Математичні Студії. Т.65, №1 Matematychni Studii. V.65, No.1
ISSN 1027-4634 (print) ISSN 2411-0620 (online)

A. I. BANDURA*, L. I. KRYSHTOPA, L. M. SHEHDA

BALL EXHAUSTION OF THE REINHARDT DOMAIN: PROPERTIES OF ANALYTIC FUNCTIONS OF BOUNDED L-INDEX IN JOINT VARIABLES

A. I. Bandura, L. I. Kryshtopa, L. M. Shehda. *Ball exhaustion of the Reinhardt domain: properties of analytic functions of bounded* **L***-index in joint variables*, Mat. Stud. **65** (2026), 22–29.

This article is a continuation of the research on the concept of an analytic function of a bounded **L**-index, conducted during 2019-2025 in the articles A. Bandura, T. Salo and O. Skaskiv, in which the construction of a complete and exhaustive theory of analytic functions of a bounded **L**-index in joint variables in an arbitrary complete Reinhardt domain was successfully started. In this article, the criterion for the boundedness of the **L**-index of a function analytic in a Reinhardt domain in terms of its maximum modulus on balls with an arbitrary center and an arbitrary radius is proved.

1. Introduction. We will need the following standard notations from the theory of holomorphic multivariate functions (see, for example, [2,6,7]). Let $\mathbb{R}^n$ and $\mathbb{C}^n$ be real and complex vector spaces, respectively, with Euclidean metric $|z-w|=\sqrt{(z_1-w_1)^2+\ldots+(z_n-w_n)^2}$ for $z=(z_1,\ldots,z_n)$, $w=(z_1,\ldots,w_n)$; $\mathbb{R}_+=[0,+\infty)$, $\mathbf{0}=(0,\ldots,0)$, $\mathbf{1}=(1,\ldots,1)$. For $A=(a_1,\ldots,a_n)\in\mathbb{R}^n$, $B=(b_1,\ldots,b_n)\in\mathbb{R}^n$, we define $AB=(a_1b_1,\cdots,a_nb_n)$, $A/B=(a_1/b_1,\ldots,a_n/b_n)$, $A^B=a_1^{b_1}a_2^{b_2}\cdot\ldots\cdot a_n^{b_n}$, $A+B=(a_1+b_1,\ldots,a_n+b_n)$, $kA=(ka_1,\ldots,ka_n)$, $rA=(ra_1,\ldots,ra_n)$, $\|A\|=\sum_{j=1}^n a_j$ and $A!=a_1!\ldots a_n!$, if each $a_j\in\mathbb{Z}_+$. And all vector inequalities are understood as coordinate inequalities. This concerns the inequalities $A<B$, $A\le B$, and so on.

The domain $\mathbb{G}\subset\mathbb{C}^n$ is called the complete Reinhardt domain ([17, 18]), if: $\forall z\in\mathbb{G}\ \forall R\in[0,1]^n$ one has $Rz\in\mathbb{G}$ and for all $(\theta_1,\ldots,\theta_n)\in[0;2\pi]^n)\,(z_1e^{i\theta_1},\ldots,z_ne^{i\theta_n})\in\mathbb{G}$.

Examples of Reinhardt domains are the ball $\mathbb{B}^n(z^0,r)=\{z\in\mathbb{C}^n\colon |z-z^0|<r\}$, $r>0$, and the polydisk $\mathbb{D}^n(z^0,R)=\{z=(z_1,\ldots,z_n)\in\mathbb{C}^n\colon\ |z_j-z_j^0|<r_j,\ j\in\{1,\ldots,n\}\}$, $R=(r_1,\ldots,r_n)$. Denote also $\mathbb{S}^n(z^0,r)=\{z\in\mathbb{C}^n\colon |z-z^0|=r\}$, $r>0$.

This paper is, in a sense, a continuation of the research carried out in papers [1–5], in which the construction of a complete and exhaustive theory of analytic functions of bounded **L**-index in joint variables in arbitrary complete Reinhardt domain was successfully initiated. From the one hand, it is well-known that every analytic function f in the complete Reinhardt domain $\mathbb{G}$ centered at $z=0$ can be represented in $\mathbb{G}$ by a multiple power series. On the other hand, the domain of convergence of every multiple power series is a logarithmically-convex complete Reinhardt domain centered at $z=0$. Our presented results are similar to the results

2020 *Mathematics Subject Classification:* 32A10, 32A17, 58C10.
Keywords: analytic function; bounded **L**-index in joint variables; Reinhardt domain; maximum modulus; boundedness criterion; function of several complex variables.
doi:10.30970/ms.65.1.22-29

*Corresponding author: A. I. Bandura

obtained for such classes of holomorphic functions: entire multivariate functions ([1]), as well as functions which are analytic in a unit ball ([10]). For a full exhaustion of the domain, balls with some radii and centers are used. Another exhaustion of multidimensional complex space by polilinear domains is recently considered in [13]. The properties of functions, which are analytic in the Reinhardt domain, are one of the cornerstones of modern multidimensional complex analysis (see other researches [14–16, 19, 20] in this direction). Our approach is valuable for its connections to the theory of value distribution ([12]) and its application to the study of the properties of analytical solutions to differential equations ([11]).

