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A. O. KURYLIAK, T. M. SALO

BOREL TYPE ASYMPTOTIC RELATION FOR ENTIRE DIRICHLET SERIES AND h-MEASURE OF AN EXCEPTIONAL SETS

A. O. Kuryliak, T. M. Salo. *Borel type asymptotic relation for entire Dirichlet series and h-measure of an exceptional sets*, Mat. Stud. **65** (2026), 15–21.

There are presented sufficient conditions for the entire Dirichlet series with monotonically increasing sequence of exponents providing validity of Borel-type relation outside some set E of finite h-measure, i.e. $m_hE = \int_E dh(x) < +\infty$ with a positive continuously differentiable on $[0, +\infty)$ function h, whose derivative h' increases to infinity. The corresponding Borel-type relation states that the logarithm of supremum of the Dirichlet series along an imaginary line behaves like as logarithm of maximal term of the Dirichlet series. The conditions are given as the convergence of some auxiliary series constructed from the values of the function h' and the sequence of exponents.

Let $\mathcal{D}(\lambda)$ be the class of entire Dirichlet series absolutely convergent in $\mathbb{C}$ of the form

$$F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n},$$

where $\lambda = (\lambda_n)$ is a sequence such that $0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(1 \le n \uparrow +\infty)$. Let us denote by $\mathcal{D} = \bigcup_\lambda \mathcal{D}(\lambda)$ the class of all entire Dirichlet series.

We denote by $\mathcal{L}$ the class of positive continuous functions $\Phi\colon \mathbb{R}_+ := [0, +\infty) \to \mathbb{R}_+$ such that $\Phi(x) \nearrow +\infty$ $(0 \le x \to +\infty)$, and by $\mathcal{L}^+$ the class of positive continuous differentiable functions h such that $h'(t) \nearrow +\infty$ $(0 \le t \nearrow +\infty)$. Let $\Phi \in L$. We introduce the class

$$\mathcal{D}(\lambda, \Phi) = \{F \in \mathcal{D}(\lambda)\colon \ln \mu(x, F) \ge x\Phi(x) \ (x \ge x_0)\}.$$

Let $h(x)$ be a positive increasing to $+\infty$ continuous function on the real ray $[0, +\infty)$. For a Lebesgue measurable set $E \subset [0, +\infty)$ we call $m_hE \equiv \int_E dh(x)$ its h-measure, at $h(x) \equiv x$ the h-measure of a set E is its Lebesgue measure.

It is known ([1, 2]) that for every entire function $F \in \mathcal{D}(\lambda)$ the relation

$$\ln M(x, F) = (1 + o(1)) \ln \mu(x, F) \tag{1}$$

holds as $x \to +\infty$ outside some set E of finite Lebesgue measure, $mE < +\infty$, if and only if

$$\sum_{n=1}^{+\infty} \frac{1}{n\lambda_n} < +\infty, \tag{2}$$

where $M(x, F) = \sup\{|F(x + iy)|\colon y \in \mathbb{R}\}$, $\mu(x, F) = \max\{|a_n|e^{x\lambda_n}\colon n \ge 0\}$.

By $\nu(x, F) = \max\{n\colon |a_n|e^{x\lambda_n} = \mu(x, F)\}$ we denote the central index of a Dirichlet series $F \in \mathcal{D}(\lambda)$.

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*Corresponding author: T.M. Salo

The finiteness of Lebesgue measure of an exceptional set E in the previous statement on the whole class $\mathcal{D}$ is the best possible description. It follows from the following statement.

Theorem 1 ([3,4])**.** *For every function $h \in \mathcal{L}^+$ there exist a sequence $\lambda = (\lambda_k)$ such that satisfy* (2)*, an entire Dirichlet series $F \in \mathcal{D}(\lambda)$, a constant $\beta > 0$ and a measurable set $E_1 \subset [0, +\infty)$ of infinite h-measure ($m_h E_1 = +\infty$) such that*

$$(\forall\ x \in E_1):\quad \ln M(x, F) > (1 + \beta) \ln \mu(x, F).$$

Due to Theorem 1 the natural question arises: *what conditions must satisfy the entire Dirichlet series in order that relation* (1) *be true as $x \to +\infty$ outside some set E_2 of finite h-measure*, i.e. $m_h E_2 < +\infty$?

In this paper we obtain the answer to this question when $h \in \mathcal{L}^+$.

Currently, in the literature, the problem of finding an unimproved description of the magnitude of an exceptional set in one or another relation from the Wiman-Valiron theory is usually called the *I. V. Ostrovskii problem*, who formulated this problem in 1995 in relation to the Wiman inequality. Later, the same issue was considered of a number of articles (for example, see [5–14]) as well, in particular, with respect to other asymptotic relations, obtained in the Wiman-Valiron theory. The described problem is important not only from the point of view of the development of the internal Wiman-Valiron theory (see also [15–20]), but also from the point of view of possible applications of its results (in this connection see, for example, [21, 22]).

We prove the following theorem.

Theorem 2. *Let $h \in \mathcal{L}^+$, $\Phi \in \mathcal{L}$. If a function $F \in \mathcal{D}(\lambda, \Phi)$ and a number $b > 0$ such that*

$$\sum_{n=1}^{+\infty} \frac{h'(\varphi(\lambda_n) + b)}{n\lambda_n} < +\infty, \tag{3}$$

then asymptotic relation (1) *holds as $x \to +\infty$ outside some set E of finite h-measure ($m_h E < +\infty$), where the function φ is an inverse function to the function Φ.*

Let us first prove a few auxiliary statements.

By $(R_j(F))_{j=0}^{+\infty}$ we denote the sequence of discontinuity points of the central index $\nu(\sigma, F)$ of an entire Dirichlet series F, numbered so that $\nu(\sigma, F) = j$ for $R_j(F) \le \sigma < R_{j+1}(F)$ and, if $\nu(R_{j+1}(F), F) = j+p$, then $R_{j+1}(F) = R_{j+2}(F) = \cdots = R_{j+p}(F) < R_{j+p+1}(F)$. The upper estimates of the general term $|a_n|e^{x\lambda_n}$ of the Dirichlet series $F \in \mathcal{D}(\lambda)$ by the maximal term $\mu(x, F)$ can be obtained from the properties of the so-called comparison series, which for the positive continuous non-decreasing on $[0, +\infty)$ function $\alpha(t)$ takes the form (see also [23–26])

$$F_\alpha(z) = \sum_{n=0}^{+\infty} a_n \exp\left\{ z\lambda_n + \int_0^{\lambda_n} \alpha(t)dt \right\}.$$

Lemma 1. *If $F_\alpha \in \mathcal{D}(\lambda)$, then for all $j \ge 0$, and $\sigma \in [R_j(F_\alpha) + \alpha(\lambda_j), R_{j+1}(F_\alpha) + \alpha(\lambda_j)]$, $n \ge 0$ one has*

$$\frac{|a_n|e^{\sigma\lambda_n}}{|a_j|e^{\sigma\lambda_j}} \le \exp\left\{ -\int_{\lambda_j}^{\lambda_n} (\alpha(t) - \alpha(\lambda_j))dt \right\}. \tag{4}$$

In the case $R_j(F_\alpha) < R_{j+1}(F_\alpha)$ we additionally have

$$\nu(\sigma, F_\alpha) = j \text{ for all } \sigma \in [R_j(F_\alpha) + \alpha(\lambda_j), R_{j+1}(F_\alpha) + \alpha(\lambda_j)).$$

Proof of Lemma 1. We denote $\alpha_n = \int_0^{\lambda_n} \alpha(t)dt$ and $R_n = R_n(F_\alpha)$ for $n \ge 0$. For $\sigma \in [R_j, R_{j+1}]$ by definition of the maximal term $\mu(\sigma, F_\alpha)$ for all $n \ge 0$ one has

$$|a_n|e^{(\sigma\lambda_n+\alpha_n)} \le \mu(\sigma, F_\alpha) = |a_j|e^{(\sigma\lambda_j+\alpha_j)}.$$

Hence, for $(\sigma - \alpha(\lambda_j)) \in [R_j, R_{j+1}]$ and $n \ge 0$ we get

$$|a_n|e^{((\sigma-\alpha(\lambda_j))\lambda_n+\alpha_n)} \le |a_j| \exp^{((\sigma-\alpha(\lambda_j))\lambda_j+\alpha_j)}.$$

So, for all $n \ge 0$ and $\sigma \in [R_j + \alpha(\lambda_j), R_{j+1} + \alpha(\lambda_j)]$ we obtain

$$\frac{|a_n|e^{\sigma\lambda_n}}{|a_j|e^{\sigma\lambda_j}} \le \exp\{-(\alpha_n - \alpha_j) + \alpha(\lambda_j)(\lambda_n - \lambda_j)\} = \exp\left\{-\int_{\lambda_j}^{\lambda_n} (\alpha(t) - \alpha(\lambda_j))dt\right\}.$$

Therefore, inequality (4) is proved.

Since, $|a_n|e^{\sigma\lambda_n+\alpha_n} < \mu(\sigma, F_\alpha)$ for all $\sigma \in (R_j, R_{j+1})$ and $n \ne j$, for all $\sigma \in (R_j(F_\alpha) + \alpha(\lambda_j), R_{j+1}(F_\alpha) + \alpha(\lambda_j))$ inequality (4) is also strict for $n \ne j$.

But $\alpha(t)$ is a non-decreasing on $[0, +\infty)$ function. Thus, for arbitrary non-negative distinct integers n and m the inequality

$$\int_{\lambda_m}^{\lambda_n} (\alpha(t) - \alpha(\lambda_m))dt \ge 0$$

holds. Then, from the inequality (4) for $\sigma \in (R_j(F_\alpha) + \alpha(\lambda_j), R_{j+1}(F_\alpha) + \alpha(\lambda_j))$ in the case $R_j(F_\alpha) < R_{j+1}(F_\alpha)$ we get

$$\frac{|a_n|e^{\sigma\lambda_n}}{|a_j|e^{\sigma\lambda_j}} < \exp\left\{-\int_{\lambda_j}^{\lambda_n} (\alpha(t) - \alpha(\lambda_j))dt\right\} \le 1$$

for all $n \ne j$. It means that $\nu(\sigma, F) = j$ for $\sigma \in (R_j(F_\alpha) + \alpha(\lambda_j), R_{j+1}(F_\alpha) + \alpha(\lambda_j))$. Since the function $\nu(\sigma, F)$ is right semi-continuous, the second statement of Lemma 1 is also proven. □

Lemma 2. *Let $A\colon [0, +\infty) \to [0, +\infty)$ be a non-decreasing function such that $A(t) \to +\infty$ $(t \to +\infty)$. If $F_\alpha \in \mathcal{D}(\Lambda)$ with a function*

$$\alpha(t) = \int_0^t A(x)\frac{d\ln n(2x)}{x} \ (t \ge 0),$$

then asymptotic relation (1) *holds as $x \to +\infty$,*

$$x \notin \bigcup_{j=1}^{+\infty} [R_j(F_\alpha) + \alpha(\lambda_{j-1}), R_j(F_\alpha) + \alpha(\lambda_j)) =: E.$$

Proof of Lemma 2. Without loss of generality, we can assume that $|a_0| = 1$. Let $R_n = R_n(F_\alpha)$ $(n \ge 0)$. By Lemma 1, for all $n \ge 0$ and $x \in [R_j + \alpha(\lambda_j), R_{j+1} + \alpha(\lambda_j))$ inequality (4) holds and $\nu(x, F) = j$ for $x \in [R_j + \alpha(\lambda_j), R_{j+1} + \alpha(\lambda_j))$. Since $\int_{\lambda_\nu}^{\lambda_n} (\alpha(t) - \alpha(\lambda_\nu))d(\lambda_n - t) = \int_{\lambda_\nu}^{\lambda_n} (\lambda_n - t)d\alpha(t)$, therefore, for all $n \ge 0$ and $x \notin E$ we have

$$\frac{|a_n|e^{x\lambda_n}}{\mu(x, F)} \le \exp\left\{-\int_{\lambda_\nu}^{\lambda_n} (\alpha(t) - \alpha(\lambda_\nu))dt\right\} = \exp\left\{-\int_{\lambda_\nu}^{\lambda_n} \frac{\lambda_n - t}{t} A(t)d\ln n(2t)\right\}, \tag{5}$$

where $\nu=\nu(x,F)$.

Then at $n=0$ for $x\notin E$ from (5) we obtain $\ln\mu(x,F)\geq\int_0^{\lambda_\nu}A(t)d\ln n(2t)$. So, for any $t_0\in(0,\lambda_\nu)$ and $x\notin E$ one has $\ln\mu(x,F)\geq A(t_0)\left(\ln n(2\lambda_\nu)-\ln n(2t_0)\right)$. By the condition $A(t_0)\to+\infty$ $(t_0\to+\infty)$, it remains to choose $t_0=t_0(x)\to+\infty$ $(x\to+\infty)$ so that the conditions $A(t_0)\to+\infty$ and $2\ln n(2t_0)\leq\ln n(2\lambda_\nu)$ at $x\to+\infty$ are simultaneously satisfied. Hence,

$$\ln n(2\lambda_\nu)=o\left(\ln\mu(x,F)\right)\quad (x\to+\infty,\ x\notin E). \tag{6}$$

We put $m=\min\{n\colon \lambda_n\geq 2\lambda_\nu\}$, $A_\nu=A(\lambda_\nu)$. One should observe that for all $n\geq m$

$$\int_{\lambda_\nu}^{\lambda_n}\frac{\lambda_n-t}{t}A(t)d\ln n(2t)\geq A_\nu\int_{\lambda_\nu}^{\frac{\lambda_n}{2}}\frac{\lambda_n-t}{t}d\ln n(2t)\geq A_\nu\left(\ln n(\lambda_n)-\ln n(2\lambda_\nu)\right).$$

Therefore, for all $x\notin E$ from inequality (5) we get

$$\Sigma_2(x)\colon=\frac{1}{\mu(x,F)}\sum_{\lambda_n\geq 2\lambda_\nu}|a_n|e^{x\lambda_n}\leq\sum_{\lambda_n\geq 2\lambda_\nu}\exp\left\{-\int_{\lambda_\nu}^{\lambda_n}\frac{\lambda_n-t}{t}A(t)d\ln n(2t)\right\}\leq$$

$$\leq n(2\lambda_\nu)^{A_\nu}\sum_{k=m+1}^{+\infty}\frac{1}{k^{A_\nu}}\leq n(2\lambda_\nu)^{A_\nu}\int_m^{+\infty}\frac{dt}{t^{A_\nu}}=n(2\lambda_\nu)^{A_\nu}\frac{1}{A_\nu-1}\frac{1}{m^{A_\nu-1}}\leq n(2\lambda_\nu).$$

So, for every $x\notin E$ one has

$$M(x,F)\leq\sum_{\lambda_n\leq 2\lambda_\nu}|a_n|e^{x\lambda_n}+\mu(x,F)\cdot\Sigma_2(x)\leq 2n(2\lambda_\nu)\mu(x,F).$$

Applying relation (6), we finally obtain

$$\frac{\ln M(x,F)}{\ln\mu(x,F)}\leq 1+\frac{\ln 2+\ln n(2\lambda_\nu)}{\ln\mu(x,F)}=1+o(1)$$

as $x\to+\infty$, $x\notin E$. □

Proof of Theorem 2. Since $\frac{1}{n}\sim\ln(n+1)-\ln n$ $(n\to+\infty)$ and $h'(\varphi(\lambda_n)+b)\geq h'(\varphi(\lambda_n/2)+b)$. Therefore, from condition (3) it follows

$$\sum_{n=1}^{+\infty}\frac{h'(\varphi(\lambda_n/2)+b)(\ln(n+1)-\ln n)}{\lambda_n}<+\infty,$$

hence,

$$\int_{\lambda_1}^{+\infty}\frac{h'(\varphi(t)+b)}{t}d\ln n(2t)=2\sum_{n=1}^{+\infty}\frac{h'(\varphi(\lambda_n/2)+b)(\ln(n+1)-\ln n)}{\lambda_n}<+\infty, \tag{7}$$

where $n(t)$ is a counting function of the sequence λ.

Let us prove that there exists a non-decreasing function $B(t)\nearrow+\infty$ as $t\to+\infty$ such that

$$\int_0^{+\infty}B(t)\frac{h'(\varphi(t)+b)}{t}d\ln n(2t)<+\infty. \tag{8}$$

Consider the function

$$\psi(t)\colon=\int_t^{+\infty}\frac{h'(\varphi(x)+b)}{x}d\ln n(2x).$$

From (7) it follows that $\psi(t)\searrow 0$ as $t\to+\infty$. Let now for $n\geq 1$ and $t\in[\lambda_n,\lambda_{n+1}]$

$$\psi_0(t)\colon =\psi(\lambda_{n-1})+\frac{\psi(\lambda_n)-\psi(\lambda_{n-1})}{\lambda_{n+1}-\lambda_n}(t-\lambda_n).$$

Obviously, $\psi_0(t)\searrow 0$ for $t\to+\infty$ and, moreover, for $n\geq 1$ we have $\psi_0(\lambda_{n+1})=\psi(\lambda_n)$. Let $B(t)\colon =(\psi_0(t))^{-\frac{1}{2}}$. Then $B(t)\nearrow+\infty$ as $t\to+\infty$ and

$$\int_0^{+\infty}B(t)\frac{h'(\varphi(t)+b)}{t}d\ln n(2t)=-\int_0^{+\infty}\frac{1}{\sqrt{\psi_0(t)}}d\psi(t)=\sum_{n=1}^{+\infty}\frac{\psi(\lambda_{n-1})-\psi(\lambda_n)}{\sqrt{\psi_0(\lambda_n)}}\leq$$

$$\leq 2\sum_{n=1}^{+\infty}(\sqrt{\psi(\lambda_{n-1})}-\sqrt{\psi(\lambda_n)})=2\sqrt{\psi(0)}<+\infty.$$

So, (8) is satisfied.

We put now

$$B\colon =\int_0^{+\infty}B(x)\frac{d\ln n(2x)}{x}.$$

Since $h\in L^+$ and (8) hold, then $B<+\infty$ and the function $A(t)\colon =bB(t)/B$ is a continuous function and $A(t)\to+\infty$ as $t\to+\infty$.

Note that

$$\alpha(t)\colon =\int_0^t A(x)\frac{d\ln n(2x)}{x}\leq\int_0^{+\infty}A(x)\frac{d\ln n(2x)}{x}=\frac{b}{B}\int_0^{+\infty}B(x)\frac{d\ln n(2x)}{x}=b. \quad (9)$$

Let us prove that the Dirichlet series F_α is absolutely convergent in the whole complex plane, that is, $F_\alpha\in\mathcal{D}(\Lambda)$. For the general term of the series for F_α we have

$$|a_n|\exp\left\{\sigma\lambda_n+\int_0^{\lambda_n}\alpha(t)dt\right\}\leq|a_n|\exp\{(\sigma+\alpha(\lambda_n))\lambda_n\}\leq|a_n|e^{(\sigma+b)\lambda_n}.$$

So, $F_\alpha\in\mathcal{D}(\Lambda)$.

We put $R_n=R_n(F)$ for $n\geq 0$. Applying Lemma 2 with the functions $\alpha(t)$ and F_α, we obtain that relation (1) holds as $x\to+\infty$, and $x\notin E=\bigcup_{n=1}^{+\infty}[R_n+\alpha(\lambda_{n-1}),R_n+\alpha(\lambda_n))$.

Let us prove that $m_hE<+\infty$. Indeed,

$$m_hE=\int_E dh(x)=\sum_{n=1}^{+\infty}\int_{R_n+\alpha(\lambda_{n-1})}^{R_n+\alpha(\lambda_n)}dh(x)=\sum_{n=1}^{+\infty}\left(h(R_n+\alpha(\lambda_n))-h(R_n+\alpha(\lambda_{n-1}))\right). \quad (10)$$

By the condition $F\in\mathcal{D}(\lambda,\Phi)$,

$$x\Phi(x)\leq\ln\mu(x,F)=\ln\mu(0,F)+\int_0^x\lambda_{\nu(t,F)}dt\leq x\lambda_{\nu(x-0,F)}$$

for all $x\geq x_0$. Hence, $x\leq\varphi(\lambda_{\nu(x-0,F)})$. Since, by Lemma 1, $\nu(R_n+\alpha(\lambda_{n-1})-0,F)\leq n-1$, we get $R_n+\alpha(\lambda_{n-1})\leq\varphi(\lambda_{n-1})$. Consequently applying the formula from Lagrange's Mean Value Theorem $h(b)-h(a)=(b-a)h'(a+\varepsilon(b-a))$ with $\varepsilon\in(0,1)$, the last inequality and inequality (9) to (10), we obtain

$$m_hE=\sum_{n=1}^{+\infty}h'\left(R_n+\alpha(\lambda_{n-1})+\varepsilon_n(\alpha(\lambda_n)-\alpha(\lambda_{n-1}))\right)(\alpha(\lambda_n)-\alpha(\lambda_{n-1}))\leq$$

$$\leq\sum_{n=1}^{+\infty}\Big(\int_{\lambda_{n-1}}^{\lambda_n}A(x)\frac{d\ln n(2x)}{x}\Big)h'\Big(\varphi(\lambda_{n-1})+\int_{\lambda_{n-1}}^{\lambda_n}A(x)\frac{d\ln n(2x)}{t}\Big)\leq$$

$$\leq\frac{b}{B}\int_0^{+\infty}B(x)\frac{h'(\varphi(x)+b)}{x}d\ln n(2x)<+\infty.$$

If all elements of the sequence (R_j) are distinct, then Theorem 2 is proved.

If there are identical points among elements of the sequence (R_j), i.e. $R_n < R_{n+1} = R_{n+2} = \cdots = R_{n+p} < R_{n+p+1}$, then

$$\int_{R_{n+1}+\alpha(\lambda_n)}^{R_{n+p}+\alpha(\lambda_{n+p})} dh(x) \le h'(R_{n+p}+\alpha(\lambda_{n+p}))(\alpha(\lambda_{n+p})-\alpha(\lambda_{n+1})) = (\alpha(\lambda_{n+p})-\alpha(\lambda_{n+1}))\times$$
$$\times h'(R_{n+1}+\alpha(\lambda_n)+\alpha(\lambda_{n+p})-\alpha(\lambda_n)) \le h'(R_{n+1}+\alpha(\lambda_n)+b)\int_{\lambda_n}^{\lambda_{n+p}} A(x)\frac{d\ln n(2x)}{x}.$$

Since, by Lemma 1, $\nu(R_{n+1}+\alpha(\lambda_n)-0,F) \le n$, then using the inequality $x \le \varphi(\lambda_{\nu(x-0,F)})$ we have

$$\int_{R_{n+1}+\alpha(\lambda_n)}^{R_{n+p}+\alpha(\lambda_{n+p})} dh(x) \le \int_{\lambda_n}^{\lambda_{n+p}} A(x)\frac{h'(\varphi(x)+b)}{x} d\ln n(2x).$$

So, in the case of existence of identical points in the sequence (R_j), we again arrive at the inequality

$$m_h E \le \frac{b}{B}\int_0^{+\infty} B(x)\frac{h'(\varphi(x)+b)}{x} d\ln n(2x) < +\infty.$$

Therefore, Theorem 2 is proved. □

Conjecture 1. *The finiteness of the h-measure of an exceptional set in the Borel relation for the class $\mathcal{D}(\lambda,\Phi)$ under the conditions of Theorem 2 is the best possible description of its magnitude.*

Conjecture 2. *Condition* (3) *in Theorem 2 can be replaced by the condition: for every sequence (b_n), $b_n \to +0$ $(n \to +\infty)$,*

$$\sum_{n=1}^{+\infty} \frac{h'(\varphi(\lambda_n)+b_n)}{n\lambda_n} < +\infty.$$

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Ivan Franko National University of Lviv
Lviv, Ukraine
andriykuryliak@gmail.com

Department of Mathematics, Lviv Polytechnic National University
Lviv, Ukraine
tetyan.salo@gmail.com

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