Математичні Студії. Т.66, №1 Matematychni Studii. V.66, No.1
ISSN 1027-4634 (print) ISSN 2411-0620 (online)

Z. O. KOVBA[1], E. O. SEVOST'YANOV*[2]

ON THE EQUICONTINUITY OF UNCLOSED ORLICZ-SOBOLEV CLASSES BY PRIME ENDS

Z. O. Kovba, E. O. Sevost'yanov. *On the equicontinuity of unclosed Orlicz-Sobolev classes by prime ends*, Mat. Stud. **66** (2026), 113–116.

We study the equicontinuity of Orlicz-Sobolev classes by prime ends. We investigate the case when the mappings are open, discrete, but not boundary preserving (closed). It is established that, under certain conditions on the cluster set, the family of above mappings is equicontinuous in the terms of prime ends whenever the corresponding Orlicz function satisfies the Calderon condition, and the inner dilatation of mappings has a majorant satisfying the integral divergence Lehto-Dini condition in the closure of a domain. A similar result was also obtained in the case where the corresponding majorant has a finite mean oscillation (FMO) at each point of the closure of a domain.

1. Introduction. This paper is devoted to the distortion estimates in some classes of mappings at the boundary points. Note that, different authors consider similar problems, in particular, for quasiconformal and quasiregular mappings, Orlicz-Sobolev classes, etc. (see, e.g., [1]–[9]). Our research in this direction concerns the case where mappings may not preserve the boundary, see, e.g., [11]–[16]. Observe that, the most of corresponding results are related to open discrete and closed mappings. At the same time, the closeness for open discrete mappings means their boundary preservation, see e.g., [10, Theorem 3.3]. If the mapping does not preserve the boundary of the domain, then it cannot be fully investigated using the modulus method. Our goal was to identify conditions on the domain and the mapping when the mapping does not preserve the boundary, but the application of the modulus method remains possible.

A Borel function $\rho\colon \mathbb{R}^n \to [0,\infty]$ is called *admissible* for Γ, abbr. $\rho\in {\rm adm}\Gamma$, if $\int_\gamma \rho(x)|dx| \geqslant 1$ for each (locally rectifiable) $\gamma\in\Gamma$. For $p\geqslant 1$, we define the quantity

$$M_p(\Gamma)=\inf_{\rho\in{\rm adm}\Gamma}\int\limits_{\mathbb{R}^n}\rho^p(x)dm(x)$$

and call $M_p(\Gamma)$ *p-modulus* of Γ [9, 6.1]; here m stands for the n-dimensional Lebesque measure.

Definitions related to Orlicz-Sobolev classes and prime ends of domains can be found in [8] and are therefore generally omitted. We also omit the concepts of a regular domain and a domain with a locally quasiconformal boundary. Let h be the chordal metric in $\overline{\mathbb{R}^n}$, and

2020 *Mathematics Subject Classification:* 30C65, 31A15, 30C62.
Keywords: equicontinuity; boundary behavior; Orlicz-Sobolev classes; continuous boundary extension.
doi:10.30970/ms.66.1.113-116
*Corresponding author: E. O. Sevost'yanov

let $h(A)$ be the chordal diameter of the set $A \subset \overline{\mathbb{R}^n}$ (see [9]). Due to [6], a domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, is called a *uniform* domain with respect to p-modulus, $p \geqslant 1$, if, for each $r > 0$, there is $\delta > 0$ such that $M_p(\Gamma(F, F^*, D)) \geqslant \delta$ whenever F and F^* are continua of D with $h(F) \geqslant r$ and $h(F^*) \geqslant r$. Let I be some set of indices. Domains D_i, $i \in I$, are said to be *equi-uniform* domains with respect to p-modulus, if, for $r > 0$, the modulus condition above is satisfied by each D_i with the same number δ. It should be noted that the proposed concept of a uniform domain has no relation to similar definition in [17]. In particular, every plane domain finitely connected on the boundary, and which has finitely many boundary components, is uniform, see e.g. [18, Corollary 6.8].

We denote by E_D the set of all prime ends in D. The completion of the domain D by its prime ends is denoted by $\overline{D}_P$. Note that, the space $\overline{D}_P = D \cup E_D$ is metric, namely, if $g\colon D_0 \to D$ is a quasiconformal mapping of a domain D_0 with a locally quasiconformal boundary onto some domain D, then for $x, y \in \overline{D}_P$ we put:

$$\rho(x, y) := |g^{-1}(x) - g^{-1}(y)|, \tag{1}$$

where the element $g^{-1}(x)$, $x \in E_D$, is to be understood as some (single) boundary point of the domain D_0 (see [3, Remark 2]). The specified boundary point exists and is unique, see [19, Theorem 4.1]. The definition of FMO class used below may be found in [5] or [8]. Given a Lebesgue measurable function $Q\colon \mathbb{R}^n \to [0, \infty]$ we put

$$Q'(x) = \begin{cases} Q(x), & Q(x) \geqslant 1; \\ 1, & Q(x) < 1. \end{cases}$$

