Математичні Студії. Т.65, №1 Matematychni Studii. V.65, No.1
ISSN 1027-4634 (print) ISSN 2411-0620 (online)

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# A DISCRETE FRÉCHET FUZZY METRIC

M. Berezkyi, O. Berezsky, M. Zarichnyi. *A discrete Fréchet fuzzy metric*, Mat. Stud. **65** (2026), 97–106.

The Fréchet distance between curves in metric spaces is known to have its fuzzy counterpart. In the present note we consider the fuzzy discrete distance between sequences of points in fuzzy metric spaces. Also, discrete curves in non-Archimedean fuzzy metric spaces are considered.

It is proved that the Fréchet distance between piecewise linear curves in the fuzzy normed spaces equals the discrete Fréchet distance between discrete curves consisting of their vertices.

The fuzzy distance between two points can be interpreted as a function of the parameter $t > 0$. We prove that the fuzzy distance between continuous curves is approximated by the distances between their close discrete curves in the topology of convergence on compact sets.

**Introduction.** Among the generalizations and modifications of metric spaces, the fuzzy metric spaces play a special role. These spaces are related to the statistical metric spaces and the Menger spaces (see, e.g., [23] and [37]).

The most important theories of fuzzy metric spaces are those by George and Veeramani ([12, 14]) and by Kramosil and Michálek ([27]).

In this article we deal with the notion of fuzzy metric space in the sense of [12, 14]. The main reason of this is because these fuzzy metrics induce metrizable topologies, which is important for applications.

The theory of fuzzy metric spaces is developing in various directions; its applications include composite indicators ([25]), fixed point theory ([16, 21]), asymptotic topology ([22, 40, 41]), color image processing ([18]), computer vision and image recognition (see, e.g., [26, 31, 32]).

Some constructions in the theory of metric spaces have their analogues for fuzzy metric spaces. These include fuzzy Hausdorff metrics on spaces of (closed) sets ([33]), fuzzy Prokhorov metrics on spaces of probability measures ([34]), fuzzy metrics on spaces of idempotent measures ([6]), etc. Recently, a fuzzy Fréchet metric on a set of curves in a fuzzy metric space has been defined ([4]).

Recall that the Fréchet distance between curves in a metric space was introduced in [11]. The Fréchet metric and its modifications found numerous applications, e.g., in hand-writing recognition ([38]), speech recognition ([28]), analysis of moving objects ([7]), molecular biology ([24]) etc. See also [42] and references therein.

The present paper is devoted to the discrete version of the fuzzy Fréchet distance between (discrete) curves in a fuzzy metric space. Separately, we consider piecewise linear curves in fuzzy normed spaces, as well as discrete curves in non-Archimedean fuzzy metric spaces.

2020 *Mathematics Subject Classification:* 54A40, 54E35, 51K05.
*Keywords:* Fréchet distance; discrete curve; fuzzy metric space; fuzzy ultrametric space.
doi:10.30970/ms.65.1.97-106

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The fuzzy distance between two points can be interpreted as a function of the parameter $t > 0$. We prove that the fuzzy distance between continuous curves is approximated by the distances between their close discrete curves in the topology of convergence on compact sets.

Finally, we formulate some open problems.

## 1. Preliminaries.

**1.1. Fuzzy metric spaces and fuzzy normed spaces.** The unit segment $[0,1]$ will be denoted by $\mathbb{I}$. A t-norm is a continuous, associatiative, commutative function $*\colon \mathbb{I}\times\mathbb{I}\to\mathbb{I}$ which is monotone (in the sense that $x\le x'$ and $y\le y'$ imply $x*y\le x'*y'$) and 1 is the unit for $*$.

Some examples of t-norms are: min (denoted by $\wedge$), $\cdot$ (i.e., multiplication), $(a,b)\mapsto \max\{0,a+b-1\}$ (the Łukasiewicz t-norm). See [37] for more examples as well as general constructions of t-norms.

We recall the definition of the fuzzy metric space in the sense of [12].

Let $X$ be a set, $*$ be a t-norm, and $\mathbb{R}_+ = (0,+\infty)$. A GV-*fuzzy metric* on $X$ (i.e. in the sense of George and Veeramani) is a pair $(M,*)$, where $M\colon X\times X\times\mathbb{R}_+\to(0,1]$ a mapping satisfies the following conditions for all $x,y,z\in X$, $s,t\in\mathbb{R}_+$:

$(1GV)$ $M(x,y,t)>0$;

$(2GV)$ $M(x,y,t)=1$ if and only if $x=y$;

$(3GV)$ $M(x,y,t)=M(y,x,t)$;

$(4GV)$ $M(x,z,t+s)\ge M(x,y,t)*M(y,z,s)$;

$(5GV)$ $M(x,y,-)\colon\mathbb{R}_+\to[0,1]$ is continuous.

If $(M,*)$ is a GV-fuzzy metric on $X$, then the triple $(X,M,*)$ is called a GV-fuzzy metric space. For the sake of brevity, in the sequel we write "fuzzy" instead of "GV-fuzzy".

