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SOME PROPERTIES OF FOURIER QUASICRYSTALS AND MEASURES ON A STRIP

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We extend certain results of the theory of Fourier quasicrystals on the real line to the case of a horizontal strip of finite width. We define the Fourier transform for measures on a strip that is a natural generalization of the Fourier transform for measures on the line. For positive or translation bounded measures μ on a strip with the Fourier transform of the form $\hat{\mu} = \sum_{\gamma\in\Gamma} b_\gamma \delta_\gamma$ we prove that the measure $\nu = \sum_{\gamma\in\Gamma} |b_\gamma|^2 \delta_\gamma$ has the exponential growth. If for some $\eta > 0$ the points of Γ in every interval of length η are linearly independent over integers then the measure $\hat{\mu}$ also has the exponential growth.

1. Introduction. A complex measure μ on a Euclidean space with discrete locally finite support is called *crystalline*, if μ is a tempered distribution and its Fourier transform in the sense of distributions $\hat{\mu}$ is also a measure with discrete locally finite support; μ is called a *Fourier quasicrystal* if both measures $|\mu|$ and $|\hat{\mu}|$ are also tempered distributions. Here and below $|\nu|(E)$ denotes the variation of the complex measure ν on the set E.

Fourier quasicrystals and crystalline measures have been studied very actively in the recent years. Many works are devoted to investigations their properties (see, for example, the papers [1, 11, 15]). In fact, Fourier quasicrystals are forms of Poisson formulas, which were used, in particular, by D. Radchenko and M. Viazovska in [17].

The first nontrivial example of Fourier quasicrystals, which are sums of unit masses

$$\mu = \sum_{\lambda\in\Lambda} \delta_\lambda, \quad \lambda \in \mathbb{R}, \tag{1}$$

was found by P. Kurasov and P. Sarnak [10]. In [16] A. Olevskii and A. Ulanovskii established a 1-1 connection between such Fourier quasicrystals and sets of zeros of real-rooted exponential polynomials. In [5] this result was generalized to the zero sets of real-rooted absolutely convergent Dirichlet series. A multidimensional version of the result by Olevskii and Ulanovskii was partially obtained in papers [1] and [15], whereas the case of an arbitrary measure on the line whose Fourier transform is a pure point measure was fully investigated in papers [8] and [9].

Measures of the form (1) with support Λ on a horizontal strip of finite width appeared in [6], where an analog of Olevskii and Ulanovskii's result was found. Also in [7], the technique

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of Fourier quasicrystals on a strip was used to provide a criterion for an entire almost periodic function to be a finite product of sines.

Returning to the definition of a Fourier quasicrystal, it is natural to ask whether we need additional conditions on the variations of the measures μ and $\hat{\mu}$ in order to prove that a crystalline measure is a Fourier quasicrystal. The answer is positive; in the paper [4] a crystalline measure on $\mathbb{R}$ was constructed that is not a Fourier quasicrystal. In the paper [3] some conditions were found for a crystalline measure to be a Fourier quasicrystal. In particular, it was proved that if a measure μ on $\mathbb{R}$ is positive or translation bounded and its Fourier transform $\hat{\mu}$ is a pure point measure

$$\hat{\mu} = \sum_{\gamma\in\Gamma} b_\gamma \delta_\gamma, \tag{2}$$

then the measure $\nu = \sum_{\gamma\in\Gamma} |b_\gamma|^2\delta_\gamma$ is also translation bounded, and if all points of Γ in every interval of length η are linearly independent over $\mathbb{Z}$, then $\hat{\mu}$ is also translation bounded.

The set Γ in (2) is called *spectrum* of the measure μ. Recall also that a measure on $\mathbb{R}$ is *translation bounded* if its variations on intervals of length 1 are uniformly bounded.

