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ESTIMATES OF THE MODULUS OF CONTINUITY OF THE LOGARITHMIC DOUBLE LAYER POTENTIAL

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For the real part of the Cauchy-type integral, which is known to be the logarithmic double layer potential, we obtain estimates of the modulus of continuity in the closure of domain bounded by an Ahlfors-regular integration curve. These estimates are more exact than the well-known Zygmund estimate for the modulus of continuity of the Cauchy-type integral. The accuracy of estimates for the logarithmic double layer potential is proved by constructing an example of a curve and an integral density for which the specified estimates are exact with respect to the order of smallness.

1. Introduction. Let γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$, and let D^+ and D^- be the interior and exterior domains bounded by γ, respectively.

The classical theory of the logarithmic double layer potential (see, e.g., J. Plemelj [1])

$$\frac{1}{2\pi}\int_\gamma g(t)\frac{\partial}{\partial \mathbf{n}_t}\left(\ln\frac{1}{|t-z|}\right)ds_t \qquad \forall z\in D^\pm \tag{1}$$

is developed in the case where the integration curve γ is a Lyapunov curve, and the integral density $g\colon \gamma\to\mathbb{R}$ taking real values is a continuous function. Here $\mathbf{n}_t$ and s_t denote the unit vector of the outward normal to the curve γ at a point $t\in\gamma$ and an arc coordinate of this point, respectively.

J. Radon [2] established the continuous extension of the logarithmic double layer potential from the domains D^+ and D^- to the boundary γ for any continuous integral density $g\colon\gamma\to\mathbb{R}$ in the case where γ is a curve of bounded rotation, i.e., a curve for which the angle between the tangent to the curve and a fixed direction is a function of bounded variation. It is known that the class of Lyapunov curves and the class of Radon curves of bounded rotation are different, i.e., each of them contains curves that do not belong to the other class (see, for example, I.I. Danilyuk [3, p. 26]).

J. Král [4] proved that logarithmic double layer potential (1) can be continuously extended from the domains D^+ and D^- to the boundary γ for all continuous functions $g\colon\gamma\to\mathbb{R}$ if and only if the curve γ satisfies the condition

$$\sup_{\xi\in\gamma}\int_0^{2\pi}\mu_\gamma(\xi,\phi)d\phi<\infty, \tag{2}$$

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where $\mu_\gamma(\xi,\phi)$ is the number of intersection points of the curve γ with the ray $\{z = \xi + re^{i\phi} \colon r > 0\}$.

Logarithmic double layer potential (1) is the real part of the Cauchy-type integral (see, for example, I.I. Danilyuk [3], F.D. Gakhov [5], N.I. Muskhelishvili [6])

$$\widetilde{g}(z) := \frac{1}{2\pi i}\int_\gamma \frac{g(t)}{t-z}dt \qquad \forall z \in D^\pm. \tag{3}$$

While the results mentioned above about the continuous extension of the logarithmic double layer potential to the boundary of a domain, which are contained in papers [1, 2, 4], are valid for arbitrary continuous functions g, the continuous extension of integral (3) to the boundary of D^+ or D^- requires additional assumptions on the density g of the integral.

The theory of the boundary properties of integral (3) is presented in the monographs by F.D. Gakhov [5] and N.I. Muskhelishvili [6] under the classical assumptions about the smoothness of the integration curve and the Hölder density of the integral. In the papers of N.A. Davydov [7], V.V. Salaev [8], T.S. Salimov [9], E.M. Dyn'kin [10], O.F. Gerus [11, 12], the theory of the Cauchy-type integral and the Cauchy singular integral is developed on an arbitrary rectifiable Jordan curve in classes that are more general than the Hölder class of the integral density, which are defined, as a rule, in terms of the modulus of continuity of the function g. In this case, for the modulus of continuity of the Cauchy-type integral and its limiting values on the boundary of a domain, both upper estimates (see [8, 11, 9, 10, 12]) and lower estimates for specially constructed functions to prove the ordinal accuracy of the specified upper estimates on subclasses of the class of closed rectifiable Jordan curves (see, for example, V.V. Salaev [8] and O.F. Gerus [13, 14]) are established.

The aim of this work is to refine the aforementioned estimates for the modulus of continuity of the real part of the Cauchy-type integral (i.e., the logarithmic double layer potential) on subclasses of the class of closed rectifiable Jordan curves.

The paper is organized as follows. In Section 2, we introduce the preliminaries and formulate the main Theorems 1 and 2, which contain the upper estimates for the modulus of continuity of the real part of the Cauchy-type integral. Section 3 contains auxiliary statements. The proofs of Theorems 1 and 2 are contained in Section 4. Finally, in Section 5, we prove the ordinal accuracy of the estimates given in Theorems 1 and 2.

2. Upper estimates for the modulus of continuity of the real part of the Cauchy-type integral in the case of Ahlfors-regular integration curves. In what follows, a closed rectifiable Jordan curve γ satisfies the condition (see V.V. Salaev [8])

$$\theta(\varepsilon) := \sup_{x\in\gamma} \theta_x(\varepsilon) = O(\varepsilon), \quad \varepsilon \to 0, \tag{4}$$

where $\theta_x(\varepsilon) := \text{mes}\gamma_\varepsilon(x)$, $\gamma_\varepsilon(x) := \{t \in \gamma \colon |t - x| \le \varepsilon\}$ and mes denotes the linear Lebesgue measure on γ. Curves satisfying condition (4) are important in solving various problems (see, for example, V.V. Salaev [8], L. Ahlfors [15], G. David [16], C. Pommerenke [17], A. Böttcher and Y.I. Karlovich [18]). Such curves are often called the *regular curves* (see, for example, [16]) or the *Ahlfors-regular curves* (see, for example, [17]), or the *Carleson curves* (see, for example, [18]).

Let $d := \max\limits_{t_1,t_2\in\gamma} |t_1 - t_2|$ be the diameter of the curve γ.