Given $x\in X$, $e\in(0,1)$, and $t>0$, one can define

$$B(x,e,t)=\{y\in X\colon\ M(x,y,t)>1-e\}.$$

It is proved that every fuzzy metric $M$ on $X$ generates a topology $\tau_M$ on $X$ which has as a base the family of open sets of the form $B(x,e,t)$, where $x\in X$, $0<e<1$, $t>0$.

A topological space $(X,\tau)$ is said to be fuzzy metrizable if there is a fuzzy metric $M$ on $X$ such that $\tau=\tau_M$. Then, it was proved ([13]) that a topological space is fuzzy metrizable if and only if it is metrizable.

The following notion is introduced in [20].

A fuzzy metric $M$ on $X$ is said to be *stationary* if $M$ does not depend on $t$, i.e., if for each $x,y\in X$, the function $M_{x,y}=M(x,y,-)$ is constant. In this case we write $M(x,y)$ instead of $M(x,y,t)$.

A triple $(X,M,\wedge)$ is called a *fuzzy ultrametric space* if $X$ is a nonempty set, and $M$ is a fuzzy set on $X\times X\times(0,\infty)$ satisfying conditions 1), 2), 3) and 5) of Definition 2 and moreover 4') $M(x,y,t)\wedge M(y,z,s)\le M(x,z,\max\{t,s\})$. It is known that 4') is equivalent to the condition $M(x,y,t)\wedge M(y,z,t)\le M(x,z,t)$ (see, e.g., [36]).

Given a topological space $X$, we denote by $\exp X$ the set of all nonempty compact subsets in $X$. Let $(X,M,*)$ be a fuzzy metric space. Given $x\in X$ and $A,B\in\exp X$, let $M(x,B,t)=\sup_{y\in B}M(x,y,t)$ and $M(A,B,t)=\inf_{x\in A}M(x,B,t)$. Then the Hausdorff fuzzy metric is a function $M_H\colon\ \exp X\times\exp X\times(0,\infty)\to(0,1]$ defined by the formula

$$M_H(A,B,t)=\min\{M(A,B,t),M(B,A,t)\}.$$

Then $M_H$ is known to be a fuzzy metric on $\exp X$ (see [33]).

Let $(X, d)$ be a metric space. Then the function $M_d\colon X \times X \times (0, \infty) \to \mathbb{R}$ defined by the formula $M_d(x, y, t) = \frac{t}{d(x,y)+t}$ is known to be a fuzzy metric for $* = \cdot$ (see [14]). We say that $M$ is the standard fuzzy metric generated by $d$.

We will need the following definition from [35]. The 3-tuple $(X, N, *)$ is said to be a fuzzy normed space if $X$ is a vector space (over $\mathbb{R}$), $*$ is a continuous t-norm and $N$ is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $t, s > 0$:

$(i)$ $N(x, t) > 0$,

$(ii)$ $N(x, t) = 1$ iff $x = 0$,

$(iii)$ $N(\alpha x, t) = N(x, t/|\alpha|)$, for all $\alpha \neq 0$,

$(iv)$ $N(x, t) * N(y, s) \leq N(x + y, t + s)$,

$(v)$ $N(x, \cdot)\colon (0, \infty) \to [0, 1]$ is continuous,

$(vi)$ $\lim_{t\to\infty} N(x, t) = 1$.

Given a fuzzy normed space $(X, N, *)$, the function $M\colon X \times X \times (0, \infty)$ defined by the formula $M(x, y, t) = N(x - y, t)$, is a fuzzy metric on $X$ (see [35]).

The structure of fuzzy normed space is tightly connected with convexity in linear spaces in the case of the norm $\wedge$ (see [1]).

**Lemma 1.** *Let $N$ be a fuzzy norm on a linear space $X$ over $\mathbb{R}$ with respect to the t-norm $\wedge$ and let $M$ be a fuzzy metric on $X$ generated by $N$. Given $x, y, z \in X$ and $\alpha \in [0, 1]$, we have*

$$M(x, \alpha y + (1 - \alpha)z, t) \geq \min\{M(x, y, t), M(x, z, t)\}.$$

*Proof.* We may assume that $0 < \alpha < 1$. Then

$$\begin{aligned} M(x, \alpha y + (1 - \alpha)z, t) &= N(\alpha y + (1 - \alpha)z - x, t) \geq \\ \geq N(\alpha y - \alpha x, \alpha t) &* N((1 - \alpha)z - (1 - \alpha)x, (1 - \alpha)t) = \\ = N(y - x, t) * N(z - x, t) &= \min\{M(x, y, t), M(x, z, t)\}. \end{aligned}$$

□

**1.2. Fréchet distance.** Let $X$ be a topological space. By $\mathcal{C}(X)$ we denote the set of all parametric curves in $X$, i.e., all continuous maps $\gamma\colon [0, 1] \to X$. The curves are considered up to monotonic change of parameter. In the sequel we are interested in the case when $X$ is a (fuzzy) metric space.

Let $(X, d)$ be a metric space. By $\mathbb{I}$ we denote the segment $[0, 1]$. By $\mathcal{H}(\mathbb{I})$ we denote the set of all nondecreasing continuous selfmaps of $\mathbb{I}$ that preserve endpoints.