In our paper we show that analogs of these results do not valid for measures on a horizontal strip of finite width. Nevertheless we prove that for non-negative or translation bounded measures μ on a strip the measures ν as above and the measure $\hat{\mu}$ in the case of locally independent set Γ have at most exponential growth. The precise formulations are presented in Section 2 after giving necessary definitions. Here we also present an analogue of the Poisson formula. An example of calculating the Fourier transform of a measure on a strip is given in Section 3. In Sections 4 and 5, we give some auxiliary results. In Section 6, we present the proofs of the main theorems.

2. Necessary notions and formulations of the main results. Let μ be a measure with support on $\mathcal{S}_H := \{z \in \mathbb{C} \colon |\mathrm{Im}\, z| \le H\}$. Denote

$$M_\mu(r) := |\mu|(\{z \in \mathcal{S}_H \colon |z| \le r\}).$$

Suppose that

$$M_\mu(r) \le C_1 \max\{1, r^\rho\}, \tag{3}$$

with some positive constants $C_0,\ \rho$. We will say that the measure μ is *translation bounded* on $\mathcal{S}_H$ if

$$\sup_{s\in\mathbb{R}} |\mu|(\{z \in \mathcal{S}_H \colon s \le \mathrm{Re}\, z \le s+1\}) = C_2 < \infty. \tag{4}$$

It is easy to check that inequality (3) with $\rho = 1,\ C_1 = 4C_2$ is valid for translation bounded measures.

Let a function φ belongs to the space $\mathcal{D}$ of all C^∞-functions in $\mathbb{R}$ with compact support. Denote by

$$\hat{\varphi}^c(z) := \int_{\mathbb{R}} \varphi(t) e^{-2\pi i z t} dt, \qquad z \in \mathbb{C} \tag{5}$$

the entire extension of the usual Fourier transform

$$\hat{\varphi}(x) := \int_{\mathbb{R}} \varphi(t) e^{-2\pi i x t} dt, \qquad x \in \mathbb{R}.$$

Clearly,

$$\hat{\varphi}^c(z) = (\widehat{\varphi(t) e^{2\pi y t}})(x), \quad z = x + iy. \tag{6}$$

It follows from the definition (5) that for all $m \in \mathbb{N}$ uniformly with respect to $|y| \le H$

$$|\hat{\varphi}^c(z)| \le C_3(\max\{1,|z|\})^{-m}. \tag{7}$$

Let a measure μ on $\mathcal{S}_H$ satisfy (3). Integrating by parts, we get

$$\int_{z\in\mathcal{S}_H} |\hat{\varphi}^c(z)||\mu|(dz) \le C_3M_\mu(1) + C_3\int_{|z|>1}|z|^{-m}|\mu|(dz) \le C_3m\int_1^\infty r^{-m-1}M_\mu(r)dr. \tag{8}$$

Therefore, for $m>\rho$ the integral

$$\int_{z\in\mathcal{S}_H}\hat{\varphi}^c(z)\mu(dz) \tag{9}$$

converges absolutely. So, *the c-Fourier transform* of a measure μ is well-defined by the equality

$$(\hat{\mu}^c,\varphi) = \int \hat{\varphi}^c(z)\mu(dz). \tag{10}$$

It is easy to see that the definition of (5) generalizes to all functions φ on $\mathbb{R}$ with compact support that have bounded derivatives up to order $m>\rho$. Consequently, (7) and (10) are also valid for such functions. In the next section we will show that the c-Fourier transform of the measure $\mu_0=\sum_{n\in\mathbb{Z}}\delta_{n+i}$ is the measure $\hat{\mu}_0^c=\sum_{n\in\mathbb{Z}}e^{2\pi n}\delta_n$. Clearly, the exact analog of results in [3] is incorrect. To formulate our generalization of these results we recall the following definition ([12, Ch.1]):

A measure ν on $\mathbb{R}$ has *a type σ with respect to the exponential growth* if

$$\overline{\lim_{r\to\infty}}\, r^{-1}\log\nu(-r,r)=\sigma.$$

In our paper we obtain the following results:

Theorem 1. *If a positive measure μ on $\mathcal{S}_H$ satisfies (3) and its c-Fourier transform $\hat{\mu}^c$ is also a measure, then μ is translation bounded.*

Theorem 2. *Let μ be a non-negative measure on $\mathcal{S}_H$ satisfying (3), and its Fourier transform $\hat{\mu}^c$ be a pure point measure of the form (2). Then the measure*

$$\nu=\sum_{\gamma\in\Gamma}|b_\gamma|^2\delta_\gamma \tag{11}$$

has the property

$$\nu(t-1,t+1)=O(e^{4\pi H|t|}) \qquad \text{as} \quad t\to\infty, \tag{12}$$

and has the type with respect to the exponential growth at most $4\pi H$. The same statement is true for any translation bounded measure μ on $\mathcal{S}_H$ with a pure point measure $\hat{\mu}^c$.

Theorem 3. *Let μ be a non-negative measure on $\mathcal{S}_H$ satisfying (3), and its Fourier transform $\hat{\mu}^c$ be a pure point measure of the form (2). If there exists $\eta>0$ such that the sets $\Gamma\cap(t-\eta,t+\eta)$ are linearly independent over $\mathbb{Z}$ for all $t\in\mathbb{R}$, then the measure $\hat{\mu}^c$ has the property*

$$|\hat{\mu}^c|(t-1,t+1)=O(e^{2\pi H|t|}) \qquad \text{as} \quad t\to\infty, \tag{13}$$

and has the type with respect to the exponential growth at most $2\pi H$. The same statement is true for any translation bounded measure μ on $\mathcal{S}_H$ with a pure point measure $\hat{\mu}^c$ satisfying the condition of the theorem.

Recall that the set $X \subset \mathbb{R}$ is *linearly independent over* $\mathbb{Z}$ if for any finite number of elements $t_1, \dots, t_n \in X$ and arbitrary $m_1, \dots, m_n \in \mathbb{Z}$ the equality $m_1 t_1 + \dots + m_n t_n = 0$ implies $m_1 = \dots = m_n = 0$.

Note that the condition on Γ is satisfied in the case $|t - t'| > \eta$ for all $t,\, t' \in \Gamma,\ t' \neq t$. Thus, the above example of the measure μ_0 shows that estimates (12) and (13) are optimal.

In conclusion, note that in the case $\mu = \sum_{\lambda\in\Lambda} a_\lambda \delta_\lambda,\ \Lambda \subset \mathcal{S}_H,\ \hat{\mu}^c = \sum_{\gamma\in\Gamma} b_\gamma \delta_\gamma,\ \Gamma \subset \mathbb{R}$, we obtain an analogue of the Poisson formula $\sum_{\gamma\in\Gamma} b_\gamma \varphi(\gamma) = \sum_{\lambda\in\Lambda} a_\lambda \hat{\varphi}^c(\lambda)$, here $\hat{\varphi}^c$ is defined in (5).

3. The example of c-Fourier transform for a measure on a strip. Let us find the c-Fourier transform of the measure $\mu_0 = \sum_{n\in\mathbb{Z}} \delta_{n+i}$ on the strip $\mathcal{S}_1 = \{z\colon |\mathrm{Im}\, z| \le 1\}$.

Let $\varphi \in \mathcal{D}$ be a test function. Since the measure μ_0 satisfy (3) and the integral (9) with $\mu = \mu_0$ converges absolutely, we can change the order of integration and summation and get

$$(\hat{\mu}_0^c, \varphi) = \int_1 \hat{\varphi}^c(z)\mu_0(dz) = \sum_{n\in\mathbb{Z}} \int_{\mathcal{S}_1} \hat{\varphi}^c(z)\delta_{n+i} = \sum_{n\in\mathbb{Z}} \hat{\varphi}^c(n+i).$$

It follows from (6) that $\hat{\varphi}^c(n+i) = (\widehat{\varphi(t)e^{2\pi t}})(n)$, therefore, according to the classical Poisson formula,

$$(\hat{\mu}_0^c, \varphi) = \sum_{n\in\mathbb{Z}} (\widehat{\varphi(t)e^{2\pi t}})(n) = \sum_{n\in\mathbb{Z}} \varphi(n)e^{2\pi n} = \sum_{n\in\mathbb{Z}} e^{2\pi n}\delta_n(\varphi),$$

and $\hat{\mu}_0^c = \sum_{n\in\mathbb{Z}} e^{2\pi n}\delta_n$.

