B$ and $(x_1, x_2, \ldots) \in X$. For $b \in B$, let the operator $g_{b,X}\colon X \to X$ be defined by

$$g_{b,X}((x_1, x_2, \ldots)) = (x_{b(1)}, x_{b(2)}, \ldots), \tag{1}$$

where $(x_1, x_2, \ldots) \in X$. Let

$$G_{B,X} = \{g_{b,X} \colon b \in B\}, \tag{2}$$

where operators $g_{b,X}$ are defined by (1). Note that if $B$ is a group with respect to the operation of composition, then $G_{B,X}$ is also a group.

**3. The main result.** Let $n \in \mathbb{N}$. By [11, Theorem 2.2, p. 52], the mapping $\mu_n \colon \mathbb{N} \to (\mathbb{Z}_+ \times \{0,\ldots,n-1\}) \setminus \{(0,0)\}$, defined by

$$\mu_n(m) = (q, r) \tag{3}$$

for $m \in \mathbb{N}$, where $q$ is the quotient and $r$ is the remainder when $m$ is divided by $n$, is a well-defined bijection. Let $\lambda_n \colon (\mathbb{Z}_+ \times \{0,\ldots,n-1\}) \setminus \{(0,0)\} \to \mathbb{N} \times \{1,\ldots,n\}$ be defined by

$$\lambda_n((q,r)) = \begin{cases} (q,n), & \text{if } r = 0; \\ (q+1,r), & \text{if } r > 0 \end{cases} \tag{4}$$

for $(q,r) \in (\mathbb{Z}_+ \times \{0,\ldots,n-1\}) \setminus \{(0,0)\}$.

**Lemma 1.** *Let $n \in \mathbb{N}$. The mapping $\lambda_n$, defined by (4), is a bijection.*

*Proof.* Let us show that $\lambda_n$ is injective. Let $(q_1,r_1),(q_2,r_2) \in (\mathbb{Z}_+ \times \{0,\ldots,n-1\}) \setminus \{(0,0)\}$ be such that $(q_1,r_1) \neq (q_2,r_2)$. Let $(j_1,s_1) = \lambda_n((q_1,r_1))$ and $(j_2,s_2) = \lambda_n((q_2,r_2))$. Let us show that

$$(j_1,s_1) \neq (j_2,s_2). \tag{5}$$

If $r_1 \neq r_2$, then, by (4), $s_1 \neq s_2$ and, consequently, (5) holds. Consider the case $r_1 = r_2$. In this case, $q_1 \neq q_2$, since $(q_1,r_1) \neq (q_2,r_2)$. By (4), $j_1 = q_1$, $j_2 = q_2$ if $r_1 = r_2 = 0$ and $j_1 = q_1 + 1$, $j_2 = q_2 + 1$ if $r_1 = r_2 > 0$, In both cases, $j_1 \neq j_2$, since $q_1 \neq q_2$. Therefore (5) holds. So, $\lambda_n$ is injective.

Let us show that $\lambda_n$ is surjective. Let $(j,s) \in \mathbb{N} \times \{1,\ldots,n\}$ and

$$(q,r) = \begin{cases} (j,0), & \text{if } s = n; \\ (j-1,s), & \text{if } s < n. \end{cases} \tag{6}$$

By (6), since $j \in \mathbb{N}$ and $s \in \{1,\ldots,n\}$, it follows that $q \in \mathbb{Z}_+$ and $r \in \{0,\ldots,n-1\}$. Also, by (6), $(q,r)$ cannot be equal to $(0,0)$. So, $(q,r) \in (\mathbb{Z}_+ \times \{0,\ldots,n-1\}) \setminus \{(0,0)\}$. By (4) and (6), $\lambda_n((q,r)) = (j,s)$. Thus, $\lambda_n$ is surjective. $\square$

Let $n \in \mathbb{N}$. Let the mapping $\varkappa_n \colon \mathbb{N} \to \mathbb{N} \times \{1,\ldots,n\}$ be defined by

$$\varkappa_n = \lambda_n \circ \mu_n, \tag{7}$$

where $\mu_n$ and $\lambda_n$ are defined by (3) and (4) resp. It can be checked that

$$m = (j-1)n + s \tag{8}$$

for every $m \in \mathbb{N}$, where $(j,s) = \varkappa_n(m)$.

We define an order on the set $\mathbb{N} \times \{1,\ldots,n\}$ as follows

$$(j_1,s_1) \prec (j_2,s_2) \Leftrightarrow j_1 < j_2 \text{ or } (j_1 = j_2 \text{ and } s_1 < s_2) \tag{9}$$

for $(j_1,s_1),(j_2,s_2) \in \mathbb{N} \times \{1,\ldots,n\}$. We will also use the following order on $\mathbb{N} \times \{1,\ldots,n\}$:

$$(j_1,s_1) \hat{\prec} (j_2,s_2) \Leftrightarrow s_1 < s_2 \text{ or } (s_1 = s_2 \text{ and } j_1 < j_2) \tag{10}$$

for $(j_1,s_1),(j_2,s_2) \in \mathbb{N} \times \{1,\ldots,n\}$. It can be checked that both "$\prec$" and "$\hat{\prec}$" are strict linear orders.

**Lemma 2.** *Let $n \in \mathbb{N}$. The mapping $\varkappa_n$, defined by (7), is a strictly increasing bijection between ordered sets $(\mathbb{N}, <)$ and $(\mathbb{N} \times \{1,\ldots,n\}, \prec)$, where "$<$" is the standard strict order on $\mathbb{N}$ and "$\prec$" is defined by (9).*

*Proof.* Since both $\mu_n$ and $\lambda_n$ are bijections, it follows that $\varkappa_n$ is a bijection. Let us show that $\varkappa_n$ is strictly increasing. Let $m_1, m_2 \in \mathbb{N}$ be such that

$$m_1 < m_2. \tag{11}$$

Let $(j_1, s_1) = \varkappa(m_1)$ and $(j_2, s_2) = \varkappa(m_2)$. Let us show that $(j_1, s_1) \prec (j_2, s_2)$. By (8),

$$m_1 = (j_1 - 1)n + s_1 \quad \text{and} \quad m_2 = (j_2 - 1)n + s_2. \tag{12}$$

By (11) and (12),

$$(j_1 - 1)n + s_1 < (j_2 - 1)n + s_2. \tag{13}$$

If $j_1 < j_2$, then, by (9), $(j_1, s_1) \prec (j_2, s_2)$. If $j_1 = j_2$, then, by (13), $s_1 < s_2$ and, consequently, by (9), $(j_1, s_1) \prec (j_2, s_2)$. Consider the case $j_1 > j_2$. We will show that this case is impossible. Since $j_1 > j_2$ and both $j_1$ and $j_2$ are integers, it follows that $j_1 \geq j_2 + 1$. Therefore $j_1 - 1 \geq (j_2 - 1) + 1$ and, consequently,

$$(j_1 - 1)n \geq (j_2 - 1)n + n. \tag{14}$$

Since $s_1$ and $s_2$ belong to $\{1, \ldots, n\}$, it follows that

$$(j_1 - 1)n + s_1 > (j_1 - 1)n, \quad (j_2 - 1)n + n \geq (j_2 - 1)n + s_2. \tag{15}$$

By (14) and (15),

$$(j_1 - 1)n + s_1 > (j_2 - 1)n + s_2. \tag{16}$$

By (12) and (16), $m_1 > m_2$, which contradicts (11). So, the case $j_1 > j_2$ is impossible. Thus, $\varkappa_n$ is strictly increasing. This completes the proof. $\square$

Let $\mathcal{S}$ be the group of all bijections that act from $\mathbb{N}$ to itself. Let $\mathrm{id}_{\{1,\ldots,n\}}$ be the identity mapping on $\{1, \ldots, n\}$. For $\sigma \in \mathcal{S}$, let $\sigma \times \mathrm{id}_{\{1,\ldots,n\}}$ be the mapping that act from $\mathbb{N} \times \{1, \ldots, n\}$ to itself defined by

$$\big(\sigma \times \mathrm{id}_{\{1,\ldots,n\}}\big)((j, s)) = (\sigma(j), s), \tag{17}$$

where $(j, s) \in \mathbb{N} \times \{1, \ldots, n\}$. Since both $\sigma$ and $\mathrm{id}_{\{1,\ldots,n\}}$ are bijections, it follows that $\sigma \times \mathrm{id}_{\{1,\ldots,n\}}$ is a bijection too.

For $\sigma \in \mathcal{S}$ and $n \in \mathbb{N}$, let $\theta_{\sigma,n} \colon \mathbb{N} \to \mathbb{N}$ be defined by

$$\theta_{\sigma,n} = \varkappa_n^{-1} \circ \big(\sigma \times \mathrm{id}_{\{1,\ldots,n\}}\big) \circ \varkappa_n, \tag{18}$$

where $\varkappa_n$ and $\sigma \times \mathrm{id}_{\{1,\ldots,n\}}$ are defined by (7) and (17) resp. Since mappings $\varkappa_n^{-1}$, $\sigma \times \mathrm{id}_{\{1,\ldots,n\}}$ and $\varkappa_n$ are bijections, it follows that $\theta_{\sigma,n}$ is a bijection too. Let

$$\Theta_n = \{\theta_{\sigma,n} \colon \sigma \in \mathcal{S}\}, \tag{19}$$

where $\theta_{\sigma,n}$ is defined by (18). By [13, Proposition 1], $\Theta_n$ is a subgroup of $\mathcal{S}$.

Let $G_{\Theta_n, \ell_p^{(\mathbb{K})}}$ be the group of operators on $\ell_p$, defined by (2), where $\Theta_n$ is defined by (19). A $G_{\Theta_n, \ell_p^{(\mathbb{K})}}$-symmetric function $f$ defined on $\ell_p^{(\mathbb{K})}$ is called $n$-block-symmetric.

For $n \in \mathbb{N}$, $p \in [1, +\infty)$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ such that $\langle k \rangle \geq p$, where $\langle k \rangle = k_1 + \ldots + k_n$, let $F_{k,n,p,\mathbb{K}} \colon \ell_p^{(\mathbb{K})} \to \mathbb{K}$ be defined by

$$F_{k,n,p,\mathbb{K}}((x_1, x_2, \ldots)) = \sum_{j=1}^{\infty} \prod_{\substack{v=1 \\ k_v > 0}}^{n} \big(x_{(j-1)n+v}\big)^{k_v}, \quad (x_1, x_2, \ldots) \in \ell_p^{(\mathbb{K})}. \tag{20}$$

Note that [13, Theorem 4] and [13, Lemma 7] imply the following theorem.

**Theorem 1.** *Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. The set of polynomials*

$$\left\{F_{k,n,p,\mathbb{C}} \colon k \in \mathbb{Z}_+^n \text{ such that } \langle k \rangle \geq p\right\},$$

*where $F_{k,n,p,\mathbb{C}}$ is defined by* (20), *is an algebraic basis of the algebra of all $n$-block-symmetric complex-valued continuous polynomials on $\ell_p^{(\mathbb{C})}$.*

Let us establish the real analog of Theorem 1.

**Theorem 2.** *Let $n \in \mathbb{N}$ and $p \in [1, +\infty)$. The set of polynomials*

$$\left\{F_{k,n,p,\mathbb{R}} \colon \quad k \in \mathbb{Z}_+^n \text{ such that } \langle k \rangle \geq p\right\},$$

*where $F_{k,n,p,\mathbb{R}}$ is defined by* (20), *is an algebraic basis of the algebra of all $n$-block-symmetric real-valued continuous polynomials on $\ell_p^{(\mathbb{R})}$.*

*Proof.* Since $\ell_p^{(\mathbb{C})}$ is the complexification of $\ell_p^{(\mathbb{R})}$ and the complex extension of every $n$-block-symmetric real-valued continuous polynomial on $\ell_p^{(\mathbb{R})}$ is an $n$-block-symmetric complex-valued continuous polynomial on $\ell_p^{(\mathbb{C})}$, taking into account Theorem 1 and [10, Corollary 1], we obtain the desired result. $\square$

Note that $n$-block-symmetric functions fail to separate points of the space. For example, $f(e_1) = f(e_{n+1})$ for every $n$-block-symmetric function $f$ defined on $\ell_p^{(\mathbb{K})}$, where

$$e_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, 0, \ldots)$$

for $j \in \mathbb{N}$. Let us consider weakly symmetric functions, which, as it will be shown in this section, separate the points of the space. Denote

$$\mathcal{G}_{2,\ell_p^{(\mathbb{K})}} = \left\{G_{\Theta_{2^n}, \ell_p^{(\mathbb{K})}} \colon \quad n \in \mathbb{Z}_+\right\}.$$

Consider $\mathcal{G}_{2,\ell_p^{(\mathbb{K})}}$-weakly symmetric continuous polynomials on $\ell_p^{(\mathbb{K})}$. Note that [14, Theorem 1] implies the following result.

**Proposition 1.** *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $p \in [1, +\infty)$. The set of all $\mathcal{G}_{2,\ell_p^{(\mathbb{K})}}$-weakly symmetric continuous polynomials on $\ell_p^{(\mathbb{K})}$ is an algebra.*

Let us establish the following analog of [2, Theorem 5].

**Theorem 3.** *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $p \in [1, +\infty)$. The set of polynomials*

$$\{F_{k,2^n,p,\mathbb{K}} \colon n \in \mathbb{Z}_+ \text{ and } k \in \mathbb{Z}_+^n \text{ such that } \langle k \rangle \geq p\}, \tag{21}$$

*where $F_{k,2^n,p,\mathbb{K}}$ is defined by* (20), *is a generating system of the algebra of all $\mathcal{G}_{2,\ell_p^{(\mathbb{K})}}$-weakly symmetric continuous $\mathbb{K}$-valued polynomials on $\ell_p^{(\mathbb{K})}$.*

*Proof.* From the definition of a weakly symmetric function it follows that the algebra of all $\mathcal{G}_{2,\ell_p^{(\mathbb{K})}}$-weakly symmetric continuous $\mathbb{K}$-valued polynomials on $\ell_p^{(\mathbb{K})}$ is the union over $n \in \mathbb{N}$ of the algebras of $2^n$-block-symmetric continuous $\mathbb{K}$-valued polynomials on $\ell_p^{(\mathbb{K})}$. Therefore for every element $f$ of this algebra there exists $n \in \mathbb{N}$ such that $f$ is $2^n$-block-symmetric and, consequently, by Theorems 1 and 2, $f$ can be represented as an algebraic combination of elements of the set

$$\left\{F_{k,2^n,p,\mathbb{K}} \colon k \in \mathbb{Z}_+^n \text{ such that } \langle k \rangle \geq p\right\}. \tag{22}$$

Therefore, since the set (22) is the subset of the set (21), it follows that $f$ is an algebraic combination of elements of the set (21). So, every element of the algebra of all $\mathcal{G}_{2,\ell_p^{(\mathbb{K})}}$-weakly symmetric continuous $\mathbb{K}$-valued polynomials on $\ell_p^{(\mathbb{K})}$ is an algebraic combination of elements of the set (21). Thus, the set (21) is a generating system of this algebra. $\square$

The following theorem shows that the set (21) is not algebraically independent, therefore it is not an algebraic basis of the algebra of all $\mathcal{G}_{2,\ell_p^{(\mathbb{K})}}$-weakly symmetric continuous $\mathbb{K}$-valued polynomials on $\ell_p^{(\mathbb{K})}$.

**Theorem 4.** *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $p \in [1, +\infty)$. Let $n \in \mathbb{Z}_+$ and $m \in \mathbb{N}$ be such that $m > 1$. Let $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ be such that $\langle k \rangle \geq p$. Then*

$$F_{k,n,p,\mathbb{K}} = \sum_{s=1}^{m} F_{k^{(s)},mn,p,\mathbb{K}}, \tag{23}$$

*where for $s \in \{1, \ldots, m\}$*

$$k^{(s)} = (\underbrace{0, \ldots, 0}_{(s-1)n}, k_1, \ldots, k_n, \underbrace{0, \ldots, 0}_{(m-s)n}). \tag{24}$$

*Proof.* Let $x = (x_1, x_2, \ldots) \in \ell_p^{(\mathbb{K})}$. By (20),

$$F_{k^{(s)},mn,p,\mathbb{K}}(x) = \sum_{j=1}^{\infty} \prod_{\substack{v=1 \\ k_v^{(s)}>0}}^{mn} \left(x_{(j-1)mn+v}\right)^{k_v^{(s)}} \tag{25}$$

for $s \in \{1, \ldots, m\}$. Note that the series in the right-hand side of (25) is absolutely convergent. By (24),

$$k_v^{(s)} = \begin{cases} k_{v-(s-1)n}, & \text{if } v \in \{(s-1)n+1, \ldots, sn\}; \\ 0, & \text{otherwise} \end{cases}$$

for $s \in \{1, \ldots, m\}$ and $v \in \{1, \ldots, mn\}$. Therefore,

$$\prod_{\substack{v=1 \\ k_v^{(s)}>0}}^{mn} \left(x_{(j-1)mn+v}\right)^{k_v^{(s)}} = \prod_{\substack{v=(s-1)n+1 \\ k_{v-(s-1)n}>0}}^{sn} \left(x_{(j-1)mn+v}\right)^{k_{v-(s-1)n}}$$

for $j \in \mathbb{N}$ and $s \in \{1, \ldots, m\}$. By making the substitution $w = v - (s-1)n$, we obtain

$$\prod_{\substack{v=(s-1)n+1 \\ k_{v-(s-1)n}>0}}^{sn} \left(x_{(j-1)mn+v}\right)^{k_{v-(s-1)n}} = \prod_{\substack{w=1 \\ k_w>0}}^{n} \left(x_{(j-1)mn+(s-1)n+w}\right)^{k_w}.$$

From the previous two equalities, taking into account the equality

$$(j-1)mn + (s-1)n = ((j-1)m + s - 1)\, n = \left(\varkappa_m^{-1}((j,s)) - 1\right) n,$$

where $\varkappa_m$ is defined by (7), we obtain $\prod_{\substack{v=1 \\ k_v^{(s)}>0}}^{mn} \left(x_{(j-1)mn+v}\right)^{k_v^{(s)}} = \prod_{\substack{w=1 \\ k_w>0}}^{n} \left(x_{\left(\varkappa_m^{-1}((j,s))-1\right)n+w}\right)^{k_w}$ for $j \in \mathbb{N}$ and $s \in \{1, \ldots, m\}$. Therefore, by (25), $F_{k^{(s)},mn,p,\mathbb{K}}(x) = \sum_{j=1}^{\infty} \prod_{\substack{w=1 \\ k_w>0}}^{n} \left(x_{\left(\varkappa_m^{-1}((j,s))-1\right)n+w}\right)^{k_w}$ for $s \in \{1, \ldots, m\}$. Consequently,

$$\sum_{s=1}^{m} F_{k^{(s)},mn,p,\mathbb{K}}(x) = \sum_{s=1}^{m} \sum_{j=1}^{\infty} \prod_{\substack{w=1 \\ k_w>0}}^{n} \left(x_{\left(\varkappa_m^{-1}((j,s))-1\right)n+w}\right)^{k_w}. \tag{26}$$

By setting $n = m$ in (9) and (10), we get the definitions of the relations "$\prec$" and "$\hat{\precsim}$", resp., on $\mathbb{N} \times \{1, \ldots, m\}$. Let us rewrite the series in the right-hand side of (26) as follows

$$\sum_{s=1}^{m} \sum_{j=1}^{\infty} \prod_{\substack{w=1 \\ k_w>0}}^{n} \left( x_{\left(\varkappa_m^{-1}((j,s))-1\right)n+w} \right)^{k_w} = \sum_{(j,s)\in(\mathbb{N}\times\{1,\ldots,n\},\hat{\precsim})} \prod_{\substack{w=1 \\ k_w>0}}^{n} \left( x_{\left(\varkappa_m^{-1}((j,s))-1\right)n+w} \right)^{k_w}. \tag{27}$$

Since the series in the right-hand side of (25) is absolutely convergent, taking into account (26) and (27), it follows that the series in the right-hand side of (27) is absolutely, and, consequently, unconditionally convergent. Therefore,

$$\sum_{(j,s)\in(\mathbb{N}\times\{1,\ldots,m\},\hat{\precsim})} \prod_{\substack{w=1 \\ k_w>0}}^{n} \left( x_{\left(\varkappa_m^{-1}((j,s))-1\right)n+w} \right)^{k_w} = \sum_{(j,s)\in(\mathbb{N}\times\{1,\ldots,m\},\prec)} \prod_{\substack{w=1 \\ k_w>0}}^{n} \left( x_{\left(\varkappa_m^{-1}((j,s))-1\right)n+w} \right)^{k_w}. \tag{28}$$

By Lemma 2, the mapping $\varkappa_m$, defined by (7), is a strictly increasing bijection between ordered sets $(\mathbb{N}, <)$ and $(\mathbb{N} \times \{1, \ldots, m\}, \prec)$. Consequently,

$$\sum_{(j,s)\in(\mathbb{N}\times\{1,\ldots,m\},\prec)} \prod_{\substack{w=1 \\ k_w>0}}^{n} \left( x_{\left(\varkappa_m^{-1}((j,s))-1\right)n+w} \right)^{k_w} = \sum_{l\in(\mathbb{N},<)} \prod_{\substack{w=1 \\ k_w>0}}^{n} \left( x_{\left(\varkappa_m^{-1}(\varkappa_m(l))-1\right)n+w} \right)^{k_w} =$$
$$= \sum_{l=1}^{\infty} \prod_{\substack{w=1 \\ k_w>0}}^{n} \left( x_{(l-1)n+w} \right)^{k_w} = F_{k,n,p,\mathbb{K}}(x). \tag{29}$$

By (26), (27), (28), and (29), the equality (23) holds. □

Let us show that $\mathcal{G}_{2,\ell_1^{(\mathbb{K})}}$-weakly symmetric continuous functionals separate points of the space $\ell_1^{(\mathbb{K})}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

**Theorem 5.** *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For every $x, y \in \ell_1^{(\mathbb{K})}$ such that $x \neq y$ there exist $n \in \mathbb{Z}_+$ and $s \in \{1, \ldots, 2^n\}$ such that $F_{k^{(s,n)},2^n,1,\mathbb{K}}(x) \neq F_{k^{(s,n)},2^n,1,\mathbb{K}}(y)$, where $F_{k^{(s,n)},2^n,1,\mathbb{K}}$ is defined by (20) and*

$$k^{(s,n)} = (\underbrace{0, \ldots, 0}_{s-1}, 1, \underbrace{0, \ldots, 0}_{2^n-s}). \tag{30}$$

*Proof.* Suppose there exist $x, y \in \ell_1^{(\mathbb{K})}$ such that $x \neq y$ and

$$F_{k^{(s,n)},2^n,1,\mathbb{K}}(x) = F_{k^{(s,n)},2^n,1,\mathbb{K}}(y) \tag{31}$$

for every $n \in \mathbb{Z}_+$ and $s \in \{1, \ldots, 2^n\}$. Let $a = (a_1, a_2, \ldots) \in \ell_1^{(\mathbb{K})}$ be defined by

$$a = x - y. \tag{32}$$

Let us show that $a = 0$. For every $n \in \mathbb{Z}_+$ and $s \in \{1, \ldots, 2^n\}$, $\langle k^{(s,n)} \rangle = 1$, and, consequently, $F_{k^{(s,n)},2^n,1,\mathbb{K}}$ is a linear functional. Therefore, by (31) and (32),

$$F_{k^{(s,n)},2^n,1,\mathbb{K}}(a) = 0 \tag{33}$$

for every $n \in \mathbb{Z}_+$ and $s \in \{1, \ldots, 2^n\}$. Let $f \colon \mathbb{D} \to \mathbb{C}$ be defined by

$$f(z) = \sum_{m=1}^{\infty} a_m z^{m-1}, \tag{34}$$

where $\mathbb{D} = \{z \in \mathbb{C} \colon |z| \leq 1\}$. Since $a \in \ell_1^{(\mathbb{K})}$, the series $\sum_{m=1}^{\infty} |a_m|$ is convergent. Consequently, by [9, Theorem 2.4, p. 52], the series in the right-hand side of (34) converges absolutely and uniformly on $\mathbb{D}$. Therefore, by [9, Theorem 2.3, p. 51], $f$ is continuous on $\mathbb{D}$.

Let us show that

$$f\left(\exp\left(2\pi i l/2^q\right)\right)=0 \tag{35}$$

for every $q\in\mathbb{Z}_+$ and $l\in\{1,\ldots,2^q\}$. Let $q\in\mathbb{Z}_+$ and $l\in\{1,\ldots,2^q\}$. By (34),

$$f\left(\exp\left(2\pi i l/2^q\right)\right)=\sum_{m=1}^{\infty}a_m\exp\left(2\pi i l(m-1)/2^q\right). \tag{36}$$

By setting $n=2^q$ in (9) and (10), we get the definitions of the relations "$\prec$" and "$\hat{\precsim}$", resp., on $\mathbb{N}\times\{1,\ldots,2^q\}$. By Lemma 2, the mapping $\varkappa_{2^q}$, defined by (7), is a strictly increasing bijection between ordered sets $(\mathbb{N},<)$ and $(\mathbb{N}\times\{1,\ldots,2^q\},\prec)$. Consequently,

$$\sum_{m=1}^{\infty}a_m\exp\left(2\pi i l(m-1)/2^q\right)=\sum_{(j,s)\in(\mathbb{N}\times\{1,\ldots,2^q\},\prec)}b_{j,s}, \tag{37}$$

where

$$b_{j,s}=a_{\varkappa_{2^q}^{-1}((j,s))}\exp\left(2\pi i l\left(\varkappa_{2^q}^{-1}((j,s))-1\right)/2^q\right). \tag{38}$$

Since the series in the left-hand side of (37) is absolutely convergent, it follows that the series in the right-hand side of (37) is absolutely, and, consequently, unconditionally convergent. Therefore,

$$\sum_{(j,s)\in(\mathbb{N}\times\{1,\ldots,2^q\},\prec)}b_{j,s}=\sum_{(j,s)\in(\mathbb{N}\times\{1,\ldots,2^q\},\hat{\precsim})}b_{j,s}. \tag{39}$$

Taking into account (10),

$$\sum_{(j,s)\in(\mathbb{N}\times\{1,\ldots,2^q\},\hat{\precsim})}b_{j,s}=\sum_{s=1}^{2^q}\sum_{j=1}^{\infty}b_{j,s}. \tag{40}$$

For $(j,s)\in(\mathbb{N}\times\{1,\ldots,2^q\}$, by (7), one has $\varkappa_{2^q}^{-1}((j,s))=(j-1)2^q+s$. Therefore, $a_{\varkappa_{2^q}^{-1}((j,s))}=a_{(j-1)2^q+s}$ and

$$\exp\left(2\pi i l\left(\varkappa_{2^q}^{-1}((j,s))-1\right)/2^q\right)=\exp\left(2\pi i l\left((j-1)2^q+s-1\right)/2^q\right)=$$
$$=\exp\left(2\pi i l(j-1)+2\pi i l(s-1)/2^q\right)=\exp\left(2\pi i l(s-1)/2^q\right),$$

consequently, by (38), $b_{j,s}=a_{(j-1)2^q+s}\exp\left(2\pi i l(s-1)/2^q\right)$. Therefore,

$$\sum_{s=1}^{2^q}\sum_{j=1}^{\infty}b_{j,s}=\sum_{s=1}^{2^q}\exp\left(2\pi i l(s-1)/2^q\right)\sum_{j=1}^{\infty}a_{(j-1)2^q+s}. \tag{41}$$

By (20), $\sum_{j=1}^{\infty}a_{(j-1)2^q+s}=F_{k^{(s,q)},2^q,1,\mathbb{K}}(a)$ for every $s\in\{1,\ldots,2^q\}$, where $k^{(s,q)}$ is defined by (30). Consequently,

$$\sum_{s=1}^{2^q}\exp\left(2\pi i l(s-1)/2^q\right)\sum_{j=1}^{\infty}a_{(j-1)2^q+s}=\sum_{s=1}^{2^q}\exp\left(2\pi i l(s-1)/2^q\right)F_{k^{(s,q)},2^q,1,\mathbb{K}}(a). \tag{42}$$

By (33), $F_{k^{(s,q)},2^q,1,\mathbb{K}}(a)=0$ for every $s\in\{1,\ldots,2^q\}$. Therefore,

$$\sum_{s=1}^{2^q}\exp\left(2\pi i l(s-1)/2^q\right)F_{k^{(s,q)},2^q,1,\mathbb{K}}(a)=0. \tag{43}$$

By (36), (37), (39), (40), (41), (42), and the (43), equality (35) holds. Every $z\in\mathbb{C}$ such that $|z|=1$ can be approximated by elements of the set

$$\left\{\exp\left(2\pi i l/2^q\right):q\in\mathbb{Z}_+\text{ and }l\in\{1,\ldots,2^q\}\right\},$$

therefore, by (35), taking into account the continuity of $f$, $f(z)=0$. The function $f$ admits power series representation which converges in the unit disc. So, it is analytic inside the unit disc. Consequently, by the Maximum Modulus Principle, $f(z)\equiv 0$ for every $z\in\mathbb{D}$. Therefore, taking into account (34), $a=0$. So, by (32), $x=y$, which is a contradiction. □

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