For each $\xi \in \gamma$, we define a branch $\arg(z - \xi)$ continuous on $\gamma \setminus \{\xi\}$ of the multivalued function $\text{Arg}(z-\xi)$ in the following way. For each positive $\delta < d/2$, we select that connected component $\gamma_{\xi,\delta}$ of the set $\gamma_\delta(\xi)$ which contains the point ξ. We choose a point $\xi_1 \in \gamma_{\xi,\delta}$ at which γ has a tangent and which does not precede the point ξ with respect to the given orientation of the curve γ. It is obvious that in the case in which there is a tangent to γ at the point ξ, we can set $\xi_1 = \xi$. Let us cut the complex plane along the curve $\Gamma_{\xi,\delta} := \gamma[\xi, \xi_1] \cup \Gamma[\xi_1, \infty]$, where $\gamma[\xi, \xi_1]$ is the arc of γ with the initial point ξ and the terminal point ξ_1, and $\Gamma[\xi_1, \infty]$ is a smooth curve that connects the points ξ_1 and ∞ and lies completely (except for its ends ξ_1 and ∞) in the domain D^-. Now, let us single out a branch $\arg_\delta(z - \xi)$ of the multivalued function $\operatorname{Arg}(z - \xi)$, which is continuous outside the cut $\Gamma_{\xi,\delta}$ with the normalization condition $\arg_\delta(z_0 - \xi) = \phi_0$, where $z_0 \in D^+$ and ϕ_0 is one of the values of the function $\operatorname{Arg}(z - \xi)$ at $z = z_0$. We shall use the fixed values z_0 and ϕ_0 for all positive $\delta < d/2$. As a result, we have the obvious equality

$$\arg_{\delta_1}(t - \xi) = \arg_\delta(t - \xi) \qquad \forall \delta_1, \delta \colon 0 < \delta_1 < \delta < d/2 \quad \forall t \in \gamma \setminus \gamma_\delta(\xi)$$

that implies the existence of the following limit

$$\arg(t - \xi) := \lim_{\delta \to 0^+} \arg_\delta(t - \xi) \qquad \forall t \in \gamma \setminus \{\xi\}.$$

By definition

$$\int_\gamma \big(g(t) - g(\xi)\big) d \arg(t - \xi) := \lim_{\delta \to 0^+} \int_{\gamma \setminus \gamma_\delta(\xi)} \big(g(t) - g(\xi)\big) d \arg(t - \xi) \qquad \forall \xi \in \gamma, \tag{5}$$

if the limit on the right-hand side of the equality exists.

Denote

$$M_\gamma(g, \varepsilon) := \sup_{\xi \in \gamma} \sup_{\delta \in (0,\varepsilon)} \left| \int_{\gamma_\varepsilon(\xi) \setminus \gamma_\delta(\xi)} \big(g(t) - g(\xi)\big) d \arg(t - \xi) \right|, \qquad \varepsilon > 0.$$

It is proved in Theorem 2 [19] that the condition

$$M_\gamma(g, \varepsilon) \to 0, \qquad \varepsilon \to 0, \tag{6}$$

is necessary and sufficient for the continuous extension to the boundary γ of the real part of Cauchy-type integral (3) from the domains D^+ and D^- in the case where γ is a closed Ahlfors-regular Jordan curve. Moreover, if condition (6) is satisfied, then the limit on the right-hand side of equality (5) exists, and for all $\xi \in \gamma$, the limiting values

$$\big(\operatorname{Re} \widetilde{g}\big)^\pm(\xi) := \lim_{z \to \xi, z \in D^\pm} \operatorname{Re} \widetilde{g}(z) \tag{7}$$

are represented by the formulas

$$\big(\operatorname{Re} \widetilde{g}\big)^+(\xi) = g(\xi) + \frac{1}{2\pi} \int_\gamma \big(g(t) - g(\xi)\big) d \arg(t - \xi), \tag{8}$$

$$\big(\operatorname{Re} \widetilde{g}\big)^-(\xi) = \frac{1}{2\pi} \int_\gamma \big(g(t) - g(\xi)\big) d \arg(t - \xi). \tag{9}$$

Let us note that in the paper of O.F. Gerus and M. Shapiro [20], an analog of the Davydov theorem [7] is proved for an appropriate Cauchy-type integral along an arbitrary rectifiable Jordan curve in $\mathbb{R}^2$, which takes values in the algebra of quaternions. This result is applied in the paper of O.F. Gerus and M. Shapiro [21] to establish sufficient conditions for the continuous extension to the boundary of a domain of metaharmonic potentials, a partial case of which is logarithmic double layer potential (1).

Continuous extensions of the function $\mathrm{Re}\widetilde{g}$ to the closures $\overline{D^+}$ and $\overline{D^-}$ will be denoted by $\big(\mathrm{Re}\widetilde{g}\big)^+$ and $\big(\mathrm{Re}\widetilde{g}\big)^-$, respectively.

For a function $f\colon E \to \mathbb{C}$ continuous on $E \subset \mathbb{C}$, we shall use its modulus of continuity

$$\omega_E(f,\varepsilon) := \sup_{t_1,t_2\in E: |t_1-t_2|\le\varepsilon} |f(t_1)-f(t_2)|\,.$$

An upper estimate for the solid modulus of continuity of the function $\big(\mathrm{Re}\widetilde{g}\big)^\pm$ on the set $\overline{D^\pm}$ in the case of an Ahlfors-regular integration curve is established in the following theorem.

Theorem 1. *Let a closed Jordan curve γ be Ahlfors-regular and let a function $g\colon \gamma \to \mathbb{R}$ be continuous on γ. If condition* (6) *is satisfied, then the following estimate is true*

$$\omega_{\overline{D^\pm}}\Big(\big(\mathrm{Re}\widetilde{g}\big)^\pm,\varepsilon\Big) \le c\left(M_\gamma(g,\varepsilon) + \varepsilon\int_\varepsilon^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta\right) \qquad \forall\varepsilon\in(0,d/8], \tag{10}$$

where the constant c depends only on γ but does not depend on ε.

Below c denotes various positive constants whose values depend only on the curve γ but are different even within one chain of inequalities.

Let us note that if the modulus of continuity of the function $g\colon \gamma\to\mathbb{R}$, which is given on a closed Ahlfors-regular Jordan curve, satisfies the Dini condition

$$\int_0^1 \frac{\omega_\gamma(g,\eta)}{\eta}d\eta < \infty, \tag{11}$$

then Cauchy-type integral (3) has the limiting values $\widetilde{g}^\pm(\xi)$ at every point $\xi\in\gamma$ from the domain $D^\pm$, and the following Zygmund estimate holds:

$$\omega_{\overline{D^\pm}}(\widetilde{g},\varepsilon) \le c\left(\int_0^\varepsilon \frac{\omega_\gamma(g,\eta)}{\eta}d\eta + \varepsilon\int_\varepsilon^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta\right), \tag{12}$$

where the constant c depends only on γ but does not depend on ε (see O.F. Gerus [11], and also V.V. Salaev [8], where the Dini condition of form (11) is given in terms of the regularized modulus of continuity using the Stechkin construction).