Having two continuous parametric curves $\gamma_i\colon \mathbb{I} \to X$, $i = 1, 2$, their Fréchet distance is defined by the formula

$$d_F(\gamma_1, \gamma_2) = \inf_{\alpha,\alpha_2\in\mathcal{H}(\mathbb{I})} \sup\{d(\gamma_1(\alpha_1(t)), \gamma_2(\alpha_2(t)))\colon t \in \mathbb{I}\}.$$

It is well known [11] that the function $d_F$ is a metric on the set $\mathcal{C}(X)$.

Let $(X, M, *)$ be a fuzzy metric space, $\gamma_i\colon \mathbb{I} \to X$, $i = 1, 2$, be parametric curves in $X$. Define

$$M_F(\gamma_1, \gamma_2, t) = \sup_{\alpha\in\mathcal{H}(\mathbb{I})} \inf_{s\in\mathbb{I}} M(\gamma_1(\alpha(s)), \gamma_2(s), t).$$

Clearly, $M_F(\gamma_1, \gamma_2, t)$ is well-defined. It immediately follows from the definition that

$$M_F(\gamma_1, \gamma_2, t) \leq M_H(\gamma_1(\mathbb{I}), \gamma_2(\mathbb{I}), t).$$

The following result is proved in [4].

**Theorem 1.** *The function $M_F$ is a fuzzy metric on the set $\mathcal{C}(X)$.*

Let $(X,d)$ be a metric space and let $(M_d,\cdot)$ denote the standard fuzzy metric generated by $d$. The following is proved in [4].

**Proposition 1.** $(M_d)_F = M_{d_F}$.

A similar result can be proved for the fuzzy metric $(M'_d,\wedge)$, where $M'_d(x,y,t) = e^{-\frac{d(x,y)}{t^n}}$, $n\in\mathbb{N}$.

**2. Discrete Fréchet fuzzy metric.** For applications, it is useful to approximate a curve by curves that consist of regular pieces like linear segments in Euclidean spaces. One can go even further and consider a curve like an ordered sequence of close enough points (this is reasonable, e.g., in computer science). In the case of metric spaces the discrete counterpart of the Fréchet distance is considered in [10]. Below we modify this approach.

Having in mind that a curve in a fuzzy metric space behaves "regularly" between its consequent chosen points, we regard a discrete curve in a fuzzy metric space $(X,M,t)$ as a map $\gamma\colon\{1,2,\dots,n\}\to X$, where $n\in\mathbb{N}$.

Given $n,m\in\mathbb{N}$, we define a coupling as a sequence $((i_1,j_1),(i_2,j_2),\dots,(i_k,j_k))$ of distinct points of $\{1,2,\dots,n\}\times\{1,2,\dots,m\}$ satisfying the following conditions:

1. $(i_1,j_1)=(1,1)$, 2. $(i_k,j_k)=(n,m)$, 3. $i_{p+1}\in\{i_p,i_p+1\}$, 4. $j_{q+1}\in\{j_q,j_q+1\}$.

The set of all such couplings will be denoted by $C(n,m)$.

Now let $\gamma_i\colon\{1,2,\dots,n_i\}\to X$, $i=1,2$, be discrete curves, where $(X,M,*)$ is a fuzzy metric space. Given a coupling

$$L\in C(n_1,n_2),\ L=((i_1,j_1),(i_2,j_2),\dots,(i_k,j_k)),$$

and $t\in(0,\infty)$, define

$$\|(L,t)\| = \min\{M(\gamma_1(i_p),\gamma_2(j_p),t)\colon p\in\{1,\dots,k\}\}.$$

Finally, the discrete fuzzy Fréchet distance is

$$M_{dF}(\gamma_1,\gamma_2,t) = \max_{L\in C(n_1,n_2)}\|(L,t)\|.$$

We denote by $d\mathcal{C}(X)$ the set of all discrete curves in $X$.

**Theorem 2.** *The function $M_{dF}$ is a fuzzy metric on the set $d\mathcal{C}(X)$.*

*Proof.* Properties (1)-(3) from the definition of a fuzzy metric are obvious.

Let $\gamma_i\colon\{1,2,\dots,n_i\}\to X$, $i=1,2,3$, be discrete curves. Let

$$L'\in C(n_1,n_2),\ L'=((i_1,j_1),(i_2,j_2),\dots,(i_p,j_p)),$$

and

$$L''\in C(n_2,n_3),\ L''=((k_1,l_1),(k_2,l_2),\dots,(k_q,l_q)),$$

be couplings such that

$$M_{dF}(\gamma_1,\gamma_2,t)=\|(L',t)\|,\ M_{dF}(\gamma_2,\gamma_3,s)=\|(L'',s)\|.$$

There exists a coupling

$$\tilde{L}\in C(p,q),\ \tilde{L}=((s_1,t_1),(s_2,t_2),\dots,(s_r,t_r)),$$

such that $\gamma_2(j_{s_\alpha})=\gamma_2(k_{t_\alpha})$ for every $\alpha\in\{1,2,\dots,r\}$ (this is implicitly used in the proof of Proposition 1 of [10]).