Beside that, we set

$$q'_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int\limits_{|x-x_0|=r} Q'(x)\, d\mathcal{H}^{n-1}. \tag{2}$$

Given a mapping $f\colon D \to \mathbb{R}^n$, we denote

$$C(f, x) := \{y \in \overline{\mathbb{R}^n}\colon \exists x_k \in D\colon x_k \to x, f(x_k) \to y, k \to \infty\}$$

and

$$C(f, \partial D) = \bigcup_{x \in \partial D} C(f, x).$$

Let

$$N(y, f, A) = \mathrm{card}\, \{x \in A\colon f(x) = y\}, \qquad N(f, A) = \sup_{y \in \mathbb{R}^n} N(y, f, A).$$

Let f be a mapping having partial derivatives almost everywhere in D. We set

$$l\,(f'(x)) = \min_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|}, \qquad J(x, f) = \det f'(x).$$

Given $\alpha \geqslant 1$ and $x \in D$ we define

$$K_{I,\alpha}(x, f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^{\alpha}}, & J(x, f) \neq 0; \\ 1, & f'(x) = 0; \\ \infty, & \text{otherwise.} \end{cases}$$

2. The main result. Given $N \in \mathbb{N}$, $\delta > 0$, $\alpha > 1$, closed sets E_*, F in $\overline{\mathbb{R}^n}$, $n \geqslant 3$, a domain $D \subset \mathbb{R}^n$, a set E in D (which is closed with a respect to D), an increasing function $\varphi\colon [0, \infty) \to [0, \infty)$ and a Lebesgue measurable function $Q\colon D \to [0, \infty]$ let us denote by $\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ the family of all bounded open discrete mappings $f\colon D \to \overline{\mathbb{R}^n} \setminus F$ in $W^{1,\varphi}_{\rm loc}(D)$ such that $K_{I,\alpha}(x, f) \leqslant Q(x)$ a.e., $N(f, D) \leqslant N$ and, in addition,

1) $C(f, \partial D) \subset E_*$,

2) for each component K of $D'_f \setminus E_*$, $D'_f := f(D)$, there is a continuum $K_f \subset K$ such that $h(K_f) \geqslant \delta$ and $h(f^{-1}(K_f), \partial D) \geqslant \delta > 0$,

3) $f^{-1}(E_*) = E$.

The following statement holds.

Theorem 1. *Let $n \geqslant 3$, let $n - 1 < \alpha \leqslant n$, let D be a regular domain in $\mathbb{R}^n$, let $Q : D \to [0, \infty]$ be a Lebesgue measurable function and let $\varphi\colon [0, \infty) \to [0, \infty)$ be an increasing function. Assume that, the following conditions are satisfied:*

1) the set E is nowhere dense in D, and D is finitely connected on E, i.e., for any $z_0 \in E$ and any neighborhood $\widetilde{U}$ of z_0 there is a neighborhood $\widetilde{V} \subset \widetilde{U}$ of z_0 such that $(D \cap \widetilde{V}) \setminus E$ consists of finite number of components;

2) for any $P \in E_D := \overline{D}_P \setminus D$ and for every neighborhood U of P in $\overline{D}_P$ there is a neighborhood $V \subset U$ in $\overline{D}_P$ of P such that $V \cap D$ is connected and $(V \cap D) \setminus E$ consists at most of m components, $1 \leqslant m < \infty$,

Let for $\alpha = n$ the set F has a positive capacity, and for $n - 1 < \alpha < n$ it is an arbitrary closed set. Suppose that, for any $x_0 \in \overline{D}$ at least one of the following conditions is satisfied: 3_1) a function $Q \in FMO(x_0)$; 3_2) $q_{x_0}(r) = O([\log \frac{1}{r}]^{n-1})$ as $r \to 0$; 3_3) the condition

$$\int\limits_0^{\delta(x_0)} \frac{dt}{t^{\frac{n-1}{\alpha-1}} q'^{\frac{1}{\alpha-1}}_{x_0}(t)} = \infty$$

holds for some $\delta(x_0) > 0$, where $q'_{x_0}(t)$ is defined in (2).

4) Assume that, the function φ satisfies Calderon condition

$$\int\limits_1^{\infty} \left(\frac{t}{\varphi(t)}\right)^{\frac{1}{n-2}} dt < \infty.$$

Let the family of all components of $D'_f \setminus E_*$ is equi-uniform over $f \in \mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ with respect to α-modulus. Then the family $\mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ is equicontinuous with respect to $\overline{D}_P$, i.e., for every $x_0 \in \overline{D}_P$ and every $\varepsilon > 0$ there is $\delta = \delta(x_0, \varepsilon) > 0$ such that $\rho(f(x), f(y)) < \varepsilon$ for every $f \in \mathfrak{R}^{E_*,E,F,N}_{\varphi,Q,\delta,\alpha}(D)$ and any $x, y \in B_\rho(x_0, \delta)$, where $B_\rho(x_0, \delta) = \{x \in D\colon \rho(x, x_0) < \delta\}$ and ρ is defined in (1). In particular, f has a continuous extension to E_D.