Condition (11) is sufficient for condition (6) to be satisfied in the case where γ is an Ahlfors-regular curve, but it is not necessary for this. Indeed, if condition (6) is satisfied, but the condition

$$\sup_{\xi\in\gamma}\sup_{\delta\in(0,\varepsilon)}\left|\int_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)}\frac{g(t)-g(\xi)}{|t-\xi|}d|t-\xi|\right| \to 0, \qquad \varepsilon\to 0,$$

is not satisfied, then condition (11) is also not satisfied, and the function $\mathrm{Im}\widetilde{g}(z)$ have no continuous extension to the boundary γ from the domains D^+ and D^-, while the function $\mathrm{Re}\widetilde{g}(z)$ is continuously extended to γ from D^+ and D^-. Thus, estimate (10) is an improvement of estimate (12) for the logarithmic double layer potential in the case where an integration curve is Ahlfors-regular.

Note that in the case of an Ahlfors-regular curve, for each $\xi \in \gamma$ and each $\delta > 0$, the function $\arg(t-\xi)$ has a bounded variation on the set $\gamma \setminus \gamma_\delta(\xi)$. However, in general, the function $\arg(t-\xi)$ can be a function of unbounded variation on γ, because, in particular, it can be unbounded in a neighborhood of the point ξ.

Consider the class of curves γ, for which the function $\arg(t-\xi)$ has a bounded variation $V_\gamma[\arg(t-\xi)]$ on $\gamma \setminus \{\xi\}$ for all $\xi \in \gamma$ and, moreover, satisfies the condition

$$\sup_{\xi\in\gamma} V_\gamma[\arg(t-\xi)] < \infty. \tag{13}$$

Condition (13) is equivalent to condition (2), which follows from the Banach indicatrix theorem (see J. Král [4, Lemma 1.2]). Therefore, curves satisfying condition (13) will be called the *Král curves*.

Every Král curve is an Ahlfors-regular curve (see [19, Proposition 4.1]).

In the case where the integration curve is a Král curve, limiting values (7) exist everywhere on γ for an arbitrary continuous function $g\colon \gamma \to \mathbb{R}$ (see J. Král [4]). Therefore, in this case, by Theorem 2 [19], condition (6) is satisfied. In addition, estimate (10) for the modulus of continuity of the limiting values of the real part of the Cauchy-type integral is specified in the following theorem.

Theorem 2. *Let a closed Jordan curve γ be a Král curve and let a function $g\colon \gamma \to \mathbb{R}$ be continuous on γ. Then the following estimate is true*

$$\omega_{\overline{D^\pm}}\Big(\big(\mathrm{Re}\widetilde{g}\big)^\pm, \varepsilon\Big) \le c\varepsilon \int_\varepsilon^{2d} \frac{\omega_\gamma(g,\eta)}{\eta^2} d\eta \qquad \forall \varepsilon \in (0, d/8], \tag{14}$$

where the constant c depends only on γ but does not depend on ε.

It will be proved in Section 5 that estimates (10) and (14) are exact with respect to the order of smallness as $\varepsilon \to 0$.

3. Auxiliary statements. Consider the following auxiliary statements.

Lemma 1. *Let a closed Jordan curve γ be Ahlfors-regular. Let a function $g\colon \gamma \to \mathbb{R}$ be continuous on γ and let condition* (6) *be satisfied. Let $\xi_1, \xi_2 \in \gamma$, $|\xi_1 - \xi_2| \le \varepsilon_1$ and $\varepsilon/2 \le \varepsilon_1 \le 5\varepsilon$ for $\varepsilon \in (0, d/8]$. Then the following estimate is true*

$$\left| \frac{1}{2\pi} \int_{\gamma_{2\varepsilon_1}(\xi_1)} \big(g(t) - g(\xi_2)\big) d\arg(t-\xi_2) \right| \le c \left(M_\gamma(g,\varepsilon) + \varepsilon \int_\varepsilon^{15\varepsilon} \frac{\omega_\gamma(g,\eta)}{\eta^2} d\eta \right),$$

where the constant c depends only on γ but does not depend on ε.

Proof. Let us evaluate the modulus of each term on the right-hand side of the equality

$$\int_{\gamma_{2\varepsilon_1}(\xi_1)} \big(g(t) - g(\xi_2)\big) d\arg(t-\xi_2) =$$

$$= \int_{\gamma_\varepsilon(\xi_2)} \big(g(t) - g(\xi_2)\big) d\arg(t-\xi_2) + \mathrm{Re}\bigg(\frac{1}{i} \int_{\gamma_{2\varepsilon_1}(\xi_1)\setminus\gamma_\varepsilon(\xi_2)} \frac{g(t) - g(\xi_2)}{t - \xi_2} dt \bigg) =: J_1 + J_2.$$

It is obvious that

$$|J_1| \le M_\gamma(g,\varepsilon).$$

Taking into account the two-sided inequality $\varepsilon/2 \le \varepsilon_1 \le 5\varepsilon$ and condition (4), we obtain a chain of inequalities

$$|J_2| \le \int_{\gamma_{2\varepsilon_1}(\xi_1)\setminus\gamma_\varepsilon(\xi_2)} \frac{|g(t)-g(\xi_1)|+|g(\xi_1)-g(\xi_2)|}{|t-\xi_2|}|dt| \le \frac{2\omega_\gamma(g,2\varepsilon_1)}{\varepsilon}\theta_{\xi_1}(2\varepsilon_1) \le$$
$$\le c\omega_\gamma(g,2\varepsilon_1) \le c\omega_\gamma(g,2\varepsilon_1)\varepsilon_1\int_{2\varepsilon_1}^{3\varepsilon_1}\frac{d\eta}{\eta^2} \le c\varepsilon\int_\varepsilon^{15\varepsilon}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta,$$

where c denotes different constants whose values depend only on the curve γ. □

Lemma 2. *Under the conditions of Lemma 1 the following estimate is true*

$$\left|\big(\mathrm{Re}\widetilde{g}\big)^{\pm}(\xi_1) - \big(\mathrm{Re}\widetilde{g}\big)^{\pm}(\xi_2)\right| \le c\left(M_\gamma(g,\varepsilon) + \varepsilon\int_\varepsilon^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta\right), \tag{15}$$

where the constant c depends only on the curve γ.

Proof. Let us prove estimate (15) for the function $\big(\mathrm{Re}\widetilde{g}\big)^{+}$ (estimate (15) for the function $\big(\mathrm{Re}\widetilde{g}\big)^{-}$ is proved similarly).

