Математичні Студії. Т.66, №1 Matematychni Studii. V.66, No.1
ISSN 1027-4634 (print) ISSN 2411-0620 (online)

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APPROXIMATION CHARACTERISTICS OF NIKOL'SKII–BESOV AND SOBOLEV CLASSES IN THE SPACE $B_{q,1}$

K. V. Pozharska, A. S. Romanyuk. *Approximation characteristics of Nikol'skii–Besov and Sobolev classes in the space* $B_{q,1}$, Mat. Stud. **66** (2026), 57–70.

We investigate Kolmogorov width and entropy numbers of the Nikol'skii–Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ and Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ of periodic functions of mixed smoothness of one and many variables in the space $B_{q,1}$. The norm in this space is "stronger" than the respective L_q-norm. The paper considers the case $2 \le p < q < \infty$, which has remained unexplored so far, since it required a significantly different method for getting upper estimates of the Kolmogorov width.

It was shown that in this particular case of relations between the parameters p and q, the exact-order estimates of the Kolmogorov widths of both of the considered classes in the space $B_{q,1}$ are not realized by an approximation by the subspace of trigonometric polynomials with "numbers" of harmonics from the step hyperbolic crosses of the corresponding dimension. Besides, in the multidimensional case ($d \ge 2$), in contrast to the one-dimensional case, the orders of the respective Kolmogorov widths and entropy numbers in the space $B_{q,1}$ differ from the corresponding orders of these approximation characteristics for the L_q-space.

1. Introduction. The paper investigates Kolmogorov widths and entropy numbers of the Nikol'skii–Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ and Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ of periodic functions of one and several variables in the space $B_{q,1}$, $2 \le p < q < \infty$. The norm in this space is "stronger" than the L_q-norm. The mentioned approximation characteristics, as well as some others, on the classes of periodic functions of one and many variables in the spaces $B_{q,1}$, $q \in \{1, \infty\}$, have been investigated already in the papers [1–3], and recently further studied, in particular, in [4,6,7]. Subsequently, the study of Kolmogorov widths and entropy numbers of the classes $B^{\boldsymbol{r}}_{p,\theta}$ and $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ was extended to the spaces $B_{q,1}$, $1 < q < \infty$, for some relations between the parameters p, q [8–10]. At the same time, the case $2 \le p < q < \infty$ remained open, since the applied method for getting upper estimates of Kolmogorov widths did not give the desired result. Thus, the results obtained in this paper complement the corresponding statements from the works [8–10].

The work consists of three parts.

The first part (Section 2) contains the definitions of the function classes $B^{\boldsymbol{r}}_{p,\theta}$ and $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ and the spaces $B_{q,1}$.

In the second part (Section 3), definitions of approximation characteristics are given and auxiliary statements are formulated.

The third part of the paper (Section 4), which is the main one, is devoted to establishing exact-order estimates of Kolmogorov widths and entropy numbers of the classes $B^{\boldsymbol{r}}_{p,\theta}$ and $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the space $B_{q,1}$, $2 \le p < q < \infty$.

2020 *Mathematics Subject Classification:* 41A46, 41A63.
Keywords: Kolmogorov width; entropy number; Sobolev class; Nikol'skii–Besov class; order estimate.
doi:10.30970/ms.66.1.57-70

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Based on the results of the research, we can make the following conclusions.

The obtained exact-order estimates of the Kolmogorov widths of the classes $B^{\boldsymbol{r}}_{p,\theta}$ and $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the space $B_{q,1}$ for $2 \le p < q < \infty$ are not realized by an approximation by the subspace of trigonometric polynomials with "numbers" of harmonics from the step hyperbolic crosses of the corresponding dimension.

It was found that in the multidimensional case ($d \ge 2$), in contrast to the one-dimensional case, the considered approximation characteristics of the classes $B^{\boldsymbol{r}}_{p,\theta}$ and $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the space $B_{q,1}$ differ in order from the corresponding characteristics in the L_q-space.

These issues will be discussed in more detail in the comments to the obtained results.

2. Definition of function classes and spaces $B_{q,1}$. Let $\mathbb{R}^d$ be a d-dimensional Euclidean space with elements $\boldsymbol{x} = (x_1, \dots, x_d)$ and $(\boldsymbol{x}, \boldsymbol{y}) = x_1y_1 + \dots + x_dy_d$. By $L_p(\mathbb{T}^d)$, $\mathbb{T}^d := \prod_{j=1}^d [0, 2\pi)$, $1 \le p \le \infty$, we denote the space of functions f, that are 2π-periodic in each variable and such that

$$\|f\|_p := \|f\|_{L_p(\mathbb{T}^d)} := \left((2\pi)^{-d} \int_{\mathbb{T}^d} |f(\boldsymbol{x})|^p d\boldsymbol{x} \right)^{1/p} < \infty, \quad 1 \le p < \infty,$$

$$\|f\|_\infty := \|f\|_{L_\infty(\mathbb{T}^d)} := \operatorname{ess\,sup}\{|f(\boldsymbol{x})| \colon \boldsymbol{x} \in \mathbb{T}^d\} < \infty.$$

Let us denote $\triangle^l_{\boldsymbol{h}} f(\boldsymbol{x}) := \triangle^l_{h_d} \dots \triangle^l_{h_1} f(x_1, \dots, x_d)$, for $\boldsymbol{h} = (h_1, \dots, h_d) \in \mathbb{R}^d$ and $l \in \mathbb{N}$, where $\triangle^l_{h_j} f(\boldsymbol{x}) = \triangle_{h_j} \triangle^{l-1}_{h_j} f(\boldsymbol{x})$, $\triangle^0_h f(\boldsymbol{x}) := f(\boldsymbol{x})$,

$$\triangle_{h_j} f(\boldsymbol{x}) \equiv \triangle^1_{h_j} f(\boldsymbol{x}) = f(x_1, \dots, x_j + h_j, \dots, x_d) - f(\boldsymbol{x}).$$

We define the full mixed pth modulus of smoothness of order l of the function $f \in L_p(\mathbb{T}^d)$ according to the formula

$$\omega_l(f, \boldsymbol{t})_p := \sup_{|h_j| \le t_j} \|\triangle^l_{\boldsymbol{h}} f\|_p, \ \boldsymbol{t} \in \mathbb{R}^d, t_j > 0, j = 1, \dots, d.$$

A function $f \in L_p(\mathbb{T}^d)$, $1 \le p \le \infty$, belongs to the space $B^{\boldsymbol{r}}_{p,\theta}$, $1 \le \theta \le \infty$, $\boldsymbol{r} \in \mathbb{R}^d$, $r_j > 0$, $j = 1, \dots, d$, $l > \max\{r_j, j = 1, \dots, d\}$, if is finite its semi-norm

$$|f|_{B^{\boldsymbol{r}}_{p,\theta}} := \Big(\int_{\mathbb{T}^d} \Big(\prod_{j=1}^d t_j^{-r_j} \, \omega_l(f, \boldsymbol{t})_p \Big)^\theta \prod_{j=1}^d \frac{dt_j}{t_j} \Big)^{1/\theta}, \text{ for } 1 \le \theta < \infty,$$

$$|f|_{B^{\boldsymbol{r}}_{p,\infty}} := \sup_{t_j > 0} \prod_{j=1}^d t_j^{-r_j} \, \omega_l(f, \boldsymbol{t})_p.$$

The norm on linear spaces $B^{\boldsymbol{r}}_{p,\theta}$ is given by the formula

$$\|f\|_{B^{\boldsymbol{r}}_{p,\theta}} := \|f\|_p + |f|_{B^{\boldsymbol{r}}_{p,\theta}}.$$

The spaces $B^{\boldsymbol{r}}_{p,\theta}$ were introduced in [11]. On the one hand, they generalize the well-known isotropic Besov spaces [12] (in particular, in the case of $\theta = \infty$, — the Nikol'skii spaces $H^{\boldsymbol{r}}_p$ [13]), and on the other hand they belong to the scale of spaces SB of mixed smoothness introduced by T.I. Amanov [14].