4. Some properties of the c-Fourier transform. The next three equalities for all $\varphi \in \mathcal{D}$ and μ satisfying (3) follow immediately from the definition of c-Fourier transform:

$$(\widehat{e^{2\pi i\tau t}\varphi(t)})^c = (\hat{\varphi}^c)(z - \tau), \qquad z \in \mathcal{S}_H, \quad \tau \in \mathbb{R}, \tag{14}$$

$$(\hat{\varphi}_\tau^c)(z) = e^{-2\pi i\tau z}\hat{\varphi}^c(z) \quad \text{for} \quad \varphi_\tau(t) = \varphi(t - \tau), \qquad z \in \mathcal{S}_H, \quad \tau \in \mathbb{R}, \tag{15}$$

$$(\widehat{\mu_\tau})^c(t) = e^{-2\pi i\tau t}\hat{\mu}^c(t) \quad \text{for} \quad \mu_\tau(E) = \mu(E - \tau), \qquad \tau \in \mathbb{R}. \tag{16}$$

Since for $s \in \mathbb{R},\ z \in \mathcal{S}_H$

$$|e^{2\pi isz}\hat{\varphi}^c(z)| \le e^{2\pi|s|H}|\hat{\varphi}^c(z)| \le C_3 e^{2\pi|s|H}(\max\{1, |z|\})^{-m},$$

we see that the same arguments as in (8) prove the estimate

$$\left| \int_{\mathcal{S}_H} e^{2\pi isz}\hat{\varphi}^c(z)\mu(dz) \right| \le C_4 e^{2\pi|s|H}, \tag{17}$$

where the constant C_4 does not depend on s.

If the measure μ is translation bounded, then for any shift of the measure $\mu_\tau(E) = \mu(E - \tau)$ we have

$$|\mu_\tau|(\{z \in \mathcal{S}_H\colon |z| \le r\}) \le 4C_2 \max\{1, r\},$$

where C_2 is the constant from (4). We obtain from (17) with $m = 2$

$$\left| \int_{\mathcal{S}_H} e^{2\pi isz}\hat{\varphi}^c(z)\mu_\tau(dz) \right| \le C_5 e^{2\pi|s|H}, \tag{18}$$

where the constant C_5 does not depend on either s or τ.

We will also use the following proposition.

Proposition. *For any $m\in\mathbb{N}$ there is a function $\Psi_m(t)$ with compact support such that its Fourier transform $\hat\Psi_m^c(z)$ satisfies* (7) *and* $Re\,\hat\Psi^c_m(z)>0$ *for all* $z\in\mathcal{S}_H$.