Then

$$M_{dF}(\gamma_1,\gamma_3,t+s)=\max_{L\in C(n_1,n_3)}\|(L,t+s)\|\ge\|(\hat{L},t+s)\|=$$

$$= \min\{M(\gamma_1(i_{s_\beta}), \gamma_3(l_{t_\beta}), t+s)\colon \beta = 1,2,\dots,r\} \geq$$
$$\geq \min\{M(\gamma_1(i_{s_\beta}), \gamma_2(j_{s_\beta}), t) * M(\gamma_2(k_{t_\beta}), \gamma_3(l_{t_\beta}), s)\colon \beta = 1,2,\dots,r\} \geq$$
$$\geq \min\{M(\gamma_1(i_{s_\beta}), \gamma_2(j_{s_\beta}), t)\colon \beta = 1,2,\dots,r\} *$$
$$* \min\{M(\gamma_2(k_{t_\beta}), \gamma_3(l_{t_\beta}), s)\colon \beta = 1,2,\dots,r\} = M_{dF}(\gamma_1,\gamma_2,t) * M_{dF}(\gamma_2,\gamma_3,s).$$

The continuity of the function $t \mapsto M_{dF}(\gamma_1,\gamma_2,t)$ is a consequence of the fact that the maxima and minima of finitely many continuous functions are continuous.

□

**Proposition 2.** *Let $M$ be a stationary fuzzy metric on a set $X$. Then the discrete fuzzy metric $M_{dF}$ is also stationary.*

*Proof.* Straightforward. □

**Proposition 3.** *Let $(X, N, \wedge)$ be a fuzzy normed space. Let $\gamma_i\colon [0,1] \to X$, $i = 1,2$, be polygonal curves. Denote by $\tilde{\gamma}_i\colon \{1,2,\dots,n_i\} \to X$ the discrete curves of vertices of $\gamma_i$, $i = 1,2$, respectively. Then $M_F(\gamma_1,\gamma_2,t) = M_{dF}(\tilde{\gamma}_1,\tilde{\gamma}_2,t)$.*

*Proof.* Fix $t > 0$. First show that $M_F(\gamma_1,\gamma_2,t) \geq M_{dF}(\tilde{\gamma}_1,\tilde{\gamma}_2,t)$. Let $L \in C(n_1,n_2)$ be such that $M_{dF}(\tilde{\gamma}_1,\tilde{\gamma}_2,t) = \|(L,t)\|$. There exist piecewise linear functions $\alpha_1,\alpha_2\colon [0,1] \to X$ that consequently connect the vertices of $\gamma_i$ and are linear between every two subsequent vertices.

Applying Lemma 1 to every couple of segments connecting consequent vertices of $\gamma_1$ and $\gamma_2$ we see that

$$\|(L,t)\| \leq \min_{s\in[0,1]} M(\gamma_1(\alpha_1(s)), \gamma_2(\alpha_2(s)), t) \leq M_F(\gamma_1,\gamma_2,t).$$

Again fix $t > 0$. Now prove that $M_F(\gamma_1,\gamma_2,t) \leq M_{dF}(\tilde{\gamma}_1,\tilde{\gamma}_2,t)$. Let $\epsilon > 0$. There exist continuous parametrizations $\alpha_i\colon [0,1] \to [0,1]$, $i = 1,2$, such that

$$\min\{M(\gamma_1(\alpha_1(s)), \gamma_1(\alpha_1(s)), t)\colon s \in [0,1]\} > M_F(\gamma_1,\gamma_2,t) - \epsilon.$$

For any $s \in [0,1]$, send $\alpha_i(s)$ to the closest predecessing vertex $v_i(s)$ of $\gamma_i$, $i = 1,2$. The obtained set of couples $\{(v_1(s), v_2(s))\colon s \in [0,1]\}$, when properly ordered and enumerated, is a coupling $L \in C(n_1,n_2)$, where $n_i$ is the number of vertices of $\gamma_i$, $i = 1,2$.

By Lemma 1,

$$\|(L,t)\| = \min\{M(\gamma_1(\alpha_1(s)), \gamma_1(\alpha_1(s)), t) > M_F(\gamma_1,\gamma_2,t) - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the inequality follows. □

Given two discrete curves $\gamma_i\colon \{1,2,\dots,n_i\} \to X$, $i = 1,2$, where $(X, M, *)$ is a fuzzy metric space, one can easily see the following.

**Proposition 4.**

$$M_{dF}(\gamma_1,\gamma_2,t) \leq H_M(\gamma_1(\{1,2,\dots,n_1\}), \gamma_2(\{1,2,\dots,n_2\}), t).$$

Recall that $M_d$ denotes the standard fuzzy metric on $X$ generated by a metric $d$ on $X$.