Denote by $\partial\mathbb{G}$ the boundary of the domain $\mathbb{G}$. For $J \in \mathbb{Z}^n_+$ we will denote

$$H^{(J)}(z) = \frac{\partial^{\|J\|} H}{\partial z^J}(z) = \frac{\partial^{j_1+j_2+\cdots+j_n} H}{\partial z_1^{j_1} \partial z_2^{j_2} \ldots \partial z_n^{j_n}}(z_1, z_2, \ldots, z_n).$$

Suppose that the function $\mathbf{L}\colon \mathbb{G} \to \mathbb{R}^n_+$ is continuous and for any $j \in \{1, 2, \ldots, n\}$ and $\beta > 1$ one has

$$l_j(z) > \beta / \inf\{(r|z_j|) - |z_j| \colon \widehat{R}_j z \in \partial\mathbb{G},\ r > 1\},$$

where $\widehat{R}_j = (1, \ldots, 1, \underbrace{r}_{j\text{-th item}}, 1 \ldots, 1)$. Denote $\mathcal{B} = (0, \beta]$, $\mathcal{B}^n = (0, \beta]^n$. Below we suppose everywhere that $\mathbb{G} \subset \mathbb{C}^n$ is the complete Reinhardt domain.

An analytic function $H\colon \mathbb{G} \to \mathbb{C}$ is called a function with *bounded* $\mathbf{L}$*-index (in joint variables)* if $\exists n_0 \in \mathbb{Z}_+$ $\forall J \in \mathbb{Z}^n_+$ $\forall z \in \mathbb{G}$:

$$\frac{|H^{(J)}(z)|}{J!\mathbf{L}^J(z)} \le \max\left\{ \frac{|H^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon\ K \in \mathbb{Z}^n_+,\ \|K\| \le n_0 \right\}.$$

The least corresponding number $N(H, \mathbf{L}, \mathbb{G}) = n_0$ is the $\mathbf{L}$*-index in joint variables*.

2. Exhaustion of the Reihardt domain by balls. Denote $\ell(z) = \min_{1\le j\le n} l_j(z)$, $\mathcal{L}(z) = \max_{1\le j\le n} l_j(z)$. Obviously, that $\ell(z) \le \mathcal{L}(z)$. By $Q'(\mathbb{G})$ we denote the class of functions $\mathbf{L}$, which satisfy the condition $(\forall r \in [0, \beta], j \in \{1, \ldots, n\})\colon\ 0 < \lambda_{1,j}(r) \le \lambda_{2,j}(r) < \infty$, where

$$\lambda_{1,j}(r) = A(z_0, z) \inf_{z^0 \in \mathbb{G}} \inf A(z_0, z),\ \ \lambda_{2,j}(r) = \sup_{z^0 \in \mathbb{G}} \sup A(z_0, z),$$

$$A(z_0, z) = \left\{ \frac{l_j(z)}{l_j(z^0)} \colon z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)\right] \right\},\ \Lambda_k(r) = (\lambda_{k,1}(r), \ldots, \lambda_{k,n}(r))\ (k \in \{1, 2\}).$$

Theorem 1. *Let* $\mathbf{L} \in Q'(\mathbb{G})$*. In order that an analytic function* F *in a domain* $\mathbb{G}$ *be of bounded* $\mathbf{L}$*-index in joint variables it is necessary that for each* $r \in (0, \beta]$ *there exist* $n_0 \in \mathbb{Z}_+$*,* $p_0 > 0$ *such that for every* $z^0 \in \mathbb{G}$ *there exists* $K^0 \in \mathbb{Z}^n_+$*,* $\|K^0\| \le n_0$*, satisfying*

$$\max\left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le n_0,\ z \in \mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right] \right\} \le p_0 \frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)} \tag{1}$$

and it is sufficient that for each $r \in (0, \beta]$ *there exist* $n_0 \in \mathbb{Z}_+$*,* $p_0 > 0$ *such that for every* $z^0 \in \mathbb{G}$ *there exists* $K^0 \in \mathbb{Z}^n_+$*,* $\|K^0\| \le n_0$*, satisfying*

$$\max\left\{ \frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le n_0,\ z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)|\right] \right\} \le p_0 \frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \tag{2}$$

Proof. Let F be of bounded $\mathbf{L}$-index in joint variables with $N = N(F, \mathbf{L}, \mathbb{G}) < \infty$. For every $r \in (0, \beta]$ we put $q = q(r) = [2(N+1)r\sqrt{n}\prod_{j=1}^{n}(\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^{N+1}] + 1$, where $[x]$ is integer part of the real number x. For $p \in \{0, \ldots, q\}$ and $z^0 \in \mathbb{G}$ we denote

$$S_p(z^0, r) = \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\},$$
$$S_p^*(z^0, r) = \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)} \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\}.$$

Using definition of $\lambda_{1,j}(r)$ and $\mathbb{B}^n[z^0, \frac{pr}{q\ell(z^0)}] \subset \mathbb{B}^n[z^0, \frac{r}{\mathcal{L}(z^0)}]$, we have

$$S_p(z^0, r) = \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\frac{\mathbf{L}^K(z^0)}{\mathbf{L}^K(z^0)} \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\} \le$$
$$\le S_p^*(z^0, r)\max\left\{\prod_{j=1}^{n}\frac{l_j^N(z^0)}{l_j^N(z)} \colon z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\} \le S_p^*(z^0, r)\prod_{j=1}^{n}(\lambda_{1,j}(r))^{-N}$$

and, using definition of $\lambda_{1,j}(r)$, we obtain

$$S_p^*(z^0, r) = \max\left\{\frac{|F^{(K)}(z)|\ \mathbf{L}^K(z)}{K!\mathbf{L}^K(z)\mathbf{L}^K(z^0)} \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\} \le$$
$$\le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}(\Lambda_2(r))^K \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\} \le S_p(z^0, r)\prod_{j=1}^{n}(\lambda_{2,j}(r))^N. \quad (3)$$

Let $K^{(p)}$ with $\|K^{(p)}\| \le N$ and $z^{(p)} \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]$ be such that

$$S_p^*(z^0, r) = \frac{|F^{(K^{(p)})}(z^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}. \quad (4)$$

Since by the Maximum Principle Modulus $z^{(p)} \in \mathbb{S}_n(z^0, \frac{pr}{q\mathcal{L}(z^0)})$, we have $z^{(p)} \ne z^0$. We choose $\widetilde{z}_j^{(p)} = z_j^0 + \frac{p-1}{p}(z_j^{(p)} - z_j^0),\ j \in \{1, \ldots, n\}$. Then we have

$$|\widetilde{z}^{(p)} - z^0| = \frac{p-1}{p}|z^{(p)} - z^0| = \frac{p-1}{p}\frac{pr}{q\mathcal{L}(z^0)}, \quad (5)$$
$$|\widetilde{z}^{(p)} - z^{(p)}| = |z^0 + \frac{p-1}{p}(z^{(p)} - z^0) - z^{(p)}| = \frac{1}{p}|z^0 - z^{(p)}| = \frac{1}{p}\frac{pr}{q\ell(z^0)} = \frac{r}{q\mathcal{L}(z^0)}. \quad (6)$$

From (5) we obtain $\widetilde{z}^{(p)} \in \mathbb{B}^n[z^0, \frac{(p-1)r}{q\mathcal{L}(z^0)}]$ and $S_{p-1}^*(z^0, r) \ge \frac{|F^{(K^{(p)})}(\widetilde{z}^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}$. From (4) it follows

$$0 \le S_p^*(z^0, r) - S_{p-1}^*(z^0, r) \le \frac{|F^{(K^{(p)})}(z^{(p)})| - |F^{(K^{(p)})}(\widetilde{z}^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)} =$$
$$= \frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\int_0^1 \frac{d}{dt}|F^{(K^{(p)})}(\widetilde{z}^{(p)} + t(z^{(p)} - \widetilde{z}^{(p)}))|dt \le$$
$$\le \frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\sum_{j=1}^{n}|z_j^{(p)} - \widetilde{z}_j^{(p)}|\left|F^{(K^{(p)}+\mathbf{1}_j)}(\widetilde{z}^{(p)} + t^*(z^{(p)} - \widetilde{z}^{(p)}))\right|, \quad (7)$$

where $0 \le t^* \le 1, \widetilde{z}^{(p)} + t^*(z^{(p)} - \widetilde{z}^{(p)}) \in \mathbb{B}(z^0, \frac{pr}{q\mathcal{L}(z^0)})$. For $z \in \mathbb{B}(z^0, \frac{pr}{q\mathcal{L}(z^0)})$ and $J \in \mathbb{Z}^n_+$, $\|J\| \le N+1$ we have

$$\frac{|F^{(J)}(z)|\mathbf{L}^J(z)}{J!\mathbf{L}^J(z^0)\mathbf{L}^J(z)} \le (\Lambda_2(r))^J \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le N\right\} \le$$
$$\le \prod_{j=1}^n (\lambda_{2,j}(r))^{N+1}(\lambda_{1,j}(r))^{-N} \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)} \colon \|K\| \le N\right\} \le$$
$$\le \prod_{j=1}^n (\lambda_{2,j}(r))^{N+1}(\lambda_{1,j}(r))^{-N} S^*_p(z^0, r).$$

From (7) and (6) we obtain

$$0 \le S^*_p(z^0,r) - S^*_{p-1}(z^0,r) \le$$
$$\le \prod_{j=1}^n (\lambda_{2,j}(r))^{N+1}(\lambda_{1,j}(r))^{-N} S^*_p(z^0,r)(N+1)\sum_{j=1}^n l_j(z^0)|z_j^{(p)} - \widetilde{z}_j^{(p)}| \le$$
$$\le \prod_{j=1}^n (\lambda_{2,j}(r))^{N+1}(\lambda_{1,j}(r))^{-N}(N+1)S^*_p(z^0,R)\sqrt{n}\mathcal{L}(z^0)|z^{(p)} - \widetilde{z}^{(p)}| =$$
$$= \prod_{j=1}^n (\lambda_{2,j}(r))^{N+1}(\lambda_{1,j}(r))^{-N}\sqrt{n}\frac{(N+1)r}{q(r)} S^*_p(z^0,R) \le \frac{1}{2}S^*_p(z^0,R).$$

This inequality implies $S^*_p(z^0,r) \le 2S^*_{p-1}(z^0,r)$, and in view of inequalities (3) and (4) we have

$$S_p(z^0,r) \le 2\prod_{j=1}^n (\lambda_{1,j}(r))^{-N} S^*_{p-1}(z^0,r) \le 2\prod_{j=1}^n (\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^N S_{p-1}(z^0,r).$$

Therefore,

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} \colon \|K\| \le N, z \in \mathbb{B}^n\left[z^0, \frac{pr}{q\mathcal{L}(z^0)}\right]\right\} = S_q(z^0,r) \le$$
$$\le 2\prod_{j=1}^n (\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^N S_{q-1}(z^0,r) \le (2\prod_{j=1}^n (\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^N)^q S_0(z^0,r) =$$
$$= \left(2\prod_{j=1}^n (\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^N\right)^q \max\left\{\frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^K(z^0)} \colon \|K\| \le N\right\}. \tag{8}$$

From (8) we obtain inequality (1) with $p_0 = (2\prod_{j=1}^n (\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^N)^q$ and some K^0 with $\|K^0\| \le N$. The necessity of condition (1) is proved.

Now we prove the sufficiency. Suppose that for every $r \in (0,\beta]$ there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 1$ such that for all $z_0 \in \mathbb{G}$ and some $K^0 \in \mathbb{Z}^n_+$, $\|K^0\| \le n_0$, the inequality (2) holds.

We write Cauchy's formula for a ball (see [8, p. 349] or [9, p. 109]) as following $\forall z^0 \in \mathbb{B}$ $\forall K \in \mathbb{Z}^n_+$ $\forall S \in \mathbb{Z}^n_+$ $\forall z \in \mathbb{G}(z^0, r/\ell(z^0))$

$$F^{(K+S)}(z) = \frac{(n+\|S\|-1)!}{(n-1)!} \int\limits_{\mathbb{S}^n(z^0, r/\ell(z^0))} \frac{|\xi - z^0|(\overline{\xi - z^0})^S F^{(K)}(\xi)}{(|\xi - z^0|^2 - \langle z - z^0, \xi - z^0\rangle)^{n+\|S\|}} d\sigma(\xi),$$

where $d\sigma(\xi)$ is the normalized surface measure on $\mathbb{S}_n$, so that $\sigma(\mathbb{S}_n(\mathbf{0},1))=1$. Put $z=z^0$:

$$F^{(K+S)}(z^0)=\frac{(n+\|S\|-1)!}{(n-1)!}\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{\overline{(\xi-z^0)}^S F^{(K)}(\xi)}{|\xi-z^0|^{2(n+\|S\|)-1}}d\sigma(\xi).$$

Therefore, applying (2), we have

$$|F^{(K+S)}(z^0)|\le\frac{(n+\|S\|-1)!}{(n-1)!}\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S||F^{(K)}(\xi)|}{|\xi-z^0|^{2(n+\|S\|)-1}}d\sigma(\xi)\le$$

$$\le\left(\frac{\ell(z^0)}{r}\right)^{2(n+\|S\|)-1}\frac{(n+\|S\|-1)!}{(n-1)!}\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S||F^{(K)}(\xi)|K!\mathbf{L}^K(\xi)}{K!\mathbf{L}^K(\xi)}d\sigma(\xi)\le$$

$$\le p_0\left(\frac{\ell(z^0)}{r}\right)^{2(n+\|S\|)-1}\frac{(n+\|S\|-1)!}{(n-1)!}\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S||F^{(K^0)}(z^0)|K!\mathbf{L}^K(z)}{K^0!\mathbf{L}^{K^0}(z^0)}d\sigma(\xi)\le$$

$$\le p_0\left(\frac{\ell(z^0)}{r}\right)^{2(n+\|S\|)-1}\frac{(n+\|S\|-1)!}{(n-1)!}\frac{|F^{(K^0)}(z^0)|K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)\mathbf{L}^K(z^0)}{K^0!\mathbf{L}^{K^0}(z^0)}\times$$

$$\times\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}|(\xi-z^0)^S|d\sigma(\xi)\le$$

$$\le p_0\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\frac{(n+\|S\|-1)!}{(n-1)!}\frac{|F^{(K^0)}(z^0)|K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)\mathbf{L}^K(z^0)}{K^0!\mathbf{L}^{K^0}(z^0)}\times$$

$$\times\int\limits_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S|}{(r/\ell(z^0))^{\|S\|}}d\sigma\left(\frac{\xi-z^0}{r/\ell(z^0)}\right)\le$$

$$\le p_0\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\frac{(n+\|S\|-1)!}{(n-1)!}\frac{|F^{(K^0)}(z^0)|K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)\mathbf{L}^K(z^0)}{K^0!\mathbf{L}^{K^0}(z^0)}\times$$

$$\times\int\limits_{\mathbb{S}^n(\mathbf{0},1)}|\xi^S|d\sigma(\xi)=p_0\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\frac{(n+\|S\|-1)!}{(n-1)!}\times$$

$$\times\frac{|F^{(K^0)}(z^0)|K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)\mathbf{L}^K(z^0)}{K^0!\mathbf{L}^{K^0}(z^0)}\frac{\Gamma(n)\prod_{j=1}^n\Gamma(s_j/2+1)}{\Gamma(n+\|S\|/2)}.$$

This implies

$$\frac{|F^{(K+S)}(z^0)|}{(K+S)!\mathbf{L}^{K+S}(z^0)}\le$$

$$\le\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}p_0\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\frac{K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)\mathbf{L}^S(z^0)}\le$$

$$\le\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}p_0\frac{K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)r^{\|S\|}}.\tag{9}$$

We choose $r>1$. Since $\|K\|\le n_0$ the quantity $p_0K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(R)$ does not depend of S. Then there exists n_1 such that

$$\frac{p_0K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)}{r^{\|S\|}}\le 1\text{ for all }\|S\|\ge n_1.\tag{10}$$

The asymptotic behavior of $\frac{(n+\|S\|-1)!\prod_{j=1}^{n}\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)r^{\|S\|}}$ is more difficult as $\|S\|\to+\infty$. Using the Stirling formula $\Gamma(m+1)=\sqrt{2\pi m}\left(\frac{m}{e}\right)^m(1+\frac{\theta}{12m})$, where $\theta=\theta(m)\in[0,1]$, we obtain

$$\frac{(n+\|S\|-1)!\prod_{j=1}^{n}\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)r^{\|S\|}}\leq\frac{(n+\|S\|-1)!\prod_{j=1}^{n}\Gamma(s_j/2+1)}{S!\Gamma(n+\|S\|/2)r^{\|S\|}}=$$

$$=\frac{\sqrt{2\pi(n+\|S\|-1)}(\frac{n+\|S\|-1}{e})^{n+\|S\|-1}\prod_{j=1}^{n}\sqrt{2\pi s_j/2}(\frac{s_j}{2e})^{s_j/2}}{\prod_{j=1}^{n}\sqrt{2\pi s_j}(\frac{s_j}{e})^{s_j}\sqrt{2\pi(n+\|S\|/2-1)}(\frac{n+\|S\|/2-1}{e})^{n+\|S\|/2-1}r^{\|S\|}}\times$$

$$\times\frac{(1+\frac{\theta(n+\|S\|-1)}{12(n+\|S\|-1)})\prod_{j=1}^{n}(1+\frac{\theta(s_j/2)}{12s_j/2})}{(1+\frac{\theta(n+\|S\|/2)}{12(n+\|S\|/2)})\prod_{j=1}^{n}(1+\frac{\theta(s_j)}{12s_j})}.$$

Denoting

$$\Theta(S)=\frac{(1+\frac{\theta(n+\|S\|-1)}{12(n+\|S\|-1)})\prod_{j=1}^{n}(1+\frac{\theta(s_j/2)}{12s_j/2})}{(1+\frac{\theta(n+\|S\|/2)}{12(n+\|S\|/2)})\prod_{j=1}^{n}(1+\frac{\theta(s_j)}{12s_j})}$$

and simplifying the previous inequality we deduce

$$\frac{(n+\|S\|-1)!\prod_{j=1}^{n}\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)r^{\|S\|}}\leq$$

$$\leq\Theta(S)\frac{2^{(1-n)/2}e^{-\|S\|/2}}{r^{\|S\|}}\left(\frac{n-1+\|S\|}{n-1+\|S\|/2}\right)^{n-1+\|S\|/2}\cdot(n-1+\|S\|)^{\|S\|/2}\times$$

$$\times\prod_{j=1}^{n}\left(\frac{e}{2s_j}\right)^{s_j/2}\leq\Theta(S)\frac{2^{(n-1+\|S\|)/2}e^{-\|S\|/2}}{r^{\|S\|}}(n-1+\|S\|)^{\|S\|/2}\prod_{j=1}^{n}\left(\frac{e}{2s_j}\right)^{s_j/2}=$$

$$=\Theta(S)\frac{2^{(n-1)/2}}{r^{\|S\|}}\left(1+\frac{n-1}{\|S\|}\right)^{\frac{\|S\|}{n-1}\cdot\frac{n-1}{2}}\cdot\|S\|^{\|S\|/2}\prod_{j=1}^{n}\frac{1}{s_j^{s_j/2}}\leq$$

$$\leq\Theta(S)(2e)^{(n-1)/2}\left(\frac{1}{r}\prod_{j=1}^{n}\left(\frac{\|S\|}{s_j}\right)^{\frac{s_j}{2\|S\|}}\right)^{\|S\|},\ s_j\to\infty. \tag{11}$$

Denote $x_j=\frac{\|S\|}{s_j}\in(1,+\infty)$, $x=(x_1,\dots,x_n)$. Obviously, $\Theta(S)\to1$ as $s_j\to\infty$, $j\in\{1,\dots,n\}$. Then (11) implies a constrained optimization problem

$$H(x):=\prod_{j=1}^{n}x_j^{1/(2x_j)}\to\max\ \text{subject to}\ \sum_{j=1}^{n}\frac{1}{x_j}=1,\ x_j\in(1,+\infty).$$

If this problem has a solution, then $H(x)$ is not greater than some H^* and we choose $r>H^*$ in (11). The optimization problem was solved in [1]. By its solution, one has $H(x)\leq\prod_{j=1}^{n}n^{1/(2n)}=\sqrt{n}$. We choose $r\geq\sqrt{n}$. For this r we have $\frac{1}{r}\prod_{j=1}^{n}\left(\frac{\|S\|}{s_j}\right)^{\frac{s_j}{2\|S\|}}\leq1$. In view of (11) it means that there exist n_2 such that

$$\frac{(n+\|S\|-1)!\prod_{j=1}^{n}\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)r^{\|S\|}}\leq1 \tag{12}$$

for all $\|S\|\geq n_2$. The asymptotic behavior of right part (9) in other cases S can be investigated similarly. Taking into account (9), (10) and (12) we have that for all $\|S\|\geq n_1+n_2$

$$\frac{|F^{(K+S)}(z^0)|}{(K+S)!\mathbf{L}^{S+K}(z^0)}\leq\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}.$$

This means that for every $J \in \mathbb{Z}^n_+$ $\frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^J(z^0)} \le \max\{\frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^K(z^0)}\colon \ \|K\| \le n_0+n_1+n_2\}$, where n_0, n_1, n_2 are independent of z_0. Therefore, the function F has bounded $\mathbf{L}$-index in joint variables with $N(F,\mathbf{L},\mathbb{G}) \le n_0+n_1+n_2$. □

If we impose some constraint by the function $\mathbf{L}$ then Theorem 1 implies the criterion

Theorem 2. *Let $\mathbf{L} \in Q'(\mathbb{G})$ be such that $\sup_{z\in\mathbb{G}} \frac{\mathcal{L}(z)}{\ell(z)} = C < \infty$. An analytic function F in $\mathbb{G}$ has bounded $\mathbf{L}$-index in joint variables if and only if for each $r \in (0,\beta]$ there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ such that for every $z^0 \in \mathbb{G}$ there exists $K^0 \in \mathbb{Z}^n_+$, $\|K^0\| \le n_0$, such that inequality* (2) *holds.*

Proof. Sufficiency is proved in Theorem 1. As for necessity we choose $q = q(R) = [2(N+1)\times \times Cr\prod_{j=1}^n(\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^{N+1}]+1$ and replace $\mathcal{L}(z^0)$ by $\ell(z^0)$ in the proof of Theorem 1. No other changes. □

We need the following lemma.

Lemma 1. *Let $\mathbf{L}_1$, $\mathbf{L}_2$ be positive continuous functions in $\mathbb{G}$ and for every $z \in \mathbb{G}$ $\mathbf{L}_1(z) \le \mathbf{L}_2(z)$. If an analytic function F in $\mathbb{G}$ has bounded $\mathbf{L}_1$-index in joint variables then F is of bounded $\mathbf{L}_2$-index in joint variables. If, in addition, for every $z \in \mathbb{G}$ $\mathcal{L}_1(z) \le \ell_2(z)$ then $N(F,\mathbf{L}_2,\mathbb{G}) \le N(F,\mathbf{L}_1,\mathbb{G})$.*

Proof. The proof of Lemma 1 is carried out using simple calculations. □

Theorem 3. *Let $\mathbf{L} \in Q'(\mathbb{G})$, a function F be analytic in $\mathbb{G}$. If there exist $r \in (0,\beta]$, $n_0 \in \mathbb{Z}_+$, $p_0 > 1$ such that for each $z^0 \in \mathbb{G}$ and for some $K^0 \in \mathbb{Z}^n_+$ with $\|K^0\| \le n_0$ the inequality* (2) *holds then F has bounded $\mathbf{L}$-index in joint variables.*

Proof. The proof of sufficiency in Theorem 1 for $r = \beta$ implies that $N(F,\mathbf{L},\mathbb{G}) < +\infty$.

Let $\mathbf{L}^*(z) = \frac{r_0\mathbf{L}(z)}{r}$, $\ell^*(z) = \frac{r_0\ell(z)}{r}$, $r^0 = \beta$ and r is radius for which (2) is true. In a general case from validity of (2) for F and $\mathbf{L}$ for $r < \beta$ we obtain

$$\max\left\{\frac{|F^{(K)}(z)|}{K!(\mathbf{L}^*(z))^K}\colon \|K\| \le n_0, z \in \mathbb{B}\left[z^0, r_0/\ell^*(z^0)\right]\right\} \le$$
$$\le \max\left\{\frac{|F^{(K)}(z)|}{K!(r_0\mathbf{L}(z)/r)^K}\colon \|K\| \le n_0, z \in \mathbb{B}\left[z^0, r_0/(r_0\ell(z^0)/r)\right]\right\} \le$$
$$\le \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\| \le n_0, z \in \mathbb{B}\left[z^0, r/\ell(z^0)\right]\right\} \le$$
$$\le \frac{p_0}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} = \frac{\beta^{\|K^0\|}p_0}{r^{\|K^0\|}K^0!}\frac{|F^{(K^0)}(z)|}{(r_0\mathbf{L}(z)/r)^{K^0}} = \frac{p_0\beta^{n_0}}{r^{n_0}}\frac{|F^{(K^0)}(z)|}{K^0!(\mathbf{L}^*(z))^{K^0}}.$$

i.e. (1) holds for F, $\mathbf{L}^*$ and $r_0 = \beta$. As above now we apply Theorem 1 to the function $F(z)$ and $\mathbf{L}^*(z) = r_0\mathbf{L}(z)/r$. This implies that F is of bounded $\mathbf{L}^*$-index in joint variables. Therefore, by Lemma 1, the function F has bounded $\mathbf{L}$-index in joint variables. □

Author Contributions. A. Bandura and L. Shehda have contributed in the conceptualization, methodology, investigation, writing. L. Kryshtopa has contributed in the validation, review and editing.

Acknowledgments. This research (A. Bandura and L. Shehda) was funded by the National Research Foundation of Ukraine (project 2025.07/0427, "Newest complex probabilistic methods for studying asymptotic properties of analytical solutions of differential equations represented by multiple random series and integrals and their potential applications").

References

1. A.I. Bandura, O.B. Skaskiv, *Exhaustion by balls and entire functions of bounded* **L***-index in joint variables*, Ufa Math. J., **11** (2019), no.1, 100–113. doi:10.13108/2019-11-1-100
2. A. Bandura, T. Salo, O. Skaskiv, *Analytic functions in a complete Reinhardt domain having bounded* **L***-index in joint variables*, Symmetry, **16** (2024), no.3, article ID 351. doi:10.3390/sym16030351
3. A.I. Bandura, T.M. Salo, *Uniform estimates for local properties of analytic functions in a complete Reinahrdt domain*, Mat. Stud., **61** (2024), no.2, 168–175. doi:10.30970/ms.61.2.168-175
4. T.M. Salo, *Hayman's theorem for analytic functions in a complete Reinhardt domain*, Mat. Stud., **63** (2025), no.2, 129–135. doi:10.30970/ms.63.2.129-135
5. A.I. Bandura, T.M. Salo, *Dominating polynomial in power series expansion for analytic functions in a complete Reinhardt domain*, Mat. Stud., **64** (2025), no.1, 42–48. doi:10.30970/ms.64.1.42-48
6. F. Nuray, *Bounded index and four dimensional summability methods*, Novi Sad Journal of Mathematics, **49** (2019), no.2, 73–85. doi:10.30755/NSJOM.08285
7. F. Nuray, R.F. Patterson, *Entire bivariate functions of exponential type*, Bull. Math. Sci., **5** (2015), no.2, 171–177. doi:10.1007/s13373-015-0066-x
8. W. Rudin, Function theory in the unit ball on $\mathbb{C}^n$, Reprint of the 1980 Edition, Springer, 2008.
9. K. Zhu, Spaces of holomorphic functions in the unit ball, Graduate Texts in Mathematics, Springer, New York, 2005.
10. A.I. Bandura, O.B. Skaskiv, *Analytic functions in the unit ball of bounded* **L***-index: asymptotic and local properties*, Mat. Stud., **48** (2017), no.1, 37–73. doi:10.15330/ms.48.1.37-73
11. A. Bandura, O. Skaskiv, L. Smolovyk, *Slice holomorphic solutions of some directional differential equations with bounded L-index in the same direction*, Demonstr. Math., **52** (2019), no.1, 482–489. doi:10.1515/dema-2019-0043
12. A.I. Bandura, *Analytic functions in the unit ball of bounded value L-distribution in a direction*, Mat. Stud., **49** (2018), no.1, 75–79. doi:10.15330/ms.49.1.75-79
13. A.O. Kuryliak, O.B. Skaskiv, S.I. Panchuk, *Bitlyan-Gol'dberg type inequality for entire functions and diagonal maximal term*, Mat. Stud., **54** (2020), no.2, 135–145. doi:10.30970/ms.54.2.135-145
14. T. Beberok, *Explicit formulas of the Bergman kernel for some Reinhardt domains*, J. Geom. Anal., **26** (2016), no.4, 2883–2892. doi:10.1007/s12220-015-9652-0.
15. G. Cho, *Comparing the Carathéodory pseudo-distance and the Kähler–Einstein distance on complete Reinhardt domains*, Asian J. Math., **26** (2022), no.1, 37–44. doi:10.4310/AJM.2022.V26.N1.A2.
16. Q. Fu, G. Deng, *The weighted reproducing kernels of the Reinhardt domain*, Compl. Var. Ell. Equ., **69** (2024), no.12, 2048–2065. doi:10.1080/17476933.2023.2272131
17. A. Kuryliak, O. Skaskiv, *Wiman's type inequality in multiple-circular domain,* Axioms, **10** (2021), no.4, 0438. doi:10.3390/axioms10040348
18. A.O. Kuryliak, O.B. Skaskiv, *Wiman-type inequality in multiple-circular domains: Lévy's phenomenon and exceptional sets*, Ukr. Math. J., **74** (2022), no.5, 743–756. doi:10.1007/s11253-022-02098-y
19. X. Sima, Z. Tu, L. Xiong, *Some results of homogeneous expansions for a class of biholomorphic mappings defined on a Reinhardt domain in* $\mathbb{C}^n$, Demonstr. Math., **56** (2023), no.1, 20220242. doi:10.1515/dema-2022-0242
20. J. Wang, Y. Zhang, *The boundary Schwarz lemma and the rigidity theorem on Reinhardt domains* B_p^n *of* $\mathbb{C}^n$, Acta Math. Sci., **44** (2024), no.2, 839–850. doi:10.1007/s10473-024-0304-y

Ivano-Frankivsk National Technical University of Oil and Gas
Ivano-Frankivsk, Ukraine

andriy.bandura@nung.edu.ua
liudmyla.kryshtopa@nung.edu.ua
liubov.shehda@nung.edu.ua

Received 04.03.2026
Revised 25.03.2026