Proof. Set $\theta(t)=1/2$ for $|t|\le 1$ and $\theta(t)=0$ for $|t|>1$. The c-Fourier transform of the function $\theta\star\theta$ equals $\sin^2(2\pi z)/(2\pi z)^2$, therefore it is non-negative for all $z\in\mathbb{R}$ and vanishes only at the points $z=k/2,\ k\in\mathbb{Z}\setminus\{0\}$. Consider the function
$$\psi(t)=e^{i\pi t/4}(\theta\star\theta)(t)+e^{-i\pi t/4}(\theta\star\theta)(t).$$
It has compact support and c-Fourier transform
$$\hat\psi^c(z)=\sin^2(2\pi z-\pi/4)/(2\pi z-\pi/4)^2+\sin^2(2\pi z+\pi/4)/(2\pi z+\pi/4)^2,$$
that is strictly positive on the real axis. The simple calculation shows that
$$\hat\psi^c(z)=\frac{i}{4}\left(e^{4\pi iz}-e^{-4\pi iz}\right)\left[\frac{1}{(2\pi z-\pi/4)^2}-\frac{1}{(2\pi z+\pi/4)^2}\right]+$$
$$+\frac14\left\{\frac{2}{(2\pi z-\pi/4)^2}+\frac{2}{(2\pi z+\pi/4)^2}\right\}.$$
The expression in the square brackets behaves like $O(z^{-3})$, and the expression in the braces behaves like $(\pi z)^{-2}$ as $z\to\infty$. Hence for $z\in\mathcal{S}_H,\ z\to\infty$,
$$\hat\psi^c(z)=O(z^{-3})+\frac{1+o(1)}{4\pi^2z^2}=\frac{1}{4\pi^2(\mathrm{Re}\,z)^2}(1+o(1)).$$
Define the convolution of $n\ge m/2$ functions $\psi(t)$ with $\Phi(t)$. Clearly, this function has compact support. Also, its c-Fourier transform
$$\hat\Phi^c(z)=\left[\sin^2(2\pi z-\pi/4)/(2\pi z-\pi/4)^2+\sin^2(2\pi z+\pi/4)/(2\pi z+\pi/4)^2\right]^n$$
satisfies (7) and strictly positive on the real axis. Therefore $\mathrm{Re}\,\hat\Phi^c(z)$ is strictly positive in a neighborhood of $\mathbb{R}$. We also have for $z\in\mathcal{S}_H,\ z\to\infty$
$$\hat\Phi^c(z)=\frac{1}{4^n\pi^{2n}(\mathrm{Re}\,z)^{2n}}(1+o(1)),$$
hence there is $N<\infty$ such that $\mathrm{Re}\,\hat\Phi^c(z)>0$ for $z\in\mathcal{S}_H,\ |\mathrm{Re}\,z|\ge N$. On the other hand, there is $\eta>0$ such that $\mathrm{Re}\,\hat\Phi^c(z)>0$ for $|\mathrm{Im}\,z|\le\eta,\ |\mathrm{Re}\,z|\le N$. Consequently, for the compactly supported function $\Psi_m(t):=(H/\eta)\Phi(Ht/\eta)$, for all $z\in\mathcal{S}_H$ we obtain
$$\mathrm{Re}\,\hat\Psi^c_m(z)=\mathrm{Re}\,\hat\Phi^c(\eta z/H)>0.$$
□

5. Some properties of almost periodic functions. The proofs of our theorems are based on some results of the classical theory of almost periodic functions (see [2], [13]).

Recall that a continuous function $f(t)$ on the real line $\mathbb{R}$ is called *almost periodic* if for any $\varepsilon>0$ the set of its ε-almost periods
$$E_\varepsilon=\left\{\tau\in\mathbb{R}\colon\ \sup_{t\in\mathbb{R}}|f(t+\tau)-f(t)|<\varepsilon\right\}$$
is relatively dense, i.e., $E_\varepsilon\cap(t,t+L)\ne\varnothing$ for all $t\in\mathbb{R}$ and some $L=L(\varepsilon)$.

Each almost periodic function on $\mathbb{R}$ is bounded, a finite sum or product of almost periodic functions is also almost periodic, and the uniform limit on $\mathbb{R}$ of almost periodic functions is almost periodic too. In particular, every absolutely convergent Dirichlet series
$$D(t)=\sum_{\omega\in\Omega}a_\omega e^{2\pi it\omega},\qquad\sum_{\omega\in\Omega}|a_\omega|<\infty,\tag{19}$$
with a countable $\Omega\subset\mathbb{R}$ is almost periodic. It is easy to check that the usual Fourier transform of $D(t)$ is a measure with a finite total mass $\hat D=\sum_{\omega\in\Omega}a_\omega\delta_\omega$.