Acknowledgments. The work was supported by the National Research Foundation of Ukraine (Project “Analogues of Carathéodory and Koebe-Bloch theorems for Orlycz-Sobolev classes”, Project number 2025.02/0010).

References

1. M. Cristea, *Local Homeomorphisms Having Local ACL^n Inverses,* Complex Variables Elliptic Equ. **53** (1) (2008), 77–99. doi:10.1080/17476930701666924
2. M. Cristea, *Open Discrete Mappings Having Local ACL^n Inverses,* Complex Variables Elliptic Equ. **55** (1–3) (2010), 61–90. doi:10.1080/17476930902998985
3. D. Kovtonyuk, V. Ryazanov, *On the Theory of Prime Ends for Space Mappings,* Ukr. Math. J. **67** (4) (2015), 528–541. doi:10.1007/s11253-015-1098-9
4. O. Martio, S. Rickman, J. Väisälä, *Distortion and Singularities of Quasiregular Mappings,* Ann. Acad. Sci. Fenn. Ser. A I Math. **465** (1970), 1–13. doi:10.5186/aasfm.1978-79.0415
5. O. Martio, V. Ryazanov, U. Srebro, E. Yakubov, Moduli in Modern Mapping Theory, Springer Monographs in Mathematics, Springer, New York etc., 2009.
6. R. Näkki, B. Palka, *Uniform Equicontinuity of Quasiconformal Mappings,* Proc. Amer. Math. Soc. **37** (2) (1973), 427–433. doi:10.2307/2039456
7. S. Rickman, Quasiregular Mappings, Springer-Verlag, Berlin, 1993.
8. E. Sevost'yanov, Mappings with Direct and Inverse Poletsky Inequalities, Developments in Mathematics, V.78, Springer Nature, Cham, 2023.
9. J. Väisälä, Lectures on n-Dimensional Quasiconformal Mappings, Lecture Notes in Math., V.229, Springer-Verlag, Berlin, 1971.
10. M. Vuorinen, *Exceptional Sets and Boundary Behavior of Quasiregular Mappings in n-Space,* Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes **11** (1976), 1–44.
11. V. Desyatka, E. Sevost'yanov, *On Boundary-Non-Preserving Mappings with Poletsky Inequality,* Canad. Math. Bull. **68** (3) (2025), 834–855. doi:10.4153/S0008439525000141
12. Z. Kovba, E. Sevost'yanov, *On Caratheodory Theorem for Non-Closed Mappings,* J. Math. Sci. **299** (2026), 48–63. doi:10.1007/s10958-026-08397-0
13. V. Desyatka, A. Halyts'ka, E. Sevost'yanov, *On Boundary Extension of Unclosed Orlicz-Sobolev Mappings,* J. Math. Sci. **302** (3) (2026), 329–348. doi:10.1007/s10958-026-08603-z
14. V. Desyatka, E. Sevost'yanov, *Caratheodory Boundary Extensions for Generalized Quasiregular Mappings,* Z. Anal. Anw. (2026). https://ems.press/journals/zaa/articles/14299657 doi:10.4171/ZAA/1824
15. Z. Kovba, E. Sevost'yanov, *On Caratheodory Theorem for Open Discrete Unclosed Mappings,* Georgian Math. J. (2026). https://www.degruyterbrill.com/document/doi/10.1515/gmj-2026-3041/html doi:10.1515/gmj-2026-3041
16. Z. Kovba, E. Sevost'yanov, *On Caratheodory Prime Ends Extension for Unclosed Orlicz-Sobolev Classes,* arXiv:2604.15026 (2026). https://arxiv.org/abs/2604.15026 doi:10.48550/arXiv.2604.15026
17. O. Martio, J. Sarvas, *Injectivity Theorems in Plane and Space,* Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1978/1979), 384–401. doi:10.5186/aasfm.1978-79.0413
18. R. Näkki, *Extension of Loewner's Capacity Theorem,* Trans. Amer. Math. Soc. **180** (1973), 229–236. doi:10.2307/1996662
19. R. Näkki, *Prime Ends and Quasiconformal Mappings,* J. Anal. Math. **35** (1979), 13–40. doi:10.1007/BF02791061

[1,2]Zhytomyr Ivan Franko State University
Zhytomyr, Ukraine
[2] Institute of Applied Mathematics and Mechanics of NAS of Ukraine
Sloviansk, Ukraine

[1]mazhydova@gmail.com
[2]esevostyanov2009@gmail.com

Received 26.04.2026
Revised 14.09.2026