We take into account equality (8) and use the following representation of the difference

$$\big(\mathrm{Re}\widetilde{g}\big)^{+}(\xi_1) - \big(\mathrm{Re}\widetilde{g}\big)^{+}(\xi_2) = g(\xi_1)-g(\xi_2)+$$
$$+\frac{1}{2\pi}\int_\gamma \big(g(t)-g(\xi_1)\big)d\arg(t-\xi_1) - \frac{1}{2\pi}\int_\gamma \big(g(t)-g(\xi_2)\big)d\arg(t-\xi_2) = g(\xi_1)-g(\xi_2)+$$
$$+\frac{1}{2\pi}\int_{\gamma_{2\varepsilon_1}(\xi_1)} \big(g(t)-g(\xi_1)\big)d\arg(t-\xi_1) - \frac{1}{2\pi}\int_{\gamma_{2\varepsilon_1}(\xi_1)} \big(g(t)-g(\xi_2)\big)d\arg(t-\xi_2)+$$
$$+\mathrm{Re}\bigg(\frac{\xi_1-\xi_2}{2\pi i}\int_{\gamma\setminus\gamma_{2\varepsilon_1}(\xi_1)}\frac{g(t)-g(\xi_1)}{(t-\xi_1)(t-\xi_2)}dt\bigg) + \mathrm{Re}\bigg(\frac{g(\xi_2)-g(\xi_1)}{2\pi i}\int_{\gamma\setminus\gamma_{2\varepsilon_1}(\xi_1)}\frac{dt}{t-\xi_2}\bigg) =:$$
$$=: g(\xi_1)-g(\xi_2)+I_1-I_2+I_3+I_4.$$

The moduli of integrals I_1 and I_2 are estimated in Lemma 1.

To estimate the modulus of the term I_3, we use the inequality $|t-\xi_2| \ge |t-\xi_1|/2$ for all $t \in \gamma\setminus\gamma_{2\varepsilon_1}(\xi_1)$, Proposition 7.2 [22] (see also the proof of Theorem 1 in the paper of O.F. Gerus [23]) and condition (4) so that we have

$$|I_3| \le \frac{|\xi_1-\xi_2|}{\pi}\int_{\gamma\setminus\gamma_{2\varepsilon_1}(\xi_1)}\frac{|g(t)-g(\xi_1)|}{|t-\xi_1|^2}|dt| \le \frac{\varepsilon_1}{\pi}\int_{[2\varepsilon_1,d]}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\theta_{\xi_1}(\eta) \le$$
$$\le \frac{2\varepsilon_1}{3\pi}\int_{\varepsilon_1}^{d}\frac{\theta_{\xi_1}(2\eta)\omega_\gamma(g,2\eta)}{\eta^3}d\eta \le c\varepsilon\int_\varepsilon^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta,$$

where the constant c depends only on the curve γ.

In addition, taking into account the inequality (see the proof of Theorem 1 in the paper of V.V. Salaev [8])

$$\left|\int_{\gamma\setminus\gamma_{2\varepsilon_1}(\xi_1)}\frac{dt}{t-\xi_2}\right| \le 4\pi,$$

we obtain the following relations

$$\left|g(\xi_1)-g(\xi_2)+I_4\right|\le \left|g(\xi_2)-g(\xi_1)\right|+\frac{\left|g(\xi_2)-g(\xi_1)\right|}{2\pi}\left|\int_{\gamma\setminus\gamma_{2\varepsilon_1}(\xi_1)}\frac{dt}{t-\xi_2}\right|\le 3\omega_\gamma(g,\varepsilon)=$$
$$=6\omega_\gamma(g,\varepsilon)\varepsilon\int_\varepsilon^{2\varepsilon}\frac{d\eta}{\eta^2}\le 6\varepsilon\int_\varepsilon^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta.$$

□

Lemma 3. *Let a closed Jordan curve γ be Ahlfors-regular. Let a function $g\colon\gamma\to\mathbb{R}$ be continuous on γ and let condition (6) be satisfied. Let $z\in D^\pm$ and let ξ_z be one of the closest points of the curve γ to the point z, and let $|z-\xi_z|\le 2\varepsilon$ with $\varepsilon\in(0,d/8]$. Then the following estimate is true*

$$\left|\big(\mathrm{Re}\widetilde{g}\big)^\pm(z)-\big(\mathrm{Re}\widetilde{g}\big)^\pm(\xi_z)\right|\le c\left(M_\gamma(g,\varepsilon)+\varepsilon\int_\varepsilon^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta\right),$$

where the constant c depends only on the curve γ.

Proof. Denote $\rho:=|z-\xi_z|$.

Taking into account the Cauchy integral theorem, the Cauchy integral formula, equalities (8) and (9), we have

$$\big(\mathrm{Re}\widetilde{g}\big)^\pm(z)-\big(\mathrm{Re}\widetilde{g}\big)^\pm(\xi_z)=$$
$$=\mathrm{Re}\bigg(\frac{1}{2\pi i}\int_\gamma\frac{g(t)-g(\xi_z)}{t-z}dt\bigg)-\frac{1}{2\pi}\int_\gamma\big(g(t)-g(\xi_z)\big)d\arg(t-\xi_z)=$$
$$=\mathrm{Re}\bigg(\frac{1}{2\pi i}\int_{\gamma_{2\rho}(\xi_z)}\frac{g(t)-g(\xi_z)}{t-z}dt\bigg)-\frac{1}{2\pi}\int_{\gamma_{2\rho}(\xi_z)}\big(g(t)-g(\xi_z)\big)d\arg(t-\xi_z)+$$
$$+\mathrm{Re}\bigg(\frac{z-\xi_z}{2\pi i}\int_{\gamma\setminus\gamma_{2\rho}(\xi_z)}\frac{g(t)-g(\xi_z)}{(t-z)(t-\xi_z)}dt\bigg)=:S_1-S_2+S_3.$$

Taking into account condition (4), we obtain the inequalities

$$|S_1|\le\frac{1}{2\pi}\int_{\gamma_{2\rho}(\xi_z)}\frac{|g(t)-g(\xi_z)|}{|t-z|}|dt|\le\frac{\omega_\gamma(g,2\rho)}{2\pi\rho}\theta_{\xi_z}(2\rho)\le c\omega_\gamma(g,2\rho)\le c\omega_\gamma(g,4\varepsilon)\le$$
$$\le c\omega_\gamma(g,4\varepsilon)\varepsilon\int_{4\varepsilon}^{5\varepsilon}\frac{d\eta}{\eta^2}\le c\varepsilon\int_{4\varepsilon}^{5\varepsilon}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta,$$

where c denotes different constants whose values depend only on the curve γ.

It is obvious that for $\rho\le\varepsilon/2$, the following inequality holds

$$|S_2|\le\frac{1}{2\pi}M_\gamma(g,\varepsilon),$$

and for $\varepsilon/2<\rho\le 2\varepsilon$, the modulus of integral S_2 is estimated in Lemma 1.

Just as when estimating $|I_3|$ in the proof of Lemma 2, we first obtain

$$|S_3|\le c\rho\int_\rho^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta,$$

where the constant c depends only on the curve γ.

Further, we have the relation

$$\rho\int_{\rho}^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta=\rho\int_{\rho}^{2\varepsilon}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta+\rho\int_{2\varepsilon}^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta\le\omega_\gamma(g,2\varepsilon)+$$
$$+2\varepsilon\int_{2\varepsilon}^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta=6\omega_\gamma(g,2\varepsilon)\varepsilon\int_{2\varepsilon}^{3\varepsilon}\frac{d\eta}{\eta^2}+2\varepsilon\int_{2\varepsilon}^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta\le 8\varepsilon\int_{2\varepsilon}^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta.$$

□

4. The proofs of Theorems 1 and 2. Consider the proofs of Theorems 1 and 2, which were presented in Section 2.

Proof of Theorem 1. Let $0<\varepsilon\le d/8$. Consider two cases for the location of the points $z_1,z_2\in\overline{D^+}$.

Case 1. Let $z_1,z_2\in\overline{D^+}$, $|z_1-z_2|\le\varepsilon$, $\min\limits_{t\in\gamma}|t-z_1|\le 2\varepsilon$ and $\min\limits_{t\in\gamma}|t-z_2|\le 2\varepsilon$. Let ξ_{z_k} be one of the closest points of the curve γ to the point z_k for $k=1,2$. It is obvious that $\xi_{z_k}=z_k$ in the case where $z_k\in\gamma$. Then we have the equality

$$\big(\mathrm{Re}\widetilde{g}\big)^+(z_1)-\big(\mathrm{Re}\widetilde{g}\big)^+(z_2)=\big(\mathrm{Re}\widetilde{g}\big)^+(z_1)-\big(\mathrm{Re}\widetilde{g}\big)^+(\xi_{z_1})+$$
$$+\big(\mathrm{Re}\widetilde{g}\big)^+(\xi_{z_1})-\big(\mathrm{Re}\widetilde{g}\big)^+(\xi_{z_2})+\big(\mathrm{Re}\widetilde{g}\big)^+(\xi_{z_2})-\big(\mathrm{Re}\widetilde{g}\big)^+(z_2).$$

Inasmuch as $|z_1-\xi_{z_1}|\le 2\varepsilon$, $|z_2-\xi_{z_2}|\le 2\varepsilon$ and $|\xi_{z_1}-\xi_{z_2}|\le 5\varepsilon$, then $|(\mathrm{Re}\widetilde{g})^+(z_1)-(\mathrm{Re}\widetilde{g})^+(\xi_{z_1})|$ and $|(\mathrm{Re}\widetilde{g})^+(z_2)-(\mathrm{Re}\widetilde{g})^+(\xi_{z_2})|$ is estimated in Lemma 3, and $|(\mathrm{Re}\widetilde{g})^+(\xi_{z_1})-(\mathrm{Re}\widetilde{g})^+(\xi_{z_2})|$ is estimated in Lemma 2. As a result, we obtain an estimate of form (15) with $\xi_1=z_1$ and $\xi_2=z_2$.

Case 2. Let $z_1,z_2\in D^+$, $|z_1-z_2|\le\varepsilon$ and $\min\limits_{t\in\gamma}|t-z_2|>2\varepsilon$. Then $\min\limits_{t\in\gamma}|t-z_1|>\varepsilon$.

Let ξ_{z_1} be one of the closest points of the curve γ to the point z_1. We use the following representation of the difference

$$\big(\mathrm{Re}\widetilde{g}\big)^+(z_1)-\big(\mathrm{Re}\widetilde{g}\big)^+(z_2)=\mathrm{Re}\bigg(\frac{z_1-z_2}{2\pi i}\int_\gamma\frac{g(t)-g(\xi_{z_1})}{(t-z_1)(t-z_2)}dt\bigg)=$$
$$=\mathrm{Re}\bigg(\frac{z_1-z_2}{2\pi i}\int_{\gamma_\varepsilon(\xi_{z_1})}\frac{g(t)-g(\xi_{z_1})}{(t-z_1)(t-z_2)}dt\bigg)+\mathrm{Re}\bigg(\frac{z_1-z_2}{2\pi i}\int_{\gamma\setminus\gamma_\varepsilon(\xi_{z_1})}\frac{g(t)-g(\xi_{z_1})}{(t-z_1)(t-z_2)}dt\bigg)=:$$
$$=:s_1+s_2.$$

Similarly to the estimate of $|S_1|$ in the proof of Lemma 3, we obtain the estimate

$$|s_1|\le c\varepsilon\int_{\varepsilon}^{2\varepsilon}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta,$$

where the constant c depends only on the curve γ.

In addition, taking into account the inequalities $|t-z_1|\ge|t-\xi_{z_1}|/2$ and $|t-z_2|\ge|t-\xi_{z_1}|/3$, which are fulfilled in this case for all $t\in\gamma$, similarly to the estimate of $|I_3|$ in the proof of Lemma 2, we obtain the estimate

$$|s_2|\le c\varepsilon\int_{\varepsilon}^{2d}\frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta,$$

where the constant c depends only on the curve γ.

An obvious corollary of the mentioned estimates for $|s_1|$ and $|s_2|$ is the inequality

$$\Big|\big(\mathrm{Re}\widetilde{g}\big)^+(z_1) - \big(\mathrm{Re}\widetilde{g}\big)^+(z_2)\Big| \le c\varepsilon \int_{\varepsilon}^{2d} \frac{\omega_\gamma(g,\eta)}{\eta^2} d\eta,$$

where the constant c depends only on the curve γ.

Since all possible cases of arrangement of the points $z_1, z_2 \in \overline{D^+}$ are considered, the proof of estimate (10) for $\omega_{\overline{D^+}}((\mathrm{Re}\widetilde{g})^+, \varepsilon)$ is complete. Estimate (10) for $\omega_{\overline{D^-}}(\big(\mathrm{Re}\widetilde{g}\big)^-, \varepsilon)$ is similarly proved. □

Proof of Theorem 2. Since condition (6) follows from Král's main result in [4] and Theorem 2 [19], estimate (10) holds.

Under condition (13) for the curve γ, the following inequalities hold

$$M_\gamma(g,\varepsilon) \le \sup_{\xi\in\gamma} \sup_{\delta\in(0,\varepsilon)} \int_{\gamma_\varepsilon(\xi)\setminus\gamma_\delta(\xi)} \big|g(t)-g(\xi)\big|\big|d\arg(t-\xi)\big| \le$$
$$\le \omega_\gamma(g,\varepsilon)\sup_{\xi\in\gamma} V_\gamma[\arg(t-\xi)] \le c\varepsilon \int_{\varepsilon}^{2\varepsilon} \frac{\omega_\gamma(g,\eta)}{\eta^2} d\eta, \tag{16}$$

where the constant c depends only on γ, and therefore estimate (10) takes form (14). □

5. A lower estimate for the modulus of continuity of the real part of the Cauchy-type integral. Let us note that the upper estimate (14) for the modulus of continuity $\omega_{\overline{D^\pm}}((\mathrm{Re}\widetilde{g})^\pm, \varepsilon)$, which is considered throughout the entire class of Král curves γ and the entire class of continuous functions $g\colon \gamma \to \mathbb{R}$, is exact with respect to the order of smallness as $\varepsilon \to 0$. The specified accuracy of estimate (14) will be proved by constructing an example of a curve γ and a continuous function $g\colon \gamma \to \mathbb{R}$ such that for the solid modulus of continuity $\omega_{\overline{D^\pm}}((\mathrm{Re}\widetilde{g})^\pm, \varepsilon)$, the lower estimate is realized by the right-hand side of inequality (14).

For fixed constants $\sigma \ge 1$ and $k \ge 0$, P.M. Tamrazov [24, p. 58] introduced the notion of a *normal majorant of the class* (σ, k) as a non-decreasing function $\mu(\eta)$ satisfying the inequality

$$\mu(\lambda\eta) \le \sigma\lambda^k \mu(\eta) \qquad \forall\lambda > 1 \quad \forall\eta > 0.$$

In the case where $\mu(\eta)$ is a normal majorant of the class $(\sigma, 1)$, the following inequality holds

$$\frac{\mu(\eta_2)}{\eta_2} \le \sigma \frac{\mu(\eta_1)}{\eta_1} \qquad \forall\eta_2 > \eta_1 > 0. \tag{17}$$

Note that in the case of the circle γ it follows from the properties of the modulus of continuity (see, for example, V.K. Dzyadyk and I.A. Shevchuk [25]) that the modulus of continuity $\omega_\gamma(g,\eta)$ of an arbitrary continuous function $g\colon \gamma \to \mathbb{R}$ is a normal majorant of the class $(2, 1)$.

Lemma 4. *Let $\gamma = \{t \in \mathbb{C}\colon |t+1| = 1\}$ and let $\mu(\eta)$ be a normal majorant of the class $(\sigma, 1)$. Then for the function*

$$\Phi_*(t) := \int_0^1 \mu(\eta)\bigg(\frac{t}{(\eta-t)^2} - \frac{t}{(\eta+t)^2}\bigg) d\eta \qquad \forall t \in \gamma,$$

for $t_1, t_2 \in \gamma$ *such that* $|t_1| \le |t_2|$, $|t_1 - t_2| \le \varepsilon$ *and* $|t_2| > 2\varepsilon$, *the following estimate is true*

$$\Big|\Phi_*(t_1) - \Phi_*(t_2)\Big| \le c\mu(\varepsilon) \qquad \forall \varepsilon \in (0,1), \tag{18}$$

where the constant c *does not depend on* ε, t_1 *and* t_2.

Proof. For $t_1, t_2 \in \gamma$, we have the equality

$$\Phi_*(t_1) - \Phi_*(t_2) = 4\big(t_1^2 - t_2^2\big)\int_0^1 \mu(\eta)\frac{\eta\big(\eta^4 - t_1^2 t_2^2\big)}{\big(\eta^2 - t_1^2\big)^2\big(\eta^2 - t_2^2\big)^2}d\eta.$$

Let $\varepsilon \in (0,1)$, $|t_1| \le |t_2|$, $|t_1 - t_2| \le \varepsilon$ and $|t_2| > 2\varepsilon$. Then $|t_1| > \varepsilon$ and $|t_2| \le 2|t_1|$. Taking also into account the inequalities

$$|\eta - t| \ge \frac{1}{\sqrt{2}}\big(\eta + |t|\big), \quad |\eta + t| \ge \frac{1}{3}\big(\eta + |t|\big) \qquad \forall \eta \in [0,1] \quad \forall t \in \gamma, \tag{19}$$

we have

$$\Big|\Phi_*(t_1) - \Phi_*(t_2)\Big| \le 36^2\varepsilon\big(|t_1| + |t_2|\big)\int_0^1 \mu(\eta)\frac{\eta\big(\eta^4 + |t_1|^2|t_2|^2\big)}{\big(\eta + |t_1|\big)^4\big(\eta + |t_2|\big)^4}d\eta \le$$

$$\le 3\cdot 36^2\varepsilon|t_1|\left(\int_0^{|t_1|} + \int_{|t_1|}^1\right)\mu(\eta)\frac{\eta\big(\eta^4 + |t_1|^2|t_2|^2\big)}{\big(\eta + |t_1|\big)^4\big(\eta + |t_2|\big)^4}d\eta = 3\cdot 36^2\big(j_1 + j_2\big), \tag{20}$$

where

$$j_1 := \varepsilon|t_1|\int_0^{|t_1|}\mu(\eta)\frac{\eta\big(\eta^4 + |t_1|^2|t_2|^2\big)}{\big(\eta + |t_1|\big)^4\big(\eta + |t_2|\big)^4}d\eta, \qquad j_2 := \varepsilon|t_1|\int_{|t_1|}^1\mu(\eta)\frac{\eta\big(\eta^4 + |t_1|^2|t_2|^2\big)}{\big(\eta + |t_1|\big)^4\big(\eta + |t_2|\big)^4}d\eta$$

and we put $j_2 = 0$ if $|t_1| > 1$.

Taking into account inequality (17), we obtain the relations

$$j_1 \le \varepsilon|t_1|\mu(|t_1|)\int_0^{|t_1|}\frac{2\eta|t_1|^2|t_2|^2}{|t_1|^4|t_2|^4}d\eta = \varepsilon\frac{|t_1|^2}{|t_2|^2}\frac{\mu(|t_1|)}{|t_1|} \le \varepsilon\sigma\frac{\mu(\varepsilon)}{\varepsilon} = \sigma\mu(\varepsilon), \tag{21}$$

$$j_2 \le \varepsilon|t_1|\int_{|t_1|}^1\mu(\eta)\frac{\eta\big(\eta^4 + 4|t_1|^2\big)}{\eta^8}d\eta \le 5\varepsilon|t_1|\int_{|t_1|}^1\frac{\mu(\eta)}{\eta^3}d\eta \le$$

$$\le 5\varepsilon|t_1|\sigma\frac{\mu(\varepsilon)}{\varepsilon}\int_{|t_1|}^1\frac{1}{\eta^2}d\eta \le 5\sigma\mu(\varepsilon). \tag{22}$$

Relations (20)–(22) imply inequality (18). □

The following statement is true.