We note that for the spaces $B^{\boldsymbol{r}}_{p,\theta}$ the following embeddings with respect to the parameter θ hold

$$B^{\boldsymbol{r}}_{p,1} \subset B^{\boldsymbol{r}}_{p,\theta_1} \subset B^{\boldsymbol{r}}_{p,\theta_2} \subset B^{\boldsymbol{r}}_{p,\infty} \equiv H^{\boldsymbol{r}}_p, \ 1 \le \theta_1 < \theta_2 \le \infty.$$

In [11] the spaces $B^{\boldsymbol{r}}_{p,\theta}$ were characterized in terms of the so-called decomposition normalization of the functions belonging to them based on their expansion into a Fourier series with respect to the trigonometric system $\{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{\boldsymbol{k} \in \mathbb{Z}^d}$. Let us formulate the result from [11] in accordance with the definitions and notations given below.

For a vector $\boldsymbol{s} \in (\mathbb{N} \cup \{0\})^d$, we set
$$\rho(\boldsymbol{s}) := \{\boldsymbol{k} \in \mathbb{Z}^d \colon\ [2^{s_j-1}] \le |k_j| < 2^{s_j},\ j = 1, \dots, d\},$$
where $[a]$ is the integer part of the number a. Then for $f \in L_p(\mathbb{T}^d)$ we define

$$\delta_{\boldsymbol{s}}(f) := \delta_{\boldsymbol{s}}(f, \boldsymbol{x}) := \sum_{\boldsymbol{k} \in \rho(\boldsymbol{s})} \widehat{f}(\boldsymbol{k}) e^{i(\boldsymbol{k}, \boldsymbol{x})}, \quad \boldsymbol{x} \in \mathbb{R}^d,$$

where $\widehat{f}(\boldsymbol{k}) = \int_{\mathbb{T}^d} f(\boldsymbol{t}) e^{-i(\boldsymbol{k}, \boldsymbol{t})} d\boldsymbol{t}$ are the Fourier coefficients of the function f.

We define
$$L_p^0(\mathbb{T}^d) := \Big\{ f \in L_p(\mathbb{T}^d) \colon \int_0^{2\pi} f(\boldsymbol{x}) dx_j = 0, j = 1, \dots, d, \text{ almost everywhere} \Big\}.$$

Let $1 < p < \infty$. In [11] it was proved that for the norm $\|f\|_{B^{\boldsymbol{r}}_{p,\theta}}$ of functions $f \in B^{\boldsymbol{r}}_{p,\theta} \cap L_p^0$ it holds

$$\|f\|_{B^{\boldsymbol{r}}_{p,\theta}} \asymp \Big(\sum_{\boldsymbol{s} \in \mathbb{N}^d} 2^{(\boldsymbol{s}, \boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}(f)\|_p^\theta \Big)^{1/\theta}, \quad 1 \le \theta < \infty, \tag{1}$$

$$\|f\|_{H^{\boldsymbol{r}}_p} \equiv \|f\|_{B^{\boldsymbol{r}}_{p,\infty}} \asymp \sup_{\boldsymbol{s} \in \mathbb{N}^d} 2^{(\boldsymbol{s}, \boldsymbol{r})} \|\delta_{\boldsymbol{s}}(f)\|_p. \tag{2}$$

Here and below, for positive quantities a and b, the notation $a \asymp b$ is used, that means that there exist positive constants C_1 and C_2 that do not depend on an essential parameter in the values a and b, such that $C_1 a \le b$ (we write $a \ll b$) and $C_2 a \ge b$ (we write $a \gg b$). All of the constants C_i, $i = 1, 2, \dots$ in this paper depend only on the parameters contained in the definition of the class, the metric in which we estimate the error of approximation, and the dimension of the space $\mathbb{R}^d$.

For a finite set $\mathfrak{N}$ by $|\mathfrak{N}|$ we denote the number of its elements.

Note that the analogs of relations (1), (2) after the corresponding modification hold also for the extreme values $p = 1$ and $p = \infty$ (see, e.g., [11, Remark 2.1]).

Let $V_l(t)$, $t \in \mathbb{R}$, $l \in \mathbb{N}$, denotes the de la Vallée-Poussin kernel of the form
$$V_l(t) := 1 + 2 \sum_{k=1}^{l} \cos kt + 2 \sum_{k=l+1}^{2l-1} \Big(1 - \frac{k-l}{l}\Big) \cos kt,$$
where for $l = 1$ we assume that the third term equals to zero. We associate each vector $\boldsymbol{s} \in \mathbb{N}^d$ with the polynomial
$$A_{\boldsymbol{s}}(\boldsymbol{x}) := \prod_{j=1}^{d} (V_{2^{s_j}}(x_j) - V_{2^{s_j-1}}(x_j)),$$
and for $f \in L_p^0(\mathbb{T}^d)$, $1 \le p \le \infty$, set

$$A_{\boldsymbol{s}}(f) := A_{\boldsymbol{s}}(f, \boldsymbol{x}) := (f * A_{\boldsymbol{s}})(\boldsymbol{x}),$$

where "$*$" denotes the operation of convolution. Thus for $1 \le p \le \infty$, $\boldsymbol{r} \in \mathbb{R}^d$, $r_j > 0$, $j = 1, \dots, d$, for $f \in B^{\boldsymbol{r}}_{p,\theta} \cap L_p^0$ it holds

$$\|f\|_{B^{\boldsymbol{r}}_{p,\theta}} \asymp \Big(\sum_{\boldsymbol{s} \in \mathbb{N}^d} 2^{(\boldsymbol{s}, \boldsymbol{r})\theta} \|A_{\boldsymbol{s}}(f)\|_p^\theta \Big)^{1/\theta}, \quad 1 \le \theta < \infty, \tag{3}$$

$$\|f\|_{H^{\boldsymbol{r}}_p} \equiv \|f\|_{B^{\boldsymbol{r}}_{p,\infty}} \asymp \sup_{\boldsymbol{s} \in \mathbb{N}^d} 2^{(\boldsymbol{s}, \boldsymbol{r})} \|A_{\boldsymbol{s}}(f)\|_p. \tag{4}$$

The class $B^{\boldsymbol{r}}_{p,\theta}$ is defined in what follows as the set of functions $f \in L^0_p$ such that $\|f\|_{B^{\boldsymbol{r}}_{p,\theta}} \le 1$, where the norm is given by either the relations (1), (2) or (3), (4), depending on the value of the parameter p.