**Proposition 5.** $(M_d)_{dF} = M_{d_{dF}}$.

*Proof.* This easily follows from the condition: given $t \in (0,\infty)$,

$$d(x,y) \leq d(x',y') \iff M(x,y,t) \geq M(x',y',t)$$

for any $x, y, x', y' \in X$. □

**Theorem 3.** *Let $\gamma_i$, $i = 1, 2$, be curves in a fuzzy metric space $(X, M, *)$. Then there exist discrete curves $\gamma_{in}$, $n \in \mathbb{N}$, $i = 1, 2$, with the following properties:*

1. *for every $n \in \mathbb{N}$, the curve $\gamma_{in}$ lies on $\gamma_i$, $i = 1, 2$;*
2. *the sequence $M(\gamma_{1n}, \gamma_{2n}, -)_{n=1}^{\infty}$ converges to $M(\gamma_1, \gamma_2, -)$ in the topology of convergence on compacta.*

*Proof.* Let $n \in \mathbb{N}$. By compactness of $[0, 1] \times [0, 1] \times [1/n, n]$ and continuity of $M$, there exist partitions

$$0 = t_0 < t_1 < t_2 < \cdots < t_{i_n} = 1, \quad 0 = s_0 < s_1 < s_2 < \cdots < s_{j_n} = 1$$

of $[0, 1]$ with the property that

$$|M(\gamma_1(t), \gamma_2(s), u) - M(\gamma_1(t'), \gamma_2(s'), u)| < \frac{1}{n},$$

whenever $t, t' \in [t_i, t_{i+1})$, $s, s' \in [s_j, s_{j+1})$, for some $i, j$, and $u \in [1/n, n]$.

Define discrete curves

$$\tilde{\gamma}_{1n} \colon \{0, 1 \ldots, i_n\} \to X, \; \tilde{\gamma}_{2n} \colon \{0, 1 \ldots, j_n\} \to X$$

by the formulas:

$$\tilde{\gamma}_{1n}(i) = \gamma_1(t_i), \; \tilde{\gamma}_{2n}(j) = \gamma_2(s_j).$$

Let $u \in [1/n, n]$ and let $\epsilon > 0$. Then from the definition of the fuzzy Fréchet distance it follows that there exist $\alpha_1, \alpha_2 \in \mathcal{H}([0, 1])$ such that

$$M_F(\gamma_1, \gamma_2, u) \leq \min_{s \in [0,1]} M(\gamma_1(\alpha_1(s)), \gamma_2(\alpha_2(s)), u) + \epsilon.$$

Let $C$ be a coupling defined as follows:

$$(i, j) \in C \iff \text{ there exists } t \in [0, 1] \text{ such that } \alpha_1(t) \in [i, i + 1), \alpha_2(t) \in [j, j + 1).$$

There exists $x_0 \in [0, 1]$ such that

$$M_F(\gamma_1, \gamma_2, u) = M(\gamma_1(\alpha_1(x_0)), \gamma_2(\alpha_2(x_0)), u).$$

There are $i, j$ such that

$$\alpha_1(x_0) \in [t_i, t_{i+1}), \alpha_2(x_0) \in [s_j, s_{j+1}).$$

Then $(i, j) \in C$ and by the choice of $x_0$,

$$|M_F(\gamma_1, \gamma_2, u) - M(\tilde{\gamma}_1(i), \tilde{\gamma}_2(j), u)| < \frac{1}{n}$$

and therefore

$$M_F(\gamma_1, \gamma_2, u) \leq M(\tilde{\gamma}_1(i), \tilde{\gamma}_2(j), u) + \frac{1}{n} + \epsilon \leq M_{dF}(\tilde{\gamma}_1, \tilde{\gamma}_2, u) + \frac{1}{n} + \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$M_F(\gamma_1, \gamma_2, u) \leq M_{dF}(\tilde{\gamma}_1, \tilde{\gamma}_2, u) + \frac{1}{n}. \tag{1}$$

On the other hand, there exists a coupling $C'$ such that

$$M_{dF}(\tilde{\gamma}_1, \tilde{\gamma}_2, u) = M(\tilde{\gamma}_1(i), \tilde{\gamma}_2(j), u)$$

for some $(i, j) \in C$. Clearly, there exist $\alpha'_1, \alpha'_2 \in \mathcal{H}([0, 1])$ such that for every $x \in [0, 1]$ there exists $(i, j) \in C'$ such that $\alpha'_1(x) \in [t_i, t_{i+1})$, $\alpha'_2(x) \in [s_j, s_{j+1})$. Then

$$M_{dF}(\tilde{\gamma}_1, \tilde{\gamma}_2, u) \leq \min_{s \in [0,1]} M(\gamma_1(\alpha_1(s)), \gamma_2(\alpha_2(s))), u) + \frac{1}{n} \leq M_F(\gamma_1, \gamma_2, u) + \frac{1}{n}. \tag{2}$$

Formulas (1) and (2) together imply

$$M_{dF}(\tilde{\gamma}_1, \tilde{\gamma}_2, u) - M_F(\gamma_1, \gamma_2, u)| \leq \frac{1}{n}$$

and we are done. □

**Theorem 4.** *Let $(X, M, \wedge)$ be a fuzzy ultrametric space. Given discrete curves $\gamma_i\colon \{1, 2, \dots, n_i\} \to X$, define*

$$M'_{dF}(\gamma_1, \gamma_2, t) = \max\{\min\{M(\gamma_1(i), \gamma_2(j), t)\colon (i,j) \in C\}\colon C \in \mathcal{C}(n_1, n_2)\}.$$

*Then $M'_{dF}$ is a fuzzy ultrametric on the set of all discrete curves in $X$.*

*Proof.* We proceed similarly to the proof of Theorem 2. Properties (1)-(3) from the definition of fuzzy ultrametric are obvious.

