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ASMA ALI, SHAKIV ALI*, MOHD TASLEEM

NONLINEAR BI-SKEW LIE TRIPLE HIGHER DERIVATIONS ON PRIME *-ALGEBRAS

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Let $\mathfrak{E}$ be a unital prime $*$-algebra. For any $U, V \in \mathfrak{E}$, the product defined by $U \diamond V = U^*V - V^*U$ is known as bi-skew Lie product of U and V. This paper establishes that if a family $\Delta = \{\zeta_n\}_{n\in\mathbb{N}}$ of mappings $\zeta_n\colon \mathfrak{E} \to \mathfrak{E}$ (not necessarily linear) on $\mathfrak{E}$ with $\zeta_0 = id_{\mathfrak{E}}$ (the identity map on $\mathfrak{E}$), satisfies the relation $\zeta_n(U \diamond V \diamond W) = \sum_{p+q+r=n} \zeta_p(U) \diamond \zeta_q(V) \diamond \zeta_r(W)$ for all $U, V, W \in \mathfrak{E}$ and for each $n \in \mathbb{N}$, then Δ is an additive $*$-higher derivation provided $\zeta_n(\frac{\beta I}{2})$ is self-adjoint for $\beta \in \{1, i\}$.

1. Introduction. Let $\mathfrak{E}$ be a $*$-algebra over the complex field $\mathbb{C}$. An algebra $\mathfrak{E}$ is called prime if $U\mathfrak{E}V = (0)$ for $U, V \in \mathfrak{E}$ implies either $U = 0$ or $V = 0$. An additive map $\zeta\colon \mathfrak{E} \to \mathfrak{E}$ is called an additive $*$-derivation if it satisfies $\zeta(UV) = \zeta(U)V + U\zeta(V)$ and $\zeta(U^*) = \zeta(U)^*$ for all $U, V \in \mathfrak{E}$. The left (resp. right) bi-skew Lie product is defined as $U \diamond V = U^*V - V^*U$ (resp. $U \circ V = UV^* - VU^*$), while the left bi-skew Jordan (resp. right bi-skew Jordan) product is defined by $U \bullet V = U^*V + V^*U$ (resp. $U \circledcirc V = UV^* + VU^*$). A map $\zeta\colon \mathfrak{E} \to \mathfrak{E}$ (not necessarily linear) is called a nonlinear bi-skew Lie derivation or bi-skew Lie triple derivation if

$$\zeta(U \diamond V) = \zeta(U) \diamond V + U \diamond \zeta(V)$$

or

$$\zeta(U \diamond V \diamond W) = \zeta(U) \diamond V \diamond W + U \diamond \zeta(V) \diamond W + U \diamond V \diamond \zeta(W)$$

for all $U, V, W \in \mathfrak{E}$. Analogously, one can define nonlinear bi-skew Jordan derivations and nonlinear bi-skew Jordan triple derivations. The properties of such derivations have been widely investigated in various works (see [2, 3, 6, 7, 8, 12]). In [12], Taghavi and Razeghi demonstrated that if $\zeta\left(\frac{I}{2}\right)$ and $\zeta\left(\frac{iI}{2}\right)$ are self-adjoint, then every nonlinear bi-skew Lie derivation on a prime $*$-algebra $\mathfrak{E}$ is an additive $*$-derivation. Later, Shahvandi and Taghavi ([11]) extended this result and prove that every nonlinear bi-skew Lie triple derivation on a prime $*$-algebra $\mathfrak{E}$ is also an additive $*$-derivation, provided that $\zeta\left(\frac{I}{2}\right)$ and $\zeta\left(\frac{iI}{2}\right)$ are self-adjoint.