Theorem 3. *Let* $\gamma = \{t \in \mathbb{C} \colon |t+1| = 1\}$ *and let* $\mu(\eta)$ *be a normal majorant of the class* $(\sigma, 1)$ *such that* $\mu(0^+) = 0$. *Then for a continuous function* $g \colon \gamma \to \mathbb{R}$ *given by the equality*

$$g(t) = 8\int_0^1 \mu(\eta)\frac{\big((\mathrm{Re}t)^2 - (\mathrm{Im}t)^2\big)(\eta^4 + |t|^4) - 2\eta^2|t|^4}{\big|\eta^2 - t^2\big|^4}\eta d\eta \qquad \forall t \in \gamma,$$

the following inequality holds

$$\omega_\gamma(g,\varepsilon) \le C\mu(\varepsilon) \qquad \forall \varepsilon \ge 0, \tag{23}$$

where the constant C does not depend on ε, and for the solid modulus of continuity of the function $\big(\mathrm{Re}\widetilde{g}\big)^{\pm}$ the following estimate is true

$$\omega_{\overline{D^\pm}}\Big(\big(\mathrm{Re}\widetilde{g}\big)^{\pm}, \varepsilon\Big) \ge c\varepsilon \int_\varepsilon^4 \frac{\mu(\eta)}{\eta^2} d\eta \qquad \forall \varepsilon \in (0, 1/4], \tag{24}$$

where a positive constant c does not depend on ε.

Proof. It is obvious that the function g is continuous on $\gamma \setminus \{0\}$. Let us prove that the function g is also continuous at the point 0. For $|t| < 1$, taking into account inequalities (17) and (19), we have the relations

$$|g(t) - g(0)| = |g(t)| \le 8 \cdot 18^2 |t|^2 \left(\int_0^{|t|} + \int_{|t|}^1 \right) \frac{\mu(\eta)\eta\big(\eta^2 + |t|^2\big)^2}{\big(\eta + |t|\big)^8} d\eta \le$$
$$\le 32 \cdot 18^2 |t|^2 \left(\int_0^{|t|} \frac{\mu(|t|)\eta}{|t|^4} d\eta + \int_{|t|}^1 \frac{\mu(\eta)}{\eta^3} d\eta \right) \le$$
$$\le 32 \cdot 18^2 \left(\frac{\mu(|t|)}{2} + \sigma \frac{\mu(|t|)}{|t|} |t|^2 \int_{|t|}^1 \frac{1}{\eta^2} d\eta \right) \le 72^2 (1 + 2\sigma)\mu(|t|), \tag{25}$$

which together with the condition $\mu(0^+) = 0$ imply the continuity of the function g at the point 0.

Further in the proof we will use the equality

$$g(t) = \int_0^1 \frac{t\mu(\eta)}{(\eta - t)^2} d\eta - \int_0^1 \frac{t\mu(\eta)}{(\eta + t)^2} d\eta + \int_0^1 \frac{\bar{t}\mu(\eta)}{(\eta - \bar{t})^2} d\eta - \int_0^1 \frac{\bar{t}\mu(\eta)}{(\eta + \bar{t})^2} d\eta =:$$
$$=: \Phi_1(t) - \Phi_2(t) + \Phi_3(t) - \Phi_4(t) \qquad \forall t \in \gamma. \tag{26}$$

To obtain estimate (23), consider two cases for the location of the points $t_1, t_2 \in \gamma$, for which $|t_1 - t_2| \le \varepsilon < 1$.

Case 1. Let $|t_1| \le |t_2| \le 2\varepsilon$. In this case, using estimate (25), by virtue of the function $\mu(\eta)$ is a normal majorant of the class $(\sigma, 1)$, we have

$$|g(t_1) - g(t_2)| \le |g(t_1) - g(0)| + |g(t_2) - g(0)| \le 2 \cdot 72^2(1 + 2\sigma)\mu(2\varepsilon) \le 144^2(1 + 2\sigma)\sigma\mu(\varepsilon).$$

Case 2. Let $|t_1| \le |t_2|$ and $|t_2| > 2\varepsilon$. In this case, taking into account equality (26) and Lemma 4, we have

$$|g(t_1) - g(t_2)| \le \Big|\Phi_*(t_1) - \Phi_*(t_2)\Big| + \Big|\Phi_*(\overline{t_1}) - \Phi_*(\overline{t_2})\Big| \le C\mu(\varepsilon),$$

where the constant C does not depend on ε.

The estimates obtained in the two considered cases yield inequality (23) for all $\varepsilon \in [0, 1)$. For $\varepsilon \ge 1$, inequality (23) follows from the relations

$$\omega_\gamma(g,\varepsilon) \le 2 \max_{t\in\gamma} |g(t)| = 2\frac{\mu(\varepsilon)}{\mu(\varepsilon)} \max_{t\in\gamma} |g(t)| \le \frac{2\max_{t\in\gamma} |g(t)|}{\mu(1)} \mu(\varepsilon).$$

Finally, to prove estimate (24), we consider the functions Φ_k, $k=\overline{1,4}$, from equality (26).

The function Φ_1 is analytic outside the segment $[0,1]$ and is continuously extended to the point 0 from the domain $\mathbb{C}\setminus[0,1]$ with $\Phi_1(0)=0$. Therefore, Φ_1 is analytic in the domain D^+ and continuous on $\overline{D^+}$.

The function Φ_2 is analytic outside the segment $[-1,0]$ and is continuously extended to the point 0 from the domain $\mathbb{C}\setminus[-1,0]$ with $\Phi_2(0)=0$. Therefore, Φ_2 is analytic in the domain D^- and continuous on $\overline{D^-}$. In addition, $\Phi_2(\infty)=0$.

Taking into account the equalities

$$\bar t=(\bar t+1)-1=\frac{1}{t+1}-1=-\frac{t}{t+1}\qquad \forall t\in\gamma,$$

we rewrite the functions Φ_3 and Φ_4 for all $t\in\gamma$ in the form

$$\Phi_3(t)=-\frac{t}{t+1}\int_0^1\frac{\mu(\eta)}{\left(\eta+\frac{t}{t+1}\right)^2}d\eta,\qquad \Phi_4(t)=-\frac{t}{t+1}\int_0^1\frac{\mu(\eta)}{\left(\eta-\frac{t}{t+1}\right)^2}d\eta. \tag{27}$$

Let us now consider the functions defined by the equalities (27), in which the variable t can take values outside the curve γ.