In the paper we also investigate the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$, that are defined in the following way. Let $F_{\boldsymbol{r}}(\boldsymbol{x},\boldsymbol{\alpha})$ be multivariate analogs of the Bernoulli kernels, i.e.,

$$F_{\boldsymbol{r}}(\boldsymbol{x},\boldsymbol{\alpha}) := 2^d \sum_{\boldsymbol{k}\in\mathbb{N}^d} \prod_{j=1}^{d} k_j^{-r_j} \cos\left(k_j x_j - \frac{\alpha_j \pi}{2}\right), \quad r_j > 0, \alpha_j \in \mathbb{R}, j = 1,\dots,d.$$

Then by $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ we denote the class of functions f of the form

$$f(\boldsymbol{x}) = \varphi(\boldsymbol{x}) * F_{\boldsymbol{r}}(\boldsymbol{x},\boldsymbol{\alpha}) := (2\pi)^{-d} \int_{\mathbb{T}^d} \varphi(\boldsymbol{y}) F_{\boldsymbol{r}}(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{\alpha}) d\boldsymbol{y},$$

with $\varphi \in L^0_p(\mathbb{T}^d)$, $\|\varphi\|_p \le 1$.

For the defined classes, the following embeddings hold:

$$B^{\boldsymbol{r}}_{p,\min\{p,2\}} \subset W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{p,\max\{p,2\}}, \quad 1 < p < \infty;$$
$$W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{p,\infty} \equiv H^{\boldsymbol{r}}_p, \quad 1 \le p \le \infty;$$
$$B^{\boldsymbol{r}}_{1,1} \subset W^{\boldsymbol{r}}_{1,\boldsymbol{\alpha}} \subset H^{\boldsymbol{r}}_1.$$

In particular, in the case $\theta = p = 2$ we have

$$W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{2,2} \subset W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}}.$$

The history overview of the study of approximation characteristics of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$, $H^{\boldsymbol{r}}_p$ and $B^{\boldsymbol{r}}_{p,\theta}$, $1 \le \theta < \infty$, in Lebesgue spaces $L_q(\mathbb{T}^d)$ $1 \le p,q \le \infty$, can be found in the monographs [15–18].

Further we will assume that coordinates of the vector $\boldsymbol{r} \in \mathbb{R}^d$ in the defined classes are ordered such that $0 < r_1 = r_2 = \dots = r_\nu < r_{\nu+1} \le \dots \le r_d$, and also that $\boldsymbol{\gamma} \in \mathbb{R}^d$ is a vector with the coordinates $\gamma_j = r_j/r_1$, $j = 1,\dots,d$. Besides, $\boldsymbol{\gamma}' \in \mathbb{R}^d$, where $\gamma'_j = \gamma_j = 1$ if $j = 1,\dots,\nu$ and $1 < \gamma'_j < \gamma_j$ if $j = \nu+1,\dots,d$.

In what follows, we define the norm $\|\cdot\|_{B_{q,1}}$, $1 \le q \le \infty$, in the subspaces $B_{q,1}$ of functions $f \in L^0_q(\mathbb{T}^d)$. For trigonometric polynomials t with respect to the multiple trigonometric system $\{e^{i(\boldsymbol{k},\boldsymbol{x})}\}_{\boldsymbol{k}\in\mathbb{Z}^d}$, the norm $\|t\|_{B_{q,1}}$, $1 \le q \le \infty$, is defined as

$$\|t\|_{B_{q,1}} := \sum_{\boldsymbol{s}\in\mathbb{N}^d} \|A_{\boldsymbol{s}}(t)\|_q$$

(the sum contains a finite number of terms).

Similarly we define the norm $\|f\|_{B_{q,1}}$, $1 \le q \le \infty$, for all functions $f \in L^0_q(\mathbb{T}^d)$, such that the series $\sum_{\boldsymbol{s}\in\mathbb{N}^d} \|A_{\boldsymbol{s}}(f)\|_q$ is convergent.

Note, that in the case $1 < q < \infty$ it holds

$$\|f\|_{B_{q,1}} \asymp \sum_{\boldsymbol{s}\in\mathbb{N}^d} \|\delta_{\boldsymbol{s}}(f)\|_q.$$

For $f \in B_{q,1}$ the following relations hold

$$\|f\|_q \ll \|f\|_{B_{q,1}}, \qquad \|f\|_{B_{1,1}} \ll \|f\|_{B_{q,1}} \ll \|f\|_{B_{\infty,1}}, \qquad 1 \le q \le \infty.$$

3. Approximation characteristics and auxiliary statements. Let $\mathcal{X}$ be a Banach space and let $B_{\mathcal{X}}(y,R)$ be a ball in $\mathcal{X}$ of radius R centered at a point y, i.e.,

$$B_{\mathcal{X}}(y,R) := \{x \in \mathcal{X}\colon\ \|x-y\|_{\mathcal{X}} \le R\}.$$

Then for a compact set $A \subset \mathcal{X}$ and $k \in \mathbb{N}$ the quantities (see, e.g., [19])

$$\varepsilon_k(A,\mathcal{X}) := \inf\Big\{\varepsilon\colon \quad \exists y^1,\dots,y^{2^k} \in \mathcal{X}\colon \quad A \subset \bigcup_{j=1}^{2^k} B_{\mathcal{X}}(y^j,\varepsilon)\Big\}. \tag{5}$$

are called the entropy numbers of the set A with respect to the space $\mathcal{X}$.

Let further $\mathcal{Y}$ be a normed space with the norm $\|\cdot\|_{\mathcal{Y}}$, $\mathcal{L}_m(\mathcal{Y})$ be the set of all subspaces of dimension at most m in the space $\mathcal{Y}$, and W be a centrally-symmetric set in $\mathcal{Y}$. Then the quantity

$$d_m(W,\mathcal{Y}) := \inf_{L_m \in \mathcal{L}_m(\mathcal{Y})} \sup_{w\in W} \inf_{u \in L_m} \|w-u\|_{\mathcal{Y}} \tag{6}$$

is called the Kolmogorov m-width of the set W in the space $\mathcal{Y}$.

The width $d_m(W,\mathcal{Y})$ was introduced in 1936 by A.N. Kolmogorov [20].

Remark 1. In further considerations, it will be more convenient for us to use the definition of the width $d_m(W,\mathcal{Y})$ in slightly different terms. Namely,

$$d_m(W,\mathcal{Y}) := \inf\{\sup\{\|w-Pw\|_{\mathcal{Y}}\colon w\in W\}\colon P\},$$

where the infimum is taken over all mappings P acting in $\mathcal{Y}$ onto linear subspaces of dimension no greater than m.

Let us recall the definition of another approximation characteristic, which we will use in the comments to the obtained results.

For $\boldsymbol{s} \in \mathbb{N}^d$, $n \in \mathbb{N}$ and $\boldsymbol{\gamma} \in \mathbb{R}^d$, $\gamma_j > 0$, $j = 1,\dots,d$, we put

$$Q_n^{\boldsymbol{\gamma}} := \bigcup_{(\boldsymbol{s},\boldsymbol{\gamma})<n} \rho(\boldsymbol{s}).$$

The set $Q_n^{\boldsymbol{\gamma}}$ is called the step hyperbolic cross. In the case $\boldsymbol{\gamma}' = (\gamma_1',\dots,\gamma_d')$, we use the notation $Q_n^{\boldsymbol{\gamma}'}$ instead of $Q_n^{\boldsymbol{\gamma}}$.