Let us check (4′). Given discrete curves $\gamma_i\colon \{1, 2, \dots, n_i\} \to X$, $i = 1, 2, 3$, let

$$L' \in C(n_1, n_2),\ L' = ((i_1, j_1), (i_2, j_2), \dots, (i_p, j_p)),$$

and

$$L'' \in C(n_2, n_3),\ L'' = ((k_1, l_1), (k_2, l_2), \dots, (k_q, l_q)),$$

be couplings such that

$$M_{dF}(\gamma_1, \gamma_2, t) = \|(L', t)\|,\ M_{dF}(\gamma_2, \gamma_3, s) = \|(L'', t)\|.$$

There exists a coupling

$$\tilde{L} \in C(p, q),\ \tilde{L} = ((s_1, t_1), (s_2, t_2), \dots, (s_r, t_r)),$$

such that $\gamma_2(j_{s_\alpha}) = \gamma_2(k_{t_\alpha})$ for every $\alpha \in \{1, 2, \dots, r\}$ (see the proof of Theorem 2).

Then

$$\begin{aligned} M_{dF}(\gamma_1, \gamma_3, t) &= \max_{L \in C(n_1,n_3)} \|(L, t)\| \geq \|(\hat{L}, t)\| = \\ &= \min\{M(\gamma_1(i_{s_\beta}), \gamma_3(l_{t_\beta}), t)\colon \beta = 1, 2, \dots, r\} \geq \\ &\geq \min\{M(\gamma_1(i_{s_\beta}), \gamma_2(j_{s_\beta}), t) \wedge M(\gamma_2(k_{t_\beta}), \gamma_3(l_{t_\beta}), t)\colon \beta = 1, 2, \dots, r\} \geq \\ &\geq \min\{M(\gamma_1(i_{s_\beta}), \gamma_2(j_{s_\beta}), t)\colon \beta = 1, 2, \dots, r\} \wedge \\ &\wedge \min\{M(\gamma_2(k_{t_\beta}), \gamma_3(l_{t_\beta}), t)\colon \beta = 1, 2, \dots, r\} = M_{dF}(\gamma_1, \gamma_2, t) * M_{dF}(\gamma_2, \gamma_3, t). \end{aligned}$$

(5) is proved similarly as in the proof of Theorem 2. □

**2.1. Fuzzy discrete Fréchet distance between closed curves.** The discrete Fréchet distance between closed curves is considered in [39].

Given a discrete curve $\gamma\colon \{1, 2, \dots, n\} \to X$, we regard $\gamma(1)$ as its starting point, i.e., a point without predecessor. If we then regard $\gamma(n)$ as the predecessor of $\gamma(1)$, we obtain the notion of closed discrete curve. Given $p \in \{0, 1, \dots, n-1\}$, define the $p$-shift $S_p(\gamma)\colon \{1, 2, \dots, n\} \to X$ of $\gamma\colon \{1, 2, \dots, n\} \to X$ by the formula:

$$S_p(\gamma)(i) = \begin{cases} \gamma(i+p), & \text{if } i + p \leq n, \\ \gamma(i+p-n), & \text{if } i + p > n. \end{cases}$$

Then define the discrete Fréchet distance between closed curves $\gamma_i\colon \{1, 2, \dots, n_i\} \to X$, $i = 1, 2$, by the formula:

$$M_{dcF}(\gamma_1, \gamma_2, t) = \max_{p_1 \in \{0,1,\dots,n_1-1\}} \max_{p_2 \in \{0,1,\dots,n_2-1\}} M_{dF}(S_{p_1}(\gamma_1), S_{p_2}(\gamma_2), t).$$

There is a counterpart of Theorem 2 for discrete closed curves in fuzzy ultrametric spaces.

## 3. Remarks.

**3.1. Modifications.** Some other modifications of the Fréchet distance are also considered in the literature. In particular, the non-monotonic Fréchet distance is defined in [3], and its discrete version is mentioned in [10]. One can define fuzzy counterpart of the non-monotonic discrete Fréchet distance.

One can define the notion of the fuzzy discrete Gromov-Fréchet distance:

$$M_{dGF}(\gamma_1,\gamma_2,t) = \sup\{M_{dF}(j_1(\gamma_1), j_2(\gamma_2), t)\colon \\ j_i\colon \gamma_i(\mathbb{I}) \to Z \text{ are isometric embeddings, } i = 1,2\}.$$

Also, given discrete curves $\gamma_1, \gamma_2$ in a fuzzy normed space $(\mathbb{R}^n, N, *)$, one defines their isometric Fréchet distance ($M$ is the fuzzy metric generated by $N$):

$$M_{dF}^{\mathrm{Iso}}(\gamma_1,\gamma_2,t) = \inf\{M_{dF}(\gamma_1, h(\gamma_2), t)\colon h\colon \mathbb{R}^n \to \mathbb{R}^n \text{ is an isometry}\}.$$

These distances can be objects of further investigations.