The function Φ_3 is analytic outside the segment $[-1/2,0]$ and is continuously extended to the point 0 from the domain $\mathbb{C}\setminus[-1/2,0]$ with $\Phi_3(0)=0$. Therefore, Φ_3 is analytic in the domain D^- and continuous on $\overline{D^-}$. In addition, the value of this function at infinity $\Phi_3(\infty)$ is real.

The function Φ_4 is analytic outside the infinite segment $[0,\infty]$ with the beginning point 0 and the end at infinity, which contains all positive numbers, and is continuously extended to the point 0 from the domain $\mathbb{C}\setminus[0,\infty]$ with $\Phi_4(0)=0$. Therefore, Φ_4 is analytic in the domain D^+ and continuous on $\overline{D^+}$.

Taking into account the Cauchy integral theorem and the Cauchy integral formula, we have

$$\widetilde{g}(z)=\widetilde{\Phi_1}(z)-\widetilde{\Phi_2}(z)+\widetilde{\Phi_3}(z)-\widetilde{\Phi_4}(z)=\begin{cases}\Phi_1(z)+\Phi_3(\infty)-\Phi_4(z) & \text{for } z\in D^+;\\ \Phi_2(z)+\Phi_3(\infty)-\Phi_3(z) & \text{for } z\in D^-,\end{cases}$$

and, in addition, $\widetilde{g}^{\pm}(0)=\Phi_3(\infty)$.

Therefore, for $0<\varepsilon\le 1/4$ we obtain

$$\omega_{\overline{D^+}}\Big(\big(\mathrm{Re}\widetilde{g}\big)^+,\varepsilon\Big)\ge\Big|\mathrm{Re}\big(\widetilde{g}(-\varepsilon)-\widetilde{g}^+(0)\big)\Big|=\Big|\mathrm{Re}\big(\Phi_1(-\varepsilon)-\Phi_4(-\varepsilon)\big)\Big|=$$
$$=\varepsilon\int_0^1\frac{\mu(\eta)}{(\eta+\varepsilon)^2}d\eta+\frac{\varepsilon}{1-\varepsilon}\int_0^1\frac{\mu(\eta)}{\left(\eta+\frac{\varepsilon}{1-\varepsilon}\right)^2}d\eta\ge\varepsilon\int_\varepsilon^1\frac{\mu(\eta)}{(\eta+\varepsilon)^2}d\eta\ge\frac14\varepsilon\int_\varepsilon^1\frac{\mu(\eta)}{\eta^2}d\eta. \tag{28}$$

To replace the upper limit of integration by 4 in the last integral, we give some relations. First, taking into account the inequality (17), we have

$$\varepsilon\int_1^4\frac{\mu(\eta)}{\eta^2}d\eta\le\varepsilon\mu(4)\int_1^4\frac{d\eta}{\eta^2}\le\varepsilon\mu(4)=4\varepsilon\frac{\mu(4)}{4}\le 4\varepsilon\sigma\frac{\mu(\varepsilon)}{\varepsilon}=4\sigma\mu(\varepsilon)$$

and, in addition, for $0<\varepsilon\le 1/4$, we obtain

$$\varepsilon\int_\varepsilon^1\frac{\mu(\eta)}{\eta^2}d\eta\ge\varepsilon\mu(\varepsilon)\int_\varepsilon^1\frac{d\eta}{\eta^2}=(1-\varepsilon)\mu(\varepsilon)\ge\frac34\mu(\varepsilon),$$

from which it follows that for $0 < \varepsilon \le 1/4$, the following inequalities hold:

$$\varepsilon\int_\varepsilon^4 \frac{\mu(\eta)}{\eta^2}d\eta \le \varepsilon\Big(\int_\varepsilon^1 + \int_1^4\Big)\frac{\mu(\eta)}{\eta^2}d\eta \le \Big(1+\frac{16\sigma}{3}\Big)\varepsilon\int_\varepsilon^1 \frac{\mu(\eta)}{\eta^2}d\eta. \tag{29}$$

Now, denoting $c_0 := \frac{1}{4}(1+\frac{16\sigma}{3})^{-1}$, as a corollary of the relations (28) and (29), we obtain estimate (24) for the solid modulus of continuity $\omega_{\overline{D^+}}((\mathrm{Re}\widetilde{g})^+,\varepsilon)$.

Estimate (24) for the solid modulus of continuity $\omega_{\overline{D^-}}((\mathrm{Re}\widetilde{g})^-,\varepsilon)$ follows from the relations

$$\omega_{\overline{D^-}}\Big(\big(\mathrm{Re}\widetilde{g}\big)^-,\varepsilon\Big) \ge \Big|\mathrm{Re}\big(\widetilde{g}(\varepsilon)-\widetilde{g}^-(0)\big)\Big| = \Big|\mathrm{Re}\big(\Phi_2(\varepsilon)-\Phi_3(\varepsilon)\big)\Big| =$$
$$=\varepsilon\int_0^1 \frac{\mu(\eta)}{(\eta+\varepsilon)^2}d\eta + \frac{\varepsilon}{1+\varepsilon}\int_0^1 \frac{\mu(\eta)}{\big(\eta+\frac{\varepsilon}{1+\varepsilon}\big)^2}d\eta \ge \varepsilon\int_\varepsilon^1 \frac{\mu(\eta)}{(\eta+\varepsilon)^2}d\eta \ge \frac{1}{4}\varepsilon\int_\varepsilon^1 \frac{\mu(\eta)}{\eta^2}d\eta$$

and inequalities (29). □

Theorem 3 and inequalities (23) and (16) imply the next statement.

Corollary. *Under the conditions of Theorem 3, for all $\varepsilon \in (0, 1/4]$, the following estimates are true:*

$$\omega_{\overline{D^\pm}}\Big(\big(\mathrm{Re}\widetilde{g}\big)^\pm,\varepsilon\Big) \ge c\varepsilon\int_\varepsilon^4 \frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta \ge c_1\left(M_\gamma(g,\varepsilon)+\varepsilon\int_\varepsilon^4 \frac{\omega_\gamma(g,\eta)}{\eta^2}d\eta\right), \tag{30}$$

where positive constants c and c_1 do not depend on ε.

Thus, inequalities (30) show that the upper estimates for the solid moduli of continuity of the logarithmic double layer potential, which are given in Theorems 1 and 2, are exact with respect to the order of smallness as $\varepsilon \to 0$.

The results of the work were announced in the preprint [26].

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