For $f \in L_1^0(\mathbb{T}^d)$ we define

$$S_{Q_n^{\boldsymbol{\gamma}}}(f) := S_{Q_n^{\boldsymbol{\gamma}}}(f,\boldsymbol{x}) := \sum_{\boldsymbol{k}\in Q_n^{\boldsymbol{\gamma}}} \widehat{f}(\boldsymbol{k}) e^{i(\boldsymbol{k},\boldsymbol{x})}, \quad \boldsymbol{x} \in \mathbb{R}^d.$$

The polynomials $S_{Q_n^{\boldsymbol{\gamma}}}(f)$ are called the step hyperbolic Fourier sums of the function f. According to the introduced above notation, they can be rewritten in the form

$$S_{Q_n^{\boldsymbol{\gamma}}}(f) = \sum_{(\boldsymbol{s},\boldsymbol{\gamma})<n} \delta_{\boldsymbol{s}}(f).$$

Let $\mathcal{X}$ be a d-dimensional, $d \ge 1$, function space defined on $\mathbb{T}^d$ equipped with the norm $\|\cdot\|_{\mathcal{X}}$. Then, for a function class $F \subset \mathcal{X}$, we set

$$\mathcal{E}_{Q_n^{\boldsymbol{\gamma}}}(F)_{\mathcal{X}} := \sup_{f\in F} \big\| f - S_{Q_n^{\boldsymbol{\gamma}}}(f)\big\|_{\mathcal{X}}.$$

Next, we formulate a number of known statements that will be used in further considerations.

Let l_p^n, $n \in \mathbb{N}$, $1 \le p \le \infty$, denotes the space $\mathbb{R}^n$ endowed with the norm

$$\|\boldsymbol{x}\|_{l_p^n} := \begin{cases} \Big(\sum\limits_{i=1}^{n} |x_i|^p \Big)^{1/p}, & 1 \le p < \infty; \\ \max\limits_{1\le i \le n} |x_i|, & p = \infty. \end{cases}$$

By B_p^n we denote the unit ball in l_p^n, that is, the set of elements $\boldsymbol{x} \in l_p^n$ for which $\|\boldsymbol{x}\|_{l_p^n} \le 1$.

If $\mathcal{T}_s$ is a subspace spanned by the exponents $e^{i(\boldsymbol{k},\boldsymbol{x})}$, $\boldsymbol{k} \in \rho(\boldsymbol{s})$, $\boldsymbol{s} \in \mathbb{N}^d$, then the following statement is true.

Lemma A ([21]). *Let $1 < p, q < \infty$, $\boldsymbol{s} \in \mathbb{N}^d$, $m_{\boldsymbol{s}} \in \mathbb{N}$ and $m_{\boldsymbol{s}} \le 2^{(\boldsymbol{s},\boldsymbol{1})}$. Then there exists a linear subspace $L_{m_s} \subset \mathcal{T}_{\boldsymbol{s}}$ of dimension $m_{\boldsymbol{s}}$ and an operator $P_{m_s} \colon \mathcal{T}_{\boldsymbol{s}} \to L_{m_s}$ such that for $f \in \mathcal{T}_{\boldsymbol{s}}$ the following order inequality holds*

$$\|f - P_{m_{\boldsymbol{s}}} f\|_q \ll d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\boldsymbol{1})}}, l_q^{2^{(\boldsymbol{s},\boldsymbol{1})}}) 2^{(\boldsymbol{s},\boldsymbol{1})(\frac{1}{p}-\frac{1}{q})} \|f\|_p. \tag{7}$$

Lemma B ([15, p. 11]). *Let $\boldsymbol{s} \in \mathbb{N}^d$, $\boldsymbol{\gamma}, \boldsymbol{\delta} \in \mathbb{R}^d$ such that $1 = \gamma_1 = \delta_1 = \dots = \gamma_\nu = \delta_\nu$ and $1 < \delta_j < \gamma_j$, $j = \nu+1, \dots, d$. Then for $\eta > 0$ it holds that*

$$\sum_{(\boldsymbol{s},\boldsymbol{\gamma})<n} 2^{\eta(\boldsymbol{s},\boldsymbol{\delta})} \ll 2^{\eta n} n^{\nu-1}. \tag{8}$$

Lemma C ([15, p. 11]). *The following order estimate holds*

$$\sum_{(\boldsymbol{s},\boldsymbol{\gamma}')\ge n} 2^{-\eta(\boldsymbol{s},\boldsymbol{\delta})} \asymp 2^{-\eta n} n^{\nu-1}, \qquad \eta > 0. \tag{9}$$

Theorem A ([22]). *Let $2 \le p < q \le \infty$, $\beta = \frac{1/p-1/q}{1-2/q}$ and $m < n$. Then it holds that*

$$d_m(B_p^n, l_q^n) \asymp \min\{1, n^{\frac{2\beta}{q}} m^{-\beta}\}. \tag{10}$$

Theorem B ([10]). *Let $r_1 > 0$, $1 \le \theta \le \infty$. Then it holds that*

$$\varepsilon_m(B_{\infty,\theta}^{\boldsymbol{r}}, B_{1,1}) \gg m^{-r_1} (\log^{\nu-1} m)^{r_1+1-\frac{1}{\theta}}. \tag{11}$$

Here and in what follows, the notation $\log m$ will be used for the logarithm with the base 2, i.e. $\log m := \log_2 m$.

The following statement (a consequence of one of Carl's inequalities (see, e.g., [23])) plays an important role in establishing upper bounds on the entropy numbers of the classes $B_{p,\theta}^{\boldsymbol{r}}$ in the space $B_{q,1}$.

Lemma D ([24]). *Let $\mathcal{K}$ be a compact set in a separable Banach space $\mathcal{X}$. Assume that for a pair of numbers (a, b), where either $a > 0$, $b \in \mathbb{R}$ or $a = 0$, $b < 0$, the relations*

$$d_m(\mathcal{K}, \mathcal{X}) \ll m^{-a} (\log m)^b, \qquad \varepsilon_m(\mathcal{K}, \mathcal{X}) \gg m^{-a} (\log m)^b$$

are true. Then, it holds

$$\varepsilon_m(\mathcal{K}, \mathcal{X}) \asymp d_m(\mathcal{K}, \mathcal{X}) \asymp m^{-a} (\log m)^b. \tag{12}$$

Theorem C ([10]). *Let $d \geq 2$, $1 < p < \infty$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ it holds*

$$\mathcal{E}_{Q_n^{\gamma'}}(B^{\boldsymbol{r}}_{p,\theta})_{B_{p,1}} \asymp 2^{-nr_1} n^{(\nu-1)(1-\frac{1}{\theta})}. \tag{13}$$

Theorem D ([10]). *Let $d \geq 2$, $1 \leq \theta \leq \infty$, $1 < p < q < \infty$. Then for $r_1 > 1/p - 1/q$ it holds*

$$\mathcal{E}_{Q_n^{\gamma}}(B^{\boldsymbol{r}}_{p,\theta})_{B_{q,1}} \asymp 2^{-n(r_1-\frac{1}{p}+\frac{1}{q})} n^{(\nu-1)(1-\frac{1}{\theta})}. \tag{14}$$

Theorem E ([10]). *Let $d \geq 2$, $1 < p < \infty$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ it holds*

$$\varepsilon_m(B^{\boldsymbol{r}}_{p,\theta}, B_{p,1}) \asymp d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{p,1}) \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+1-\frac{1}{\theta}}. \tag{15}$$

Remark 2. The order of the Kolmogorov width $d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{p,1})$ in Theorem E is realized by approximating the functions $f \in B^{\boldsymbol{r}}_{p,\theta}$ by their Fourier sums $S_{Q_n^{\gamma'}}(f)$ (see Theorem C) under the condition $m \asymp 2^n n^{\nu-1}$.