**3.2. Completability.** A sequence $(x_n)$ in a fuzzy metric space $(X, M)$ is said to be Cauchy, if for each $\epsilon \in (0,1)$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. $X$ is called complete if every Cauchy sequence in $X$ is convergent in topology on $X$ induced by $M$ (see, e.g., [12]).

A fuzzy metric completion of $(X, M)$ is a complete fuzzy metric space $(\tilde{X}, \tilde{M})$ such that $(X, M)$ isometrically embeds in $\tilde{X}$ as a dense subspace. $X$ is called completable if it admits a fuzzy metric completion.

It is proved in [29] that the hyperspace $\mathrm{Fin}(X)$ of nonempty finite subsets of a fuzzy metric space is completable if and only if $X$ is completable. We conjecture that the same is true for the space of discrete closed curves in $X$.

**3.3. Connections to the DTW distance.** The dynamic time warping (DTW) distance is the integral counterpart of the discrete Fréchet distance [5]. In the fuzzy metric setting, its version looks as follows. Given two sequences, $x = (x_i)_{i=1}^m, y = (y_i)_{i=1}^n$, in a fuzzy metric space $(X, M, *)$, we first define a warping path $w$ connecting $x$ and $y$ as a sequence $w = (w_s = (x_{i(s)}, y_{j(s)}))_{s=1}^k$ satisfying:

1. $w_1 = (x_1, y_1)$, $w_k = (x_m, y_n)$;
2. $|i(s) - i(s-1)| \leq 1$ and $|j(s) - j(s-1)| \leq 1$;
3. $i(s) \geq i(s-1)$ and $j(s) \geq j(s-1)$.

The set of all warping paths connecting $x$ and $y$ is denoted by $\mathcal{W}(x, y)$.

By the definition, the fuzzy DWP distance is

$$\mathrm{fDTW}(x,y,t) = \sup_w\{*_{s=1}^k M(x_{i(s)}, y_{j(s)}, t)\colon w \in \mathcal{W}(x,y)\}.$$

It is known that the DWP distance does not satisfy the triangle inequality, i.e., is not a metric ([9]). Modifying the example from [9] one can show that the fDTW does not satisfy property (iv) from the definition of the fuzzy metric.

In [30], a version of the DTW distance, which is a combination of the DTW and the so-called symbolic edit distance (the so-called time warped edit distance), is defined. We formulate as an open problem the finding and investigating of a fuzzy analogue of this metric.

**Acknowledgements.** The authors express their gratitude to the referee for valuable comments.

# REFERENCES


1. C. Alegreand, S. Romaguera, *The Hahn-Banach extension theorem for fuzzy normed spaces revisited*, Abstract and Applied Analysis, **2014**, Article ID 151472. doi:10.1155/2014/151472
2. H. Alt, M. Buchin, *Can we compute the similarity between surfaces?*, Discrete and Computational Geometry, **43** (2010), 78–99, arXiv:cs.CG/0703011, doi:10.1007/s00454-009-9152-8
3. H. Alt, M. Godau, *Computing the Fréchet distance between two polygonal curves*, International Journal of Computational Geometry & Applications, **5** (1995), no.1, 75–91.
4. L. Bazylevych, O. Berezsky, M. Zarichnyi, *Fréchet fuzzy metric*, Mat. Stud., **57** (2022), no.2, 210–215. doi:10.30970/ms.57.2.210-215
5. R. Bellman, R. Kalaba, *On adaptive control processes*, Automatic Control, IRE Transactions on, **4** (1959), no.2, 19p [Online]. Available: http://ieeexplore.ieee.org/xpls/abs all.jsp? arnumber=1104847
6. V. Brydun, A. Savchenko, M. Zarichnyi, *Fuzzy metrization of the spaces of idempotent measures*, European Journal of Mathematics, **6** (2020), 98–109.
7. K. Buchin, M. Buchin, J. Gudmundsson, M. Löffler, Jun Luo, *Detecting commuting patterns by clustering subtrajectories*, Int. J. Comput. Geometry Appl., **21** (2011), no.3, 253–282.
8. E.W. Chambers, É.C. de Verdière, J. Erickson, S. Lazard, F. Lazarus, S. Thite, *Homotopic Fréchet distance between curves, or Walking your dog in the woods in polynomial time*, Computational Geometry: Theory and Applications, **43** (2009), no.3, 295–311. doi:10.1016/j.comgeo.2009.02.008
9. L. Chen, R. Ng. *On the marriage of $L_p$-norm and edit distance*, Proc. 30th Int'l Conf. on Very Large Data Bases, (2004), 792–801.
10. T. Eiter, H. Mannila, Computing discrete Fréchet distance, Tech. Report CD-TR 94/64, Christian Doppler Laboratory for Expert Systems, 1994, TU Vienna, Austria.
11. M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rend. Circolo Mat. Palermo, **74** (1906), 1–74.
12. A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Syst., **64** (1994), 395–399.
13. A. George, P. Veeramani, *Some theorems in fuzzy metric spaces*, The Journal of Fuzzy Mathematics, **3** (1995), 933–940.
14. A. George, P. Veeramani, *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets Syst., **90** (1997), 365–368.
15. M. Godau, Die Fréchet-Metrik fur Polygonzüge-Algorithmen zur Abstandsmessung und Approximation, PhD thesis, Fachbereich Mathematik, 1991.
16. M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, **27** (1988), no.3, 385–389. doi:10.1016/0165-0114(88)90064-4
17. V. Gregori, J.-J. Miñana, B. Roig, A. Sapena, *On completeness and fixed point theorems in fuzzy metric spaces*, Mathematics, **12** (2024), 287. doi:10.3390/math12020287
18. V. Gregori, S. Morillas, A. Sapena, *Examples of fuzzy metrics and applications*, Fuzzy Sets and Systems, **170** (2011), no.1, 95–111.
19. V. Gregori, S. Romaguera, *On completion of fuzzy metric spaces*, Fuzzy Sets and Systems, **130** (2002), no.3, 399–404.
20. V. Gregori, S. Romaguera, *Characterizing completable fuzzy metric spaces*, Fuzzy Sets and Systems, **144** (2004), no.3, 411–420.
21. V. Gregori, A Sapena, *On fixed-point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems, **125** (2002), no.2, 245–252.
22. P. Grzegrzolka, *Asymptotic dimension of fuzzy metric spaces*, Fuzzy Sets Syst., **437** (2022), 20–34.
23. O. Hadžić, E. Pap, Fixed point theory in probabilistic metric spaces, Kluwer Academic Publishers, Dordrecht, 2001.
24. Minghui Jiang, Ying Xu, Binhai Zhu, *Protein structure structure alignment with discrete Frechet distance*, Journal of bioinformatics and computational biology, **6** (2008), 51–64. doi:10.1142/S0219720008003278
25. E. Jiménez-Fernández, A. Sánchez, E.A. Sánchez Pérez, *Unsupervised machine learning approach for building composite indicators with fuzzy metrics*, Expert Systems With Applications, **200** (2022), 116927. doi:10.1016/j.eswa.2022.116927
26. Kamran Alam Khan, *Generalized fuzzy metric spaces with an application to Colour image filtering*, Global Journal of Pure and Applied Mathematics, **13** (2107), no.7, 3601–3616.
27. I. Kramosil, J. Michálek, *Fuzzy metrics and statistical metric spaces*, Kybernetika, **11** (1975), no.5, 336–344.

28. Sam Kwong, Qianhua He, Kim-Fung Man, Chak-Wai Chau, Kit Sang Tang, *Parallel genetic-based hybrid pattern matching algorithm for isolated word recognition*, IJPRAI, **12** (1998), no.4, 573–594.
29. Li Changqing, Y.L. Zhang, K.X. Zhang, *A note on completeness of the Hausdorff fuzzy metric spaces*, Italian Journal of Pure and Applied Mathematics, **37**, (2017), 49–58.
30. P.-F. Marteau, Time warp edit distance, arXiv:0802.3522v5.
31. S. Morillas, V. Gregori, G. Peris-Fajarnés, P. Latorre, *A new vector median filter based on fuzzy metrics*, ICIAR05, Lecture Notes in Computer Science, **3656** (2005), 81–90.
32. N.M. Ralević, D. Karaklić, N. Pištinjat, *Fuzzy metric and its applications in removing the image noise*, Soft Computing, **23** (2019), no.22, 12049–12061.
33. J. Rodríguez-López, S. Romaguera, *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets and Systems, **147** (2004), 273–283.
34. D. Repovš, A. Savchenko, M. Zarichnyi, *Fuzzy Prokhorov metric on the set of probability measures*, Fuzzy Sets and Systems, **175** (2011), 96–104.
35. R. Saadati, S.M. Vaezpour, *Some results on fuzzy Banach spaces*, J. Appl. Math. & Computing, **17** (2005), no.1-2, 475–484.
36. A. Sapena, *A contribution to the study of fuzzy metric spaces*, Applied General Topology, **2** (2001), no.1, 63–75.
37. B. Schweizer, A. Sklar, *Associative functions and abstract semigroups*, Publ. Math. Debrecen, **10** (1963), 69–81.
38. E. Sriraghavendra, K. Karthik, C. Bhattacharyya, *Fréchet distance based approach for searching online handwritten documents*, 9th International Conference on Document Analysis and Recognition (ICDAR 2007), Curitiba, Parana, Brazil, (2007), 461–465.
39. E. Vodolazskiy, *Discrete Fréchet distance for closed curves*, Computational Geometry: Theory and Applications, **111** (2023), 101967. doi:10.1016/j.comgeo.2022.101967
40. Yeong Chyuan Chung, *Property A and coarse embeddability for fuzzy metric spaces*, Fuzzy Sets and Systems, **444**, 2022, 156–171.
41. M. Zarichnyi, *Coarse structures and fuzzy metrics*, Mat. Stud., **32** (2009), no.2, 180–184. doi:10.30970/ms.32.2.180-184
42. M. Zarichnyi, O. Berezsky, M. Berezkyi, V. Teslyuk, *Method and algorithms for computing fuzzy Fréchet and Hausdorff distance*, Mathematics, **14** (2026), 892. doi:10.3390/math14050892

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*Received 20.10.2025*
*Revised 12.03.2026*