For every almost periodic $f(t)$ and every $\omega \in \mathbb{R}$ one considers the Fourier coefficients $c_\omega(f) \in \mathbb{C}$ as

$$c_\omega(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-2\pi i t\omega} dt.$$

The set $\{\omega\colon c_\omega(f) \neq 0\}$ is at most countable. It is easy to check that the uniform convergence of the series (19) implies $c_\omega(D) = a_\omega$ for all $\omega \in \mathbb{R}$. Next, the Parseval identity

$$\sum_{\omega\in\mathbb{R}} |c_\omega(f)|^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt \tag{20}$$

holds.

Note that by Y. Meyer (cf. [14, Theorem 3.8]), if the Fourier transform of the measure $f(t)dt$ for an almost periodic function f is also a measure, then

$$\sum_{|\omega|<r} |c_\omega(f)| < \infty, \quad \forall\, r < \infty, \qquad \text{and} \quad \widehat{f(t)dt} = \sum_{\omega\in\Omega} c_\omega(f)\delta_\omega.$$

We will also use the Kroneker lemma (see [13, Chapter II]):

Let $x_1, \ldots, x_N$ be linearly independent over $\mathbb{Z}$ real numbers and $\theta_1, \ldots, \theta_N$ be arbitrary real numbers. Then for every $\varepsilon > 0$ there exist $t \in \mathbb{R}$ and $p_j \in \mathbb{Z}$, $j \in \{1, \ldots, N\}$, such that

$$|tx_j - \theta_j - p_j| < \varepsilon, \quad j \in \{1, \ldots, N\}.$$

6. Proofs of theorems. *Proof of Theorem 1.* Let the measure $\mu \geq 0$ be satisfied (3) with some $\rho < \infty$, and Ψ_m be the function from the Proposition with $m > \rho$. Set $\eta = \inf_{z\in\mathcal{S}_H\colon 0\le \operatorname{Re} z\le 1} \operatorname{Re} \hat\Psi^c_m(z)$. Using (16), we get for any fixed $s \in \mathbb{R}$

$$\int_{z\in\mathcal{S}_H\colon s\le \operatorname{Re} z\le s+1} \mu(dz) \le \eta^{-1} \int_{z\in\mathcal{S}_H\colon 0\le \operatorname{Re} z\le 1} \operatorname{Re} \hat\Psi^c_m(z)\mu_{-s}(dz) \le \eta^{-1} \int_{\mathcal{S}_H} \operatorname{Re} \hat\Psi^c_m(z)\mu_{-s}(dz) =$$

$$= \eta^{-1}\operatorname{Re} \int_{\mathcal{S}_H} \hat\Psi^c_m(z)\mu_{-s}(dz) \le \eta^{-1} \left| \int_{\mathcal{S}_H} \hat\Psi^c_m(z)\mu_{-s}(dz) \right| = \eta^{-1} \left| \int_{\mathbb{R}} \Psi_m(t)\hat\mu^c_{-s}(dt) \right| =$$

$$\eta^{-1} \left| \int_{\mathbb{R}} \Psi_m(t) e^{2\pi i s t} \hat\mu^c(dt) \right| \le \eta^{-1} \int_{\mathbb{R}} |\Psi_m(t)||\hat\mu^c|(dt).$$

Since Ψ_m is compactly supported and $|\hat\mu^c|$ is a measure, we see that the last integral is finite, and the first integral is bounded by the independent on s constant. □

Proof of Theorem 2. By Theorem 1, it is enough to prove Theorem 2 for any translation bounded measure μ.

Let $\varphi \in \mathcal{D}$ be a non-negative even function such that $\varphi(t) \equiv 1$ for $|t| < 1$. Note that $\hat\varphi^c(z)$ is also even, and for any $\tau \in \mathbb{R}$: $\hat\varphi^c(\tau - z) = \hat\varphi^c(z - \tau)$.

Using (14) and the definition of the Fourier transform of measure on $\mathcal{S}_H$, we get

$$(\mu \star \hat\varphi^c)(\tau) = \int_{z\in\mathcal{S}_H} \hat\varphi^c(\tau - z)\mu(dz) = \int_{z\in\mathcal{S}_H} (\widehat{e^{2\pi i\tau t}\varphi(t)})^c(z)\mu(dz) =$$

$$= \int_{\mathbb{R}} \varphi(t) e^{2\pi i\tau t} \hat\mu^c(dt) = \sum_{\gamma\in\operatorname{supp}\varphi} b_\gamma \varphi(\gamma) e^{2\pi i\tau\gamma}$$

for every $\tau \in \mathbb{R}$. Since $\hat\mu^c$ is a measure, we see that $\sum_{\gamma\in\operatorname{supp}\varphi} |b_\gamma| < \infty$ and the last sum converges absolutely. Therefore, $(\mu \star \hat\varphi^c)(\tau)$ is an almost periodic function on $\mathbb{R}$. The above equality also implies (see Section 5) that the usual Fourier transform of this function is a measure

$$\widehat{\mu \star \hat\varphi^c} = \sum_{\gamma\in\operatorname{supp}\varphi} \varphi(\gamma) b_\gamma \delta_\gamma.$$

By Meyer's result, the Fourier coefficients of the function $(\mu\star\hat\varphi^c)(\tau)$ are
$$c_\gamma(\mu\star\hat\varphi^c)=b_\gamma\varphi(\gamma).$$
Using Parseval's equality (20), we obtain that mass of the measure ν from (11) on $[-1,1]$ can be estimated as follows
$$\nu[-1,1]=\sum_{|\gamma|<1}|b_\gamma|^2\le\sum_{\gamma\in\mathbb{R}}|\varphi(\gamma)b_\gamma|^2=\sum_{\gamma\in\mathbb{R}}|c_\gamma(\mu\star\hat\varphi^c)|^2$$
$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|(\mu\star\hat\varphi^c)(\tau)|^2d\tau\le\sup_{\tau\in\mathbb{R}}|(\mu\star\hat\varphi^c)(\tau)|^2=\sup_{\tau\in\mathbb{R}}\Big|\int_{\mathcal{S}_H}\hat\varphi^c(z)\mu_{-\tau}(dz)\Big|^2.$$
The measures μ is translation bounded, hence it follows from (18) with $s=0$ that
$$\Big|\int_{\mathcal{S}_H}\hat\varphi^c(z)\mu_{-\tau}(dz)\Big|\le C_5.$$
If we replace $\varphi(t)$ with $\varphi(t-t')$ with $t'\in\mathbb{R}$ then by (15) we have to replace $\hat\varphi^c(z)$ with $e^{-2\pi it'z}\hat\varphi^c(z)$. By (18), we obtain $\nu[t'-1,t'+1]\le C_5^2e^{4\pi|t'|H}$. We have proved (12).

Then, each interval $(-r,r)$ can be covered by $\lfloor r\rfloor+1$ intervals of the form $(t-1,t+1)$ with $t<r$, where $\lfloor r\rfloor$ is an integer part of the number r. Therefore, $\nu(-r,r)\le C_5^2(\lfloor r\rfloor+1)e^{4\pi rH}$, and $\overline{\lim_{r\to\infty}}\, r^{-1}\log\nu(-r,r)\le 4\pi H$. □

Proof of Theorem 3. By Theorem 1, it is enough to prove Theorem 3 for any translation bounded measure μ.