In this paper, we extend the above result [11] to the case of higher derivations. Let $\mathbb{N}$ denote the set of all non-negative integers, and consider a family $\Delta = \{\zeta_n\}_{n\in\mathbb{N}}$ of linear mappings $\zeta_n\colon \mathfrak{E} \to \mathfrak{E}$ on the $*$-algebra $\mathfrak{E}$, where $\zeta_0 = \mathrm{id}_{\mathfrak{E}}$ (the identity map on $\mathfrak{E}$). Then we say that Δ is:
(a) a higher derivation if for all $U, V \in \mathfrak{E}$ and $n \in \mathbb{N}$

$$\zeta_n(UV) = \sum_{p+q=n} \zeta_p(U)\zeta_q(V);$$

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*Corresponding author: S. Ali

(b) a bi-skew Lie higher derivation if for all $U, V \in \mathfrak{E}$ and $n \in \mathbb{N}$

$$\zeta_n(U \diamond V) = \sum_{p+q=n} \zeta_p(U) \diamond \zeta_q(V);$$

(c) a bi-skew Lie triple higher derivation if for all $U, V, W \in \mathfrak{E}$ and $n \in \mathbb{N}$

$$\zeta_n(U \diamond V \diamond W) = \sum_{p+q+r=n} \zeta_p(U) \diamond \zeta_q(V) \diamond \zeta_r(W).$$

If the linearity assumption is relaxed, Δ is termed a nonlinear higher derivation, a nonlinear bi-skew Lie higher derivation and a nonlinear bi-skew Lie triple higher derivation, respectively. It is evident that for $n = 1$, higher derivation, bi-skew Lie higher derivation and bi-skew Lie triple higher derivation are usual derivation, bi-skew Lie derivation and bi-skew Lie triple derivation respectively. In recent years, different types of higher derivations have been studied on various algebras (see [5], [4], [10], [13], [14]). Zhang and coauthors ([15]), proved that a nonlinear $*$-Lie higher derivation on factor von neumann algebras is an additive $*$-higher derivation. In [1], Ali et al. characterized the nonlinear mixed bi-skew Jordan triple higher derivations on prime $*$-algberas.

Motivated by the work on higher derivations, we obtain the structure of nonlinear bi-skew Lie triple higher derivations on prime $*$-algebras. In fact we prove that every nonlinear bi-skew Lie triple higher derivation on a prime $*$-algebra $\mathfrak{E}$ is an additive $*$-higher derivation, provided $\zeta_n\left(\frac{\beta I}{2}\right)$ is self-adjoint.

2. Preliminaries and main result. Throughout the article, unless mentioned otherwise, $\mathfrak{E}$ stands for a prime $*$-algebra over the complex field $\mathbb{C}$. Let H be a complex Hilbert space and $\mathfrak{B}(H)$ refer to the algebra of all bounded linear operators on H. Let $P \in \mathfrak{B}(H)$ be a projection operator, then $P^2 = P$ and $P^* = P$.

Main Theorem. Let $\mathfrak{E}$ be a prime $*$-algebra with unity I and a nontrivial projection P. Let $\Delta = \{\zeta_n\}_{n\in\mathbb{N}}$ be a nonlinear bi-skew Lie triple higher derivation on $\mathfrak{E}$ i.e.,

$$\zeta_n(U \diamond V \diamond W) = \sum_{p+q+r=n} \zeta_p(U) \diamond \zeta_q(V) \diamond \zeta_r(W)$$

for all $U, V, W \in \mathfrak{E}$ and $n \in \mathbb{N}$. Then Δ is an additive $*$-higher derivation on $\mathfrak{E}$, provided $\zeta_n\left(\frac{\beta I}{2}\right)$ is self-adjoint for $\beta \in \{1, i\}$.

Let $P_1 = P$ be a nontrivial projection in $\mathfrak{E}$ and $P_2 = I - P$. Write $\mathfrak{E}_{ij} = P_i\mathfrak{E}P_j, 1 \leq i, j \leq 2$. Then by the Peirce's decomposition, we can write $\mathfrak{E} = \mathfrak{E}_{11} \oplus \mathfrak{E}_{12} \oplus \mathfrak{E}_{21} \oplus \mathfrak{E}_{22}$ and thus any operator $U \in \mathfrak{E}$ can be written as $U = U_{11} + U_{12} + U_{21} + U_{22}$.
The proof of our main theorem relies on several key lemmas, which we now present.

Lemma 1. *$\zeta_n(0) = 0$ for each $n \in \mathbb{N}$.*

Proof. We employ an inductive argument on positive integers $n \geq 1$. The initial case ($n = 1$) is an immediate consequence of Claim 2 in [11]. Proceeding inductively, we suppose $\zeta_k(0) = 0$ holds whenever $1 \leq k < n$. The remaining task consists in verifying $\zeta_n(0) = 0$. Then

$$\zeta_n(0) = \zeta_n(0 \diamond 0 \diamond 0) = \sum_{p+q+r=n} \zeta_p(0) \diamond \zeta_q(0) \diamond \zeta_r(0) =$$

$$= \zeta_n(0) \diamond 0 \diamond 0 + 0 \diamond \zeta_n(0) \diamond 0 + 0 \diamond 0 \diamond \zeta_n(0) + \sum_{\substack{p+q+r=n \\ 0\leq p,q,r\leq n-1}} \zeta_p(0) \diamond \zeta_q(0) \diamond \zeta_r(0) = 0.$$

□

Lemma 2. *$\zeta_n(\frac{I}{2}) = 0$; $\zeta_n(\frac{-I}{2}) = 0$ and $\zeta_n(\frac{iI}{2}) = 0$ for each $n \in \mathbb{N}$ with $n \geq 1$.*

Proof. The statement holds for $n = 1$, as established by Claim 3 in [11]. Suppose the statement is valid for all $k < n$, i.e., $\zeta_k(\frac{I}{2}) = 0$; $\zeta_k(\frac{-I}{2}) = 0$ and $\zeta_k(\frac{iI}{2}) = 0$. Our goal is to prove that this holds for $k = n$. Given that $\frac{iI}{2} = \frac{iI}{2} \diamond \frac{I}{2} \diamond \frac{I}{2}$, it follows that

$$\zeta_n\Big(\frac{iI}{2}\Big) = \zeta_n\Big(\frac{iI}{2} \diamond \frac{I}{2} \diamond \frac{I}{2}\Big) = \sum\nolimits_{p+q+r=n} \zeta_p\Big(\frac{iI}{2}\Big) \diamond \zeta_q\Big(\frac{I}{2}\Big) \diamond \zeta_r\Big(\frac{I}{2}\Big) = \zeta_n\Big(\frac{iI}{2}\Big) \diamond \frac{I}{2} \diamond \frac{I}{2}+$$
$$+\frac{iI}{2} \diamond \zeta_n\Big(\frac{I}{2}\Big) \diamond \frac{I}{2} + \frac{iI}{2} \diamond \frac{I}{2} \diamond \zeta_n\Big(\frac{I}{2}\Big) + \sum\nolimits_{\substack{p+q+r=n\\0\le p,q,r\le n-1}} \zeta_p\Big(\frac{iI}{2}\Big) \diamond \zeta_q\Big(\frac{I}{2}\Big) \diamond \zeta_r\Big(\frac{I}{2}\Big) =$$
$$= \frac{1}{2}\zeta_n\Big(\frac{iI}{2}\Big) - \frac{1}{2}\zeta_n\Big(\frac{iI}{2}\Big)^* + \frac{i}{2}\zeta_n\Big(\frac{I}{2}\Big) + \frac{i}{2}\zeta_n\Big(\frac{I}{2}\Big)^* + \frac{i}{2}\zeta_n\Big(\frac{I}{2}\Big) + \frac{i}{2}\zeta_n\Big(\frac{I}{2}\Big)^*.$$

This implies that

$$\frac{1}{2}\zeta_n\Big(\frac{iI}{2}\Big) + \frac{1}{2}\zeta_n\Big(\frac{iI}{2}\Big)^* - i\zeta_n\Big(\frac{I}{2}\Big) - i\zeta_n\Big(\frac{I}{2}\Big)^* = 0. \tag{1}$$

It follows from (1) that

$$\frac{1}{2}\zeta_n\Big(\frac{iI}{2}\Big) + \frac{1}{2}\zeta_n\Big(\frac{iI}{2}\Big)^* + i\zeta_n\Big(\frac{I}{2}\Big) + i\zeta_n\Big(\frac{I}{2}\Big)^* = 0. \tag{2}$$

On adding (1) and (2), we get

$$\zeta_n\Big(\frac{iI}{2}\Big) + \zeta_n\Big(\frac{iI}{2}\Big)^* = 0. \tag{3}$$

Since, by the hypothesis, $\zeta_n(\frac{iI}{2})$ is self-adjoint, then we have

$$\zeta_n\Big(\frac{iI}{2}\Big)^* = \zeta_n\Big(\frac{iI}{2}\Big). \tag{4}$$

From (3) and (4), we get

$$\zeta_n\Big(\frac{iI}{2}\Big) = 0. \tag{5}$$

Again using the fact that $\zeta_n(\frac{I}{2})$ is self adjoint and $\zeta_n(\frac{iI}{2}) = 0$ in (1), we get $\zeta_n(\frac{I}{2}) = 0$.
We now demonstrate that $\zeta_n(\frac{-I}{2}) = 0$. Utilizing the relation $\frac{iI}{2} = \frac{iI}{2} \diamond \frac{-I}{2} \diamond \frac{-I}{2}$ and condition $\zeta_n(\frac{iI}{2}) = 0$, we can write

$$0 = \zeta_n\Big(\frac{iI}{2}\Big) = \zeta_n\Big(\frac{iI}{2} \diamond \frac{-I}{2} \diamond \frac{-I}{2}\Big) = \sum\nolimits_{p+q+r=n} \zeta_p\Big(\frac{iI}{2}\Big) \diamond \zeta_q\Big(\frac{-I}{2}\Big) \diamond \zeta_r\Big(\frac{-I}{2}\Big) =$$
$$= \zeta_n\Big(\frac{iI}{2}\Big) \diamond \frac{-I}{2} \diamond \frac{-I}{2} + \frac{iI}{2} \diamond \zeta_n\Big(\frac{-I}{2}\Big) \diamond \frac{-I}{2} + \frac{iI}{2} \diamond \frac{-I}{2} \diamond \zeta_n\Big(\frac{-I}{2}\Big)+$$
$$+\sum\nolimits_{\substack{p+q+r=n\\0\le p,q,r\le n-1}} \zeta_p\Big(\frac{iI}{2}\Big) \diamond \zeta_q\Big(\frac{-I}{2}\Big) \diamond \zeta_r\Big(\frac{-I}{2}\Big) = -i\zeta_n\Big(\frac{-I}{2}\Big)^* - i\zeta_n\Big(\frac{-I}{2}\Big).$$

This gives

$$\zeta_n\Big(\frac{-I}{2}\Big)^* = -\zeta_n\Big(\frac{-I}{2}\Big). \tag{6}$$

Next using the fact that $\frac{-I}{2} \diamond \frac{I}{2} \diamond \frac{I}{2} = 0$ and Lemma 1, we obtain

$$0 = \zeta_n\Big(\frac{-I}{2} \diamond \frac{I}{2} \diamond \frac{I}{2}\Big) = \sum\nolimits_{p+q+r=n} \zeta_p\Big(\frac{-I}{2}\Big) \diamond \zeta_q\Big(\frac{I}{2}\Big) \diamond \zeta_r\Big(\frac{I}{2}\Big) =$$
$$= \zeta_n\Big(\frac{-I}{2}\Big) \diamond \frac{I}{2} \diamond \frac{I}{2} + \frac{-I}{2} \diamond \zeta_n\Big(\frac{I}{2}\Big) \diamond \frac{I}{2} + \frac{-I}{2} \diamond \frac{I}{2} \diamond \zeta_n\Big(\frac{I}{2}\Big)+$$
$$+\sum\nolimits_{\substack{p+q+r=n\\0\le p,q,r\le n-1}} \zeta_p\Big(\frac{-I}{2}\Big) \diamond \zeta_q\Big(\frac{I}{2}\Big) \diamond \zeta_r\Big(\frac{I}{2}\Big) = \frac{1}{2}\zeta_n\Big(\frac{-I}{2}\Big) - \frac{1}{2}\zeta_n\Big(\frac{-I}{2}\Big)^*.$$

This implies that $\zeta_n(\frac{-I}{2})^* = \zeta_n(\frac{-I}{2})$. Hence, using equality (6), we get $\zeta_n(\frac{-I}{2}) = 0$. $\square$

Lemma 3. *For any $U \in \mathfrak{E}$ and $n \in \mathbb{N}$, we have*

$$(i)\ \zeta_n(-iU) = -i\zeta_n(U); \quad (ii)\ \zeta_n(iU) = i\zeta_n(U).$$

Proof. (i) To verify this, observe that $\zeta_n\big(-iU \diamond \frac{I}{2} \diamond \frac{I}{2}\big) = \zeta_n\big(U \diamond \frac{iI}{2} \diamond \frac{I}{2}\big)$. By using Lemma 2.2, we get $\zeta_n\big(-iU\big) \diamond \frac{I}{2} \diamond \frac{I}{2} = \zeta_n\big(U\big) \diamond \frac{iI}{2} \diamond \frac{I}{2}$. It implies that

$$\zeta_n(-iU) - \zeta_n(-iU)^* = -i\zeta_n(U)^* - i\zeta_n(U). \tag{7}$$

Additionally, it is straightforward to confirm that $\zeta_n\big(-iU \diamond \frac{iI}{2} \diamond \frac{I}{2}\big) = \zeta_n\big(U \diamond \frac{-I}{2} \diamond \frac{I}{2}\big)$, which leads to $\zeta_n\big(-iU\big) \diamond \frac{iI}{2} \diamond \frac{I}{2} = \zeta_n\big(U\big) \diamond \frac{-I}{2} \diamond \frac{I}{2}$. Consequently, we obtain

$$-i\zeta_n(-iU)^* - i\zeta_n(-iU) = \zeta_n(U)^* - \zeta_n(U). \tag{8}$$

Rewriting (8) equivalently, we establish

$$\zeta_n(-iU)^* + \zeta_n(-iU) = i\zeta_n(U)^* - i\zeta_n(U). \tag{9}$$

Adding (7) and (9), we get $\zeta_n(-iU) = -i\zeta_n(U)$.

A similar argument can be used to show that $\zeta_n(iU) = i\zeta_n(U)$. □

Lemma 4. *For any $U_{11} \in \mathfrak{E}_{11}, U_{12} \in \mathfrak{E}_{12}, U_{21} \in \mathfrak{E}_{21}$ and $U_{22} \in \mathfrak{E}_{22}$, we have:*

$$(i)\ \zeta_n(U_{11} + U_{12}) = \zeta_n(U_{11}) + \zeta_n(U_{12}); \quad (ii)\ \zeta_n(U_{21} + U_{22}) = \zeta_n(U_{21}) + \zeta_n(U_{22}).$$

Proof. (i) We proceed by induction on $n \in \mathbb{N}$ with $n \geq 1$. By Claim 5 in [11], the result holds true for $n = 1$. Assume that for all $k < n$, the result holds, i.e.,

$$\zeta_k(U_{11} + U_{12}) = \zeta_k(U_{11}) + \zeta_k(U_{12}).$$

Let $T = \zeta_n(U_{11} + U_{12}) - \zeta_n(U_{11}) - \zeta_n(U_{12})$. We now demonstrate that $T = 0$. For any $W_{21} \in \mathfrak{E}_{21}$, observe that $U_{11} \diamond W_{21} \diamond I = 0$ and in view of Lemma 1, we find that

$$\begin{aligned}
\zeta_n((U_{11} + U_{12}) \diamond W_{21} \diamond I) &= \zeta_n(U_{11} \diamond W_{21} \diamond I) + \zeta_n(U_{12} \diamond W_{21} \diamond I) = \\
&= \zeta_n(U_{11}) \diamond W_{21} \diamond I + U_{11} \diamond \zeta_n(W_{21}) \diamond I + U_{11} \diamond W_{21} \diamond \zeta_n(I) + \\
&+ \sum_{\substack{p+q+r=n \\ 0 \leq p,q,r \leq n-1}} \zeta_p(U_{11}) \diamond \zeta_q(W_{21}) \diamond \zeta_r(I) + \zeta_n(U_{12}) \diamond W_{21} \diamond I + U_{12} \diamond \zeta_n(W_{21}) \diamond I + \\
&+ U_{12} \diamond W_{21} \diamond \zeta_n(I) + \sum_{\substack{p+q+r=n \\ 0 \leq p,q,r \leq n-1}} \zeta_p(U_{12}) \diamond \zeta_q(W_{21}) \diamond \zeta_r(I) = \\
&= (\zeta_n(U_{11}) + \zeta_n(U_{12})) \diamond W_{21} \diamond I + (U_{11} + U_{12}) \diamond \zeta_n(W_{21}) \diamond I + \\
&+ (U_{11} + U_{12}) \diamond W_{21} \diamond \zeta_n(I) + \sum_{\substack{p+q+r=n \\ 0 \leq p,q,r \leq n-1}} (\zeta_p(U_{11}) + \zeta_p(U_{12})) \diamond \zeta_q(W_{21}) \diamond \zeta_r(I).
\end{aligned}$$

On the other hand, by induction hypothesis, we also have

$$\begin{aligned}
\zeta_n((U_{11} + U_{12}) \diamond W_{21} \diamond I) &= \zeta_n(U_{11} + U_{12}) \diamond W_{21} \diamond I + (U_{11} + U_{12}) \diamond \zeta_n(W_{21}) \diamond I + \\
&+ (U_{11} + U_{12}) \diamond W_{21} \diamond \zeta_n(I) + \sum_{\substack{p+q+r=n \\ 0 \leq p,q,r \leq n-1}} \zeta_p(U_{11} + U_{12}) \diamond \zeta_q(W_{21}) \diamond \zeta_r(I) = \\
&= \zeta_n(U_{11} + U_{12}) \diamond W_{21} \diamond I + (U_{11} + U_{12}) \diamond \zeta_n(W_{21}) \diamond I + (U_{11} + U_{12}) \diamond W_{21} \diamond \zeta_n(I) + \\
&+ \sum_{\substack{p+q+r=n \\ 0 \leq p,q,r \leq n-1}} (\zeta_p(U_{11}) + \zeta_p(U_{12})) \diamond \zeta_q(W_{21}) \diamond \zeta_r(I).
\end{aligned}$$

By comparing the above two equations, we deduce that $T \diamond W_{21} \diamond I = 0$, which gives

$$W_{21}^* T - T^* W_{21} = 0. \tag{10}$$

Following the same procedure as above and replacing W_{21} by iW_{21}, we obtain

$$W_{21}^*T + T^*W_{21} = 0. \tag{11}$$

From (10) and (11) and using the primeness of $\mathfrak{E}$, we get $T_{21} = T_{22} = 0$.
For any $W_{12} \in \mathfrak{E}_{12}$, applying Lemma 1 and utilizing the condition $U_{12} \diamond W_{12} \diamond P_1 = 0$, we obtain

$$\begin{aligned}
\zeta_n((U_{11}+U_{12}) \diamond W_{12} \diