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ANALOG OF THE WIMAN INEQUALITY FOR TAYLOR-DIRICHLET TYPE SERIES

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There are presented sufficient conditions for the Taylor-Dirichlet series of the form $F(x)=\sum_{n=0}^{+\infty} a_n e^{x\lambda_n+\tau(x)\beta_n}$, and $\lambda=(\lambda_n)$, $\beta=(\beta_n)$ be positive sequences, $\tau\colon \mathbb{R}_+\to\mathbb{R}_+$ be a differentiable function such that $\tau'(x)\geq 0$ $(x\geq x_0)$ providing validity of Wiman-type inequality outside some set E of finite Lebesgue measure. We prove the following statement: If for the sequences $\lambda=(\lambda_n)$, $\beta=(\beta_n)$ and a twice differentiable function $\tau\colon \mathbb{R}_+\to\mathbb{R}_+$ such that $\tau'(x)\geq 0$ $(x\geq x_0)$, $\tau''(x)\geq 0$ $(x\geq x_0)$ there exists a function $\psi\in\mathcal{L}_1$ that the condition

$$\alpha_0 := \sup_{x>0} \varlimsup_{u\to+\infty} \frac{\ln \nu\{n\geq 0\colon u-\sqrt{\psi(u)}<\lambda_n+\tau'(x)\beta_n\leq u+\sqrt{\psi(u)}\}}{\ln u}<+\infty$$

holds, where $\nu(D)=\#\{n\geq 0\colon \lambda_n+\tau'(x)\beta_n\in D\}$, then there exists a set E of finite Lebesgue measure such that for arbitrary $\delta>0$ and $x\in\mathbb{R}_+\setminus E$ we have

$$F(x)\leq \mu(x,F)(\ln\mu(x,F))^{\alpha_0+\delta}.$$

For every non-costant entire functions of the form $f(z)=\sum_{n=0}^{+\infty} a_n z^n$ and $r>0$ denote $\mu_f(r)=\max\{|a_n|r^n\colon n\geq 0\}$ and $M_f(r)=\max\{|f(z)|:\ |z|=r\}$. It is well known that (see [1–4], [5, Part IV, Ch. 1, § 3, Problem 56]) the (Wiman) inequality

$$M_f(r)\leq C\cdot\mu_f(r)\big(\ln\mu_f(r)\big)^{1/2+\varepsilon}$$

holds for $r\in[0,+\infty)\setminus E$, where $\varepsilon>0$ is arbitrary given and the set E has finite logarithmic mesure, i.e. ln-meas $E:=\int_{E\cap[1,+\infty)} d\ln r<+\infty$. Statements about analogues of this Wiman's theorem have been repeatedly proven for the entire Dirichlet series ([6–11]). In [6, p.71] it is proved that for every entire Dirichlet series $F\in\mathcal{D}(\lambda)$ of the form

$$F(z)=\sum_{n=0}^{+\infty} F_n\exp\{z\lambda_n\},\quad 0=\lambda_0<\lambda_n<\lambda_{n+1}\uparrow+\infty\ (1\leq n\uparrow+\infty),$$

whose exponents satisfy the condition $\lambda_{n+1}-\lambda_n\geq h>0$ $(n\geq 0)$, for each $\varepsilon>0$ and for all $x\in[0,+\infty)\setminus E$ (E is some set of finite Lebesgue measure) the following

$$M(x,F)\leq\mu(x,F)(\ln\mu(x,F))^{1/2+\varepsilon}$$

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holds; here $M(x,F)=\sup\{|F(x+iy)|\colon y\in\mathbb{R}\}$, $\mu(x,F)=\max\{|F_n|e^{x\lambda_n}\colon n\ge 0\}$. In the paper [9] we find the following statement: if the counting function $n(t)=\sum_{\lambda_k\le t}1$ of a sequence $\lambda=(\lambda_k)$ satisfy the condition

$$\varlimsup_{t\to+\infty}\frac{\ln\big(n(t+\sqrt{\psi(t)})-n(t-\sqrt{\psi(t)})\big)}{\ln t}\le p_1<+\infty \tag{1}$$

for some positive function $\psi\colon\mathbb{R}_+\to\mathbb{R}_+$ such that $\int_0^{+\infty}dt/\psi(t)<+\infty$, then for every entire Dirichlet series $F\in\mathcal{D}(\lambda)$ and any $\varepsilon>0$ there exists a set $E=E(\varepsilon,F)\subset[0,+\infty)$ of finite Lebesgue measure such that inequality

$$M(x,F)\le\mu(x,F)(\ln\mu(x,F))^{p_1+\varepsilon}$$

holds as $x\to+\infty$ ($x\notin E$). Previously, a similar statement was proven in [10] (see also [6, p. 62], [7,11]) with $p_1=\rho-\frac{1}{2}$, $\rho\ge\frac{1}{2}$, by the following condition

$$(\exists\triangle,D\in(0,+\infty))(\exists\varrho\in[1/2,1])(\exists t_0>0)(\forall t>t_0)\colon|n(t)-\triangle t^{\varrho}|\le D. \tag{2}$$

We note that condition (2) implies condition (1) with $p_1=\rho-\frac{1}{2}$.

A similar result was also obtained for the Laplace-Stieltjes integrals in [12, Theorem 1].

Let $\mathcal{TD}(\lambda,\beta,\tau)$ be the class of positive convergent for all $x\ge 0$ series of the form

$$F(x)=\sum_{n=0}^{+\infty}a_ne^{x\lambda_n+\tau(x)\beta_n},$$

and $\lambda=(\lambda_n)$, $\beta=(\beta_n)$ be positive sequences, $\tau\colon\mathbb{R}_+\to\mathbb{R}_+$ be a differentiable function such that $\tau'(x)\ge 0$ ($x\ge x_0$). The following statement is well known.

Theorem (see [17,18])**.** *If for a sequence $\lambda=(\lambda_n)$ such that $0=\lambda_0<\lambda_n\uparrow+\infty$ ($1\le n\uparrow+\infty$), the condition*

$$\sum_{n=1}^{+\infty}\frac{1}{n\lambda_n}<+\infty \tag{3}$$

holds, then for every function $F\in\mathcal{TD}(\lambda,\beta,\tau)$ there exists a set E of finite Lebesgue measure such that the Borel relation

$$\ln F(x)=(1+o(1))\ln\mu(x,F) \tag{4}$$

holds as $x\to+\infty$ outside some set E of finite Lebesgue measure; here

$$\mu(x,F)=\max\big\{a_ne^{x\lambda_n+\tau(x)\beta_n}\colon n\ge 0\big\}.$$

Remark ([15,16]), condition (3) is also sufficient and necessary in order that Borel relation (4) holds for each entire Dirichlet series $F\in\mathcal{D}(\lambda)$ as $x\to+\infty$ outside of some set E of finite Lebesgue measure. The statements just cited show the complete similarity of the conditions that ensure the fulfillment of the Borel relation outside a set of finite Lebesgue measure for both entire Dirichlet series and Taylor-Dirichlet series. Similar result was also obtained for the series by a system of functions in [13,14].

We note that analogues of the Wiman inequality for Taylor-Dirichlet series are currently completely absent in the literature. The purpose of this paper is to attempt to fill this gap regarding analogues of the Wiman inequality for Taylor-Dirichlet series.

We denote by $\mathcal{L}$ the class of positive continuous functions $\Phi\colon \mathbb{R}_+ := [0, +\infty) \to \mathbb{R}_+$ such that $\Phi(x) \nearrow +\infty$ $(0 \le x \to +\infty)$. Denote by $\mathcal{L}_1$ the class of function $\psi \in \mathcal{L}$ such that $\int_{x_0}^{+\infty} dt/\psi(t) < +\infty$ for some $t_0 > 0$.

We prove the following theorem

Theorem 1. *If for the sequences $\lambda = (\lambda_n)$, $\beta = (\beta_n)$ and a twice differentiable function $\tau\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\tau'(x) \ge 0$ $(x \ge x_0)$, $\tau''(x) \ge 0$ $(x \ge x_0)$ there exists a function $\psi \in \mathcal{L}_1$ that the condition*

$$\alpha_0 := \sup_{x>0} \varlimsup_{u\to+\infty} \frac{\ln \nu\big\{n \ge 0\colon u - \sqrt{\psi(u)} < \lambda_n + \tau'(x)\beta_n \le u + \sqrt{\psi(u)}\big\},}{\ln u} < +\infty \tag{5}$$

holds, where $\nu(D) = \#\{n \ge 0\colon \lambda_n + \tau'(x)\beta_n \in D\}$, then for every function $F \in \mathcal{TD}(\lambda, \beta, \tau)$ there exists a set E of finite Lebesgue measure such that for arbitrary $\delta > 0$ and $x \in \mathbb{R}_+ \setminus E$ we have

$$F(x) \le \mu(x, F)(\ln \mu(x, F))^{\alpha_0+\delta}.$$

For the proof we need the following lemma.

Lemma 1 ([16],Lemma 1)**.** *Let $\psi \in \mathcal{L}_1$, and $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable non-negative and non-decreasing function. Then the set*

$$E = \{x \ge 0\colon g'(x) \ge \psi(g(x))\}$$

has finite Lebesgue measure.

Proof of Theorem 1. We consider the functions $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ represented for all $x \ge 0$ by integrals of the form

$$F(x) = \int_0^{+\infty} a(t) \exp\{tx + \beta(t)\tau(x)\}\nu(dt),$$

where $\nu(dt)$ is Borel measure, $\beta(t), a(t)$ are the positive Borel measurable functions, and a differentiable function $\tau\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\tau'(x) \ge 0$ $(x \ge x_0)$.

For a given function $\psi \in \mathcal{L}_1$ and $x > 0$ we denote

$$\Delta(u, x) := \nu\big\{t > 0\colon u - \sqrt{\psi(u)} < t + \tau'(x)\beta(t) \le u + \sqrt{\psi(u)}\big\},$$

and also

$$\alpha(x) = \varlimsup_{u\to+\infty} \frac{\ln \Delta(u, x)}{\ln u}.$$

We suppose that $\alpha := \sup\{\alpha(x)\colon x \ge 0\} < +\infty$. Without loss of generality, we can assume that

$$(\forall x > 0)(\forall u \ge 1)\colon\ \frac{\ln \Delta(u, x)}{\ln u} \le \alpha_0 < +\infty.$$

Denote $g(x) = \ln F(x)$. Let

$$P_x(dt) = \frac{a(t)e^{xt+\tau(x)\beta(t)}}{F(x)}\nu(dt)$$

be probability measure on the σ-algebra of Borel sets on $\mathbb{R}_+$. Then for the mean of random variable $\xi = t + \tau'(x)\beta(t)$ we get

$$\mathbf{M}\xi = \int_0^{+\infty} \xi P_x(dt) = \int_0^{+\infty} a(t)e^{xt+\tau(x)\beta(t)} \frac{(t+\tau'(x)\beta(t))}{F(x)}\nu(dt) = g'(x),$$

and also in the case when $\tau''(x)$ exist for all $x > 0$,

$$\mathbf{M}\xi^2 = \int_0^{+\infty} \xi^2 P_x(dt) = \int_0^{+\infty} a(t)e^{xt+\tau(x)\beta(t)} \frac{(t+\tau'(x)\beta(t))^2}{F(x)}\nu(dt) = \frac{F''(x)}{F(x)} - \tau''(x),$$

therefore,

$$D\xi = M\xi^2 - (M\xi)^2 = \frac{F''(x)}{F(x)} - \tau''(x) - \Big(\frac{F'(x)}{F(x)}\Big)^2 = g''(x) - \tau''(x)$$

in the case $\tau''(x) \geq 0$ we obtain

$$D\xi \leq g''(x).$$

Apply now the Tschebyschov inequality at $a = \sqrt{2D\xi}$

$$\frac{1}{F(x)} \int\limits_{|\xi - M\xi| \geq a} a(t)e^{tx+\beta(t)\tau(x)}\nu(dt) = \int\limits_{|\xi - M\xi| \geq a} P_x(dt) = P\{|\xi - M\xi| \geq a\} \leq D\xi/a^2 = \frac{1}{2}.$$

Then, we get

$$F(x) = F(x)\Big(\int_{|\xi - M\xi|>a} + \int_{|\xi - M\xi| \leq a}\Big)P_x(dt) \leq \frac{F(x)}{2} + F(x)\int_{|\xi - M\xi| \leq a} P_x(dt).$$

So,

$$F(x) \leq 2F(x)\int_{|\xi - M\xi| \leq a} P_x(dt) = 2\int_{|\xi - M\xi| \leq \sqrt{2D\xi}} a(t)e^{tx+\beta(t)\tau(x)}\nu(dt).$$

Using Lemma 1 to function g' with the function $\psi(x)/2$ instead the function $\psi(x)$, for all x outside of some set finite Lebesgue measure we obtain

$$F(x) \leq 2\int_{|t+\tau'(x)\beta(t)\xi - g'(x)| \leq \sqrt{2g''(x)}} a(t)e^{xt+\tau(x)\beta(t)} P_x(dt) \leq 2\mu(x, F)\times$$
$$\times\nu\big\{t > 0\colon u - \sqrt{\psi(u)} < t + \tau'(x)\beta(t) \leq u + \sqrt{\psi(u)}\big\}\Big|_{u=g'(x)} \leq 2(g'(x))^{\alpha_0}\mu(x, F). \qquad (6)$$

Applying Lemma 1 to the function $g(x)$ with the function $\psi(x) = x\ln^{1+\varepsilon} x$, we get

$$g'(x) \leq g(x)\ln^{1+\varepsilon} g(x) \qquad (7)$$

and $\ln g'(x) \leq (1+\varepsilon)\ln g(x) = o(g(x))$ as $x \to +\infty$ outside a set of finite Lebesgue measure.

Therefore, applying inequality (7) to the inequality (6) we obtain

$$F(x) \leq 2\mu(x, F)(g(x))^{\alpha_0} \ln^{\alpha_0(1+\varepsilon)} g(x)$$

as $x \to +\infty$ outside a set of finite Lebesgue measure. Hence, finally, for arbitrary $\delta > 0$ we have

$$F(x) \leq \mu(x, F)(\ln\mu(x, F))^{\alpha_0+\delta}$$

as $x \to +\infty$ outside a set of finite Lebesgue measure.

□

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