Let $\varphi\in\mathcal{D}$ be such that $0\le\varphi(t)\le1$, $\operatorname{supp}\varphi\subset(-\eta,\eta)$, $\varphi(t)\equiv1$ for $|t|<\eta/2$, and C_5 is the constant from (18) that corresponds to this φ. Suppose that (13) does not valid. Then there are intervals $I_n=(t_n-\eta/2,t_n+\eta/2)$ such that $t_n\to\infty$ and
$$\sum_{\gamma\in\Gamma\cap I_n}|b_\gamma|=|\hat\mu^c|(I_n)>5C_5e^{2\pi H|t_n|}.$$
If the set $\Gamma\cap(t_n-\eta,t_n+\eta)$ is finite for some $n\in\mathbb{N}$, then we set $A_n=\Gamma\cap(t_n-\eta,t_n+\eta)$. Otherwise, taking into account that $\hat\mu$ is a measure and $|\hat\mu|(t_n-\eta,t_n+\eta)<\infty$, we choose A_n to be a finite subset of $\Gamma\cap(t_n-\eta,t_n+\eta)$ such that
$$\sum_{\gamma\in\Gamma\cap(t_n-\eta,t_n+\eta)\setminus A_n}|b_\gamma|<C_5e^{2\pi H|t_n|}.\tag{21}$$
With this setting, we have
$$\sum_{\gamma\in A_n\cap I_n}|b_\gamma|>4C_5e^{2\pi H|t_n|}.$$
Using the Kroneker lemma, we can find $s_n\in\mathbb{R}$ and $m_\gamma^{(n)}\in\mathbb{Z}$ such that for all $\gamma\in A_n$ $|s_n\gamma+\frac{\arg b_\gamma}{2\pi}-m_\gamma^{(n)}|<1/6$. Therefore, for all $\gamma\in A_n$ we have $\mathrm{Re}\,e^{2\pi is_n\gamma}b_\gamma=|b_\gamma|\cos(2\pi s_n\gamma+\arg b_\gamma)>|b_\gamma|/2>0$, and
$$\mathrm{Re}\sum_{\gamma\in A_n\cap I_n}e^{2\pi is_n\gamma}b_\gamma>2C_5e^{2\pi H|t_n|}.\tag{22}$$
Let $\varphi\in\mathcal{D}$ be the function defined at the beginning of the proof. It follows from (21), (22), and the inequality $\mathrm{Re}\,e^{2\pi is_n\gamma}b_\gamma>0$ for all $\gamma\in A_n$ that
$$\Big|\int e^{2\pi is_nt}\varphi(t-t_n)\hat\mu^c(dt)\Big|\ge\Big|\sum_{\gamma\in A_n}\varphi(\gamma-t_n)e^{2\pi is_n\gamma}b_\gamma\Big|-\sum_{\gamma\in\Gamma\cap(t_n-\eta,t_n+\eta)\setminus A_n}|b_\gamma|\ge$$
$$\ge\mathrm{Re}\sum_{\gamma\in A_n\cap I_n}e^{2\pi is_n\gamma}b_\gamma+\mathrm{Re}\sum_{\gamma\in A_n\setminus I_n}\varphi(\gamma-t_n)e^{2\pi is_n\gamma}b_\gamma-C_5e^{2\pi H|t_n|}>C_5e^{2\pi H|t_n|}.$$

The c-Fourier transform of the function $e^{2\pi i s_n t}\varphi(t-t_n)$ equals $e^{-2\pi i t_n(z-s_n)}\hat{\varphi}^c(z-s_n)$. Hence, by definition of the c-Fourier transform of the measure,

$$\int_{\mathbb{R}} e^{2\pi i s_n t}\varphi(t-t_n)\hat{\mu}^c(dt) = \int_{\mathcal{S}_H} e^{-2\pi i t_n(z-s_n)}\hat{\varphi}^c(z-s_n)\mu(dz) = \int_{\mathcal{S}_H} e^{-2\pi i t_n z}\hat{\varphi}^c(z)\mu_{-s_n}(dz).$$

By (18), the modulus of the last integral does not exceed $C_5 e^{2\pi|t_n|H}$. This bound contradicts to what was received earlier, hence (13) is true. Arguing as in the proof of Theorem 2, we obtain that the measure $\hat{\mu}^c$ has the type with respect to the exponential growth at most $2\pi H$. $\square$

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