Theorem F ([21,25,26]). *Let $d \geq 2$, $2 \leq p < q < \infty$, $1 \leq \theta \leq \infty$. Then for $r_1 > \beta := \frac{1/p-1/q}{1-2/q}$ it holds*

$$d_m(B^{\boldsymbol{r}}_{p,\theta}, L_q) \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+(\frac{1}{2}-\frac{1}{\theta})_+}, \tag{16}$$

where $a_+ = \max\{a, 0\}$.

Theorem G. *Let $d \geq 1$, $1 \leq \theta \leq \infty$. Then for $r_1 > 0$ it holds*

$$\varepsilon_m(B^{\boldsymbol{r}}_{\infty,\theta}, L_1) \gg m^{-r_1}(\log^{\nu-1} m)^{r_1+\frac{1}{2}-\frac{1}{\theta}}. \tag{17}$$

Estimate (17) for $\theta = \infty$ was proved in [1], and for $1 \leq \theta < \infty$ in [27].

4. Main results. First, we establish the orders of entropy numbers and Kolmogorov widths of the Nikol'skii–Besov classes $B^{\boldsymbol{r}}_{p,\theta}$ in the space $B_{q,1}$, $2 \leq p < q < \infty$, in the multidimensional case ($d \geq 2$).

Theorem 1. *Let $d \geq 2$, $2 \leq p < q < \infty$, $1 \leq \theta \leq \infty$. Then for $r_1 > \beta = \frac{1/p-1/q}{1-2/q}$ the following relations hold*

$$\varepsilon_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \asymp d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+1-\frac{1}{\theta}}. \tag{18}$$

Proof. In order to use Lemma D, we first establish an upper estimate for the Kolmogorov width $d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1})$.

So, let $f \in B^{\boldsymbol{r}}_{p,\theta}$. Consider the operator $P_m \colon L_q(\mathbb{T}^d) \to L_m$, which we define for $f \in L_q(\mathbb{T}^d)$ by the formula

$$P_m f := \sum_{\boldsymbol{s} \in \mathbb{N}^d} P_{m_{\boldsymbol{s}}} \delta_{\boldsymbol{s}}(f),$$

where $P_{m_{\boldsymbol{s}}} \colon \mathcal{T}_{\boldsymbol{s}} \to L_{m_{\boldsymbol{s}}}$, the subspaces $L_{m_{\boldsymbol{s}}}$ from Lemma A and $\bigcup_{\boldsymbol{s}\in\mathbb{N}^d} L_{m_{\boldsymbol{s}}} = L_m$.

Then, taking into account the definition of the norm $\|\cdot\|_{B_{q,1}}$, $1 < q < \infty$, and using Lemma A, we can write

$$\|f - P_m f\|_{B_{q,1}} \asymp \sum_{\boldsymbol{s}\in\mathbb{N}^d} \|\delta_{\boldsymbol{s}}(f - P_m f)\|_q = \sum_{\boldsymbol{s}\in\mathbb{N}^d} \|\delta_{\boldsymbol{s}}(f) - P_{m_{\boldsymbol{s}}}\delta_{\boldsymbol{s}}(f)\|_q \ll$$
$$\ll \sum_{\boldsymbol{s}\in\mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{1})(\frac{1}{p}-\frac{1}{q})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\boldsymbol{1})}}, l_q^{2^{(\boldsymbol{s},\boldsymbol{1})}}) \|\delta_{\boldsymbol{s}}(f)\|_p = J_1. \tag{19}$$

To simplify further calculations, we introduce the notation $\boldsymbol{\alpha} := \boldsymbol{r} - (1/p - 1/q)\mathbf{1}$.

Let us consider three cases.

Let $\theta \in (1, \infty)$. Applying to J_1 the Hölder inequality with exponent θ, and $1/\theta' + 1/\theta = 1$, we obtain

$$J_1 \le \Big(\sum_{\boldsymbol{s} \in \mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})\theta} \|\delta_{\boldsymbol{s}}(f)\|_p^\theta \Big)^{\frac{1}{\theta}} \Big(\sum_{\boldsymbol{s} \in \mathbb{N}^d} (2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}))^{\theta'} \Big)^{1/\theta'} \ll$$

$$\ll \|f\|_{B^{\boldsymbol{r}}_{p,\theta}} \Big(\sum_{\boldsymbol{s} \in \mathbb{N}^d} (2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}))^{\theta'} \Big)^{1/\theta'} \le \Big(\sum_{\boldsymbol{s} \in \mathbb{N}^d} (2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}))^{\theta'} \Big)^{1/\theta'}. \tag{20}$$

Let $\theta = 1$. Then after some transformations we will have

$$J_1 = \sum_{\boldsymbol{s} \in \mathbb{N}^d} (2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}) 2^{(\boldsymbol{s},\boldsymbol{r})} \|\delta_{\boldsymbol{s}}(f)\|_p \le$$

$$\le \sup_{\boldsymbol{s} \in \mathbb{N}^d} 2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}) \sum_{\boldsymbol{s} \in \mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})} \|\delta_{\boldsymbol{s}}(f)\|_p \asymp$$

$$\asymp \sup_{\boldsymbol{s} \in \mathbb{N}^d} 2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}) \, \|f\|_{B^{\boldsymbol{r}}_{p,1}} \le \sup_{\boldsymbol{s} \in \mathbb{N}^d} 2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}). \tag{21}$$

Let $\theta = \infty$. Given that in this case $\|f\|_{B^{\boldsymbol{r}}_{p,\infty}} \ll 2^{-(\boldsymbol{s},\boldsymbol{r})}$, we can write

$$J_1 \ll \sup_{\boldsymbol{s} \in \mathbb{N}^d} 2^{(\boldsymbol{s},\boldsymbol{r})} \|\delta_{\boldsymbol{s}}(f)\|_p \sum_{\boldsymbol{s} \in \mathbb{N}^d} 2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}) \asymp$$

$$\asymp \|f\|_{B^{\boldsymbol{r}}_{p,\infty}} \sum_{\boldsymbol{s} \in \mathbb{N}^d} 2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}) \le \sum_{\boldsymbol{s} \in \mathbb{N}^d} 2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}). \tag{22}$$

Thus, combining the estimates (20)–(22) and taking into account Remark 1, we have

$$d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \ll \begin{cases} \Big(\sum\limits_{\boldsymbol{s} \in \mathbb{N}^d} (2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}))^{\theta'} \Big)^{1/\theta'}, & 1 < \theta \le \infty; \\ \sup\limits_{\boldsymbol{s} \in \mathbb{N}^d} 2^{-(\boldsymbol{s},\boldsymbol{\alpha})} d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\mathbf{1})}}, l_q^{2^{(\boldsymbol{s},\mathbf{1})}}), & \theta = 1. \end{cases} \tag{23}$$

Here we note that the estimate (22) is actually contained in (20), since $\theta' = 1$ for $\theta = \infty$.

For making further evaluation of the right-hand sides of (23), we define the numbers $m_{\boldsymbol{s}}$ using the formulas

$$m_{\boldsymbol{s}} := \begin{cases} 2^{(\boldsymbol{s},\mathbf{1})}, & (\boldsymbol{s},\boldsymbol{\gamma}) \le \mu; \\ [2^{\mu(1+\varepsilon\alpha_1) - \varepsilon(\boldsymbol{s},\boldsymbol{\alpha})}], & (\boldsymbol{s},\boldsymbol{\gamma}) > \mu, \end{cases}$$

where $[a]$ is the integer part of a, $\varepsilon > 0$, $\alpha_1 = r_1 - 1/p + 1/q$ and the number μ is chosen from the condition $2^\mu \mu^{\nu-1} \asymp m$. We will show that with such a choice of numbers $m_{\boldsymbol{s}}$, the following relation holds

$$\sum_{\boldsymbol{s} \in \mathbb{N}^d} m_{\boldsymbol{s}} \ll m.$$

In view of the definition of $m_{\boldsymbol{s}}$, we have

$$\sum_{\boldsymbol{s} \in \mathbb{N}^d} m_{\boldsymbol{s}} \le \sum_{(\boldsymbol{s},\boldsymbol{\gamma}) \le \mu} 2^{(\boldsymbol{s},\mathbf{1})} + \sum_{(\boldsymbol{s},\boldsymbol{\gamma}) > \mu} 2^{\mu(1+\varepsilon\alpha_1) - \varepsilon(\boldsymbol{s},\boldsymbol{\alpha})}. \tag{24}$$

The first term on the right-hand side of (24) is estimated using Lemma B, i.e.

$$\sum_{(\boldsymbol{s},\boldsymbol{\gamma})\le\mu} 2^{(\boldsymbol{s},\boldsymbol{1})} \ll 2^{\mu}\mu^{\nu-1}. \tag{25}$$

Moving to evaluation of the second term in (24), we write

$$\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} 2^{\mu(1+\varepsilon\alpha_1)-\varepsilon(\boldsymbol{s},\boldsymbol{\alpha})} = 2^{\mu(1+\varepsilon\alpha_1)}\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} 2^{-\varepsilon(\boldsymbol{s},\boldsymbol{\alpha})} = 2^{\mu(1+\varepsilon\alpha_1)}\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} 2^{-\varepsilon\alpha_1(\boldsymbol{s},\widetilde{\boldsymbol{\gamma}})}, \tag{26}$$

where $\widetilde{\boldsymbol{\gamma}} = (1,\dots,1,\widetilde{\gamma}_{\nu+1},\dots,\widetilde{\gamma}_d)$ and $\widetilde{\gamma}_j = \frac{\alpha_j}{\alpha_1} = \frac{r_j-1/p+1/q}{r_1-1/p+1/q}$, $j=\nu+1,\dots,d$. It is easy to verify that $\widetilde{\gamma}_j > \gamma_j = r_j/r_1$, $j=\nu+1,\dots,d$, and therefore using Lemma C we obtain

$$\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} 2^{-\varepsilon\alpha_1(\boldsymbol{s},\widetilde{\boldsymbol{\gamma}})} \ll 2^{-\varepsilon\alpha_1\mu}\mu^{\nu-1}. \tag{27}$$

Substituting (27) into (26) we have

$$\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} 2^{\mu(1+\varepsilon\alpha_1)-\varepsilon(\boldsymbol{s},\boldsymbol{\alpha})} \ll 2^{\mu(1+\varepsilon\alpha_1)}2^{-\varepsilon\alpha_1\mu}\mu^{\nu-1} = 2^{\mu}\mu^{\nu-1}. \tag{28}$$

Finally, taking into account the estimates (28), (25) and (24), we arrive at the required relation

$$\sum_{\boldsymbol{s}\in\mathbb{N}^d} m_{\boldsymbol{s}} \ll 2^{\mu}\mu^{\nu-1} \asymp m.$$

Returning now directly to the continuation of estimates (23), first of all we note that according to the choice of numbers $m_{\boldsymbol{s}}$ we have

$$d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\boldsymbol{1})}}, l_q^{2^{(\boldsymbol{s},\boldsymbol{1})}}) = 0, \quad (\boldsymbol{s},\boldsymbol{\gamma})\le\mu. \tag{29}$$

Using further Theorem A, for the remaining widths in (23) we can write

$$d_{m_{\boldsymbol{s}}}(B_p^{2^{(\boldsymbol{s},\boldsymbol{1})}}, l_q^{2^{(\boldsymbol{s},\boldsymbol{1})}}) \ll 2^{\frac{2\beta}{q}(\boldsymbol{s},\boldsymbol{1})} m_{\boldsymbol{s}}^{-\beta}, \quad \beta = \frac{1/p-1/q}{1-2/q},\ (\boldsymbol{s},\boldsymbol{\gamma})>\mu. \tag{30}$$

Thus, taking into account (29) and (30), the relation (23) will take the form

$$d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \ll \begin{cases} \Big(\sum\limits_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} (2^{-(\boldsymbol{s},\boldsymbol{\alpha})}2^{\frac{2\beta}{q}(\boldsymbol{s},\boldsymbol{1})} m_{\boldsymbol{s}}^{-\beta})^{\theta'}\Big)^{1/\theta'}, & 1<\theta\le\infty; \\ \sup\limits_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} 2^{-(\boldsymbol{s},\boldsymbol{\alpha})}2^{\frac{2\beta}{q}(\boldsymbol{s},\boldsymbol{1})} m_{\boldsymbol{s}}^{-\beta}, & \theta=1. \end{cases} \tag{31}$$

Therefore, substituting in (31) the values of m_s and taking into account that $1/p-1/q+2\beta/q=\beta$, we obtain

$$d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \ll \begin{cases} \Big(\sum\limits_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} (2^{(\boldsymbol{s},-\boldsymbol{r}+(\beta+\beta\varepsilon\alpha)\boldsymbol{1})-\beta\mu(1+\varepsilon\alpha_1)})^{\theta'}\Big)^{1/\theta'}, & 1<\theta\le\infty; \\ \sup\limits_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} 2^{(\boldsymbol{s},-\boldsymbol{r}+(\beta+\beta\varepsilon\alpha)\boldsymbol{1})-\beta\mu(1+\varepsilon\alpha_1)}, & \theta=1. \end{cases} \tag{32}$$

Next, we consider separately the cases $\theta = 1$ and $1 < \theta \le \infty$.

Let $\theta = 1$. Then

$$d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \ll \sup_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} 2^{-(\boldsymbol{s},\boldsymbol{r}-(\beta+\beta\varepsilon\alpha)\mathbf{1})-\beta\mu(1+\varepsilon\alpha_1)} = \sup_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} 2^{-(\boldsymbol{s},\overline{\boldsymbol{\gamma}})(r_1-\beta-\beta\varepsilon\alpha_1)-\beta\mu(1+\varepsilon\alpha_1)}, \quad (33)$$

where $\overline{\boldsymbol{\gamma}} = (1, \dots, 1, \overline{\gamma}_{\nu+1}, \dots, \overline{\gamma}_d)$, $\overline{\gamma}_j = \frac{r_j-\beta-\beta\varepsilon\alpha_j}{r_1-\beta-\beta\varepsilon\alpha_1}$, $j = \nu+1, \dots, d$.

Using elementary transformations, it is easy to verify that for a certain $\varepsilon > 0$ the inequalities $r_1 - \beta - \beta\varepsilon\alpha_1 > 0$ and $\overline{\gamma}_j > \gamma_j$, $j = \nu+1, \dots, d$, that is also $(\boldsymbol{s}, \overline{\boldsymbol{\gamma}}) > (\boldsymbol{s}, \boldsymbol{\gamma}) > \mu$, are true. Taking into account this inequalities, the estimate (33) takes the form

$$d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \ll 2^{-\mu(r_1-\beta-\beta\varepsilon\alpha_1)-\beta\mu(1+\varepsilon\alpha_1)} = 2^{-\mu r_1} \asymp (m^{-1}\log^{\nu-1} m)^{r_1}. \quad (34)$$

Let $1 < \theta \le \infty$. Then, having performed certain transformations on the right-hand side of (32) and applying Lemma C, we have

$$d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \ll 2^{-\beta\mu(1+\varepsilon\alpha_1)} \Big(\sum_{(\boldsymbol{s},\boldsymbol{\gamma})>\mu} 2^{-(\boldsymbol{s},\overline{\boldsymbol{\gamma}})(r_1-\beta-\beta\varepsilon\alpha_1)\theta'} \Big)^{1/\theta'} \ll$$

$$\ll 2^{-\beta\mu(1+\varepsilon\alpha_1)} 2^{-\mu(r_1-\beta-\beta\varepsilon\alpha_1)} \mu^{\frac{\nu-1}{\theta'}} = 2^{-\mu r_1} \mu^{(\nu-1)(1-\frac{1}{\theta})} \asymp m^{-r_1} (\log^{\nu-1} m)^{r_1+1-\frac{1}{\theta}}. \quad (35)$$

Thus, combining (34) and (35), we arrive at an upper estimate of the Kolmogorov width $d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1})$:

$$d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \ll m^{-r_1} (\log^{\nu-1} m)^{r_1+1-\frac{1}{\theta}}. \quad (36)$$

Moving on to the lower estimate of the entropy numbers $\varepsilon_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1})$, we note that $B^{\boldsymbol{r}}_{\infty,\theta} \subset B^{\boldsymbol{r}}_{p,\theta}$, $1 \le p \le \infty$, and $\|\cdot\|_{B_{q,1}} \gg \|\cdot\|_{B_{1,1}}$. Therefore, taking into account these relations and using Theorem B, we obtain

$$\varepsilon_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1}) \gg \varepsilon_m(B^{\boldsymbol{r}}_{\infty,\theta}, B_{1,1}) \gg m^{-r_1} (\log^{\nu-1} m)^{r_1+1-\frac{1}{\theta}}. \quad (37)$$

Hence, applying Lemma D to estimates (36) and (37), we obtain (18). $\square$

In order to comment on the result obtained in Theorem 1, we first make some remarks regarding the estimates of the corresponding quantities in the L_q-space.

On the one hand, the estimate of the Kolmogorov width $d_m(B^{\boldsymbol{r}}_{p,\theta}, L_q)$ is known (see Theorem E), which for $2 \le p < q < \infty$, $r_1 > \beta = \frac{1/p-1/q}{1-2/q}$ and $1 \le \theta \le \infty$ has the form

$$d_m(B^{\boldsymbol{r}}_{p,\theta}, L_q) \asymp m^{-r_1} (\log^{\nu-1} m)^{r_1+(\frac{1}{2}-\frac{1}{\theta})_+}, \quad (38)$$

where $a_+ = \max\{a, 0\}$.

On the other hand, using the result of Theorem G, we can write a lower bound for the entropy numbers $\varepsilon_m(B^{\boldsymbol{r}}_{p,\theta}, L_q)$ as follows:

$$\varepsilon_m(B^{\boldsymbol{r}}_{p,\theta}, L_q) \gg \varepsilon_m(B^{\boldsymbol{r}}_{\infty,\theta}, L_1) \gg m^{-r_1} (\log^{\nu-1} m)^{r_1+\frac{1}{2}-\frac{1}{\theta}}, \quad (39)$$

where $r_1 > 0$, $1 \le \theta \le \infty$.

Therefore, applying Lemma D to the bounds (38) and (39), we obtain the following statement.

Theorem 2. *Let $d \geq 2$, $2 \leq \theta \leq \infty$, $2 \leq p < q < \infty$. Then for $r_1 > \beta = \frac{1/p-1/q}{1-2/q}$ it holds*

$$\varepsilon_m(B^{\boldsymbol{r}}_{p,\theta}, L_q) \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+\frac{1}{2}-\frac{1}{\theta}}. \tag{40}$$

Thus, comparing relations (18), (38) and (40) we come to the following conclusion.

The estimates of the corresponding characteristics of the classes $B^{\boldsymbol{r}}_{p,\theta}$, $2 \leq \theta \leq \infty$, in the spaces $B_{q,1}$ and L_q, except for the case $\nu = 1$, are different in order.

In the one-dimensional case ($d = 1$) the situation is different. For this, we give the corresponding statements in the spaces $B_{q,1}$ and L_q.

Theorem 1′. *Let $d = 1$, $2 \leq p < q < \infty$, $1 \leq \theta \leq \infty$. Then for $r_1 > \beta = \frac{1/p-1/q}{1-2/q}$ we have*

$$\varepsilon_m(B^{r_1}_{p,\theta}, B_{q,1}) \asymp d_m(B^{r_1}_{p,\theta}, B_{q,1}) \asymp m^{-r_1}. \tag{41}$$

Proof. The upper bound for the Kolmogorov width $d_m(B^{r_1}_{p,\theta}, B_{q,1})$ is established by modifying the arguments that were used in proving Theorem 1.

As a result, we will have

$$d_m(B^{r_1}_{p,\theta}, B_{q,1}) \ll m^{-r_1}. \tag{42}$$

The lower bound for the entropy numbers $\varepsilon_m(B^{r_1}_{p,\theta}, B_{q,1})$ is a consequence of relation (11) for $\nu = 1$, i.e.

$$\varepsilon_m(B^{r_1}_{p,\theta}, B_{q,1}) \gg \varepsilon_m(B^{r_1}_{\infty,\theta}, B_{1,1}) \gg m^{-r_1}. \tag{43}$$

Using Lemma G in relation to the bounds (42) and (43), we arrive at (41). □

The statement corresponding to Theorem 1′ in the space L_q has the following form.

Theorem 2′. *Let $d = 1$, $1 \leq \theta \leq \infty$, $2 \leq p < q < \infty$. Then for $r_1 > \beta = \frac{1/p-1/q}{1-2/q}$ the following relations hold*

$$\varepsilon_m(B^{r_1}_{p,\theta}, L_q) \asymp d_m(B^{r_1}_{p,\theta}, L_q) \asymp m^{-r_1}. \tag{44}$$

Proof. The upper bound for the Kolmogorov width $d_m(B^{r_1}_{p,\theta}, L_q)$ follows from (42) according to the inequality $\|\cdot\|_q \ll \|\cdot\|_{B_{q,1}}$, i.e.

$$d_m(B^{r_1}_{p,\theta}, L_q) \ll d_m(B^{r_1}_{p,\theta}, B_{q,1}) \ll m^{-r_1}. \tag{45}$$

The lower bound for the entropy numbers $\varepsilon_m(B^{r_1}_{p,\theta}, L_q)$ is obtained from (17) as follows

$$\varepsilon_m(B^{r_1}_{p,\theta}, L_q) \gg \varepsilon_m(B^{r_1}_{\infty,\theta}, L_1) \gg m^{-r_1}. \tag{46}$$

Applying Lemma D to (45) and (46), we obtain (44). □

So, comparing the results of Theorems 1′ and 2′, we observe that the estimates of the corresponding characteristics of the classes $B^{r_1}_{p,\theta}$ in the spaces $B_{q,1}$ and L_q are of the same order. In addition, unlike the multidimensional case, these estimates do not depend on the parameter θ.

In addition to the above comments, we will make one more important remark, that has an opposite to Remark 2 sense.

Remark 3. Comparing the results of Theorems 1 and D, we come to the following conclusion. The order of the width $d_m(B^{\boldsymbol{r}}_{p,\theta}, B_{q,1})$, $2 \le p < q < \infty$, is not realized by a subspace of trigonometric polynomials with "numbers" of harmonics from step hyperbolic crosses Q^{γ}_n under the condition $m \asymp 2^n n^{d-1}$. A similar situation is observed in the one-dimensional case, where the subspace of polynomials

$$T_m := \Big\{t\colon\ t(x) = \sum_{k=-m}^{m} c_k e^{ikx}\Big\},\ c_k \in \mathbb{C},$$

also does not realize the order of the width $d_m(B^{r_1}_{p,\theta}, B_{q,1})$, $2 \le p < q < \infty$.

Next, we establish the order of the Kolmogorov widths and entropy numbers of the Sobolev classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ also in the space $B_{q,1}$ for $2 \le p < q < \infty$.

Theorem 3. *Let $d \ge 2$, $2 \le p < q < \infty$, $\boldsymbol{\alpha} \in \mathbb{R}^d$. Then for $r_1 > \frac{1}{2}$ the following estimates hold*

$$\varepsilon_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \asymp d_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+\frac{1}{2}}. \tag{47}$$

Proof. Let us first establish an upper estimate of the Kolmogorov width $d_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1})$. Since $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}} \subset W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}}$, $2 \le p < \infty$, it suffices to obtain an upper estimate of the quantity $d_m(W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}}, B_{q,1})$. Note that in this case $\beta = 1/2$ and accordingly $r_1 > 1/2$.

Therefore, taking into account that $W^{\boldsymbol{r}}_{2,\boldsymbol{\alpha}} \subset B^{\boldsymbol{r}}_{2,2}$, and using the result of Theorem 1 for $p = \theta = 2$, we have

$$d_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \ll d_m(W^{\boldsymbol{r}}_{2,\alpha}, B_{q,1}) \ll d_m(B^{\boldsymbol{r}}_{2,2}, B_{q,1}) \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1+\frac{1}{2}}. \tag{48}$$

On the other hand, using the known estimate (see [27])

$$\varepsilon_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{1,1}) \gg m^{-r_1}(\log^{\nu-1} m)^{r_1+\frac{1}{2}}, \quad 1 \le p < \infty, r_1 > 0, \boldsymbol{\alpha} \in \mathbb{R}^d, d \ge 1, \tag{49}$$

we get

$$\varepsilon_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1}) \gg \varepsilon_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{1,1}) \gg m^{-r_1}(\log^{\nu-1} m)^{r_1+\frac{1}{2}}. \tag{50}$$

Applying Lemma D to estimates (48) and (50), we obtain (47). $\square$

In addition to Theorem 3, we give the corresponding statement in the one-dimensional case.

Theorem 3$'$. *Let $d = 1$, $2 \le p < q < \infty$, $\alpha \in \mathbb{R}$. Then for $r_1 > 1/2$ the following relations hold*

$$\varepsilon_m(W^{r_1}_{p,\alpha}, B_{q,1}) \asymp d_m(W^{r_1}_{p,\alpha}, B_{q,1}) \asymp m^{-r_1}. \tag{51}$$

Proof. The upper bound for the quantity $d_m(W^{r_1}_{p,\alpha}, B_{q,1})$ is obtained from Theorem 1$'$ using the same considerations that were used to establish the bound (48).

We have

$$d_m(W^{r_1}_{p,\alpha}, B_{q,1}) \ll d_m(W^{r_1}_{2,\alpha}, B_{q,1}) \ll d_m(B^{r_1}_{2,2}, B_{q,1}) \asymp m^{-r_1}. \tag{52}$$

The lower bound for the entropy numbers $\varepsilon_m(W^{r_1}_{p,\alpha}, B_{q,1})$ follows from (49) for $\nu = 1$, i.e.

$$\varepsilon_m(W^{r_1}_{p,\alpha}, B_{q,1}) \gg \varepsilon_m(W^{r_1}_{p,\alpha}, B_{1,1}) \gg m^{-r_1}. \tag{53}$$

Using Lemma D to the estimates (52) and (53) we arrive at (51). $\square$

In order to comment on the results of Theorems 3, $3'$ we present analogues of relations (47) and (51) in the space L_q (see [18, Theorems 4.3.1, 6.2.1]).

Therefore, for $d \geq 1$, $2 \leq p < q < \infty$, $r_1 > 1/2$, $\boldsymbol{\alpha} \in \mathbb{R}^d$ the following relations hold:

$$\varepsilon_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_q) \asymp d_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, L_q) \asymp m^{-r_1}(\log^{\nu-1} m)^{r_1}. \tag{54}$$

Thus, comparing relations (47) and (54) for $d \geq 2$, we observe that for $\nu \neq 1$ the orders of the corresponding characteristics of the classes $W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}$ in the spaces $B_{q,1}$ and L_q are different.

In the case of $d = 1$ the situation is different. Namely, as relations (51) and (54) show for $\nu = 1$, the considered characteristics in the spaces $B_{q,1}$ and L_q are the same in order.

In conclusion, we note that the subspaces of trigonometric polynomials discussed in the comments to Theorem 1 also do not realize the orders of Kolmogorov widths $d_m(W^{\boldsymbol{r}}_{p,\boldsymbol{\alpha}}, B_{q,1})$, $2 \leq p < q < \infty$.

Acknowledgements. KP would like to acknowledge support by the budget program of the NAS of Ukraine "Support for the development of priority areas of research" (KPKVK 6541230) and the German Research Foundation (DFG 403/4-1).

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Received 21.04.2026
Revised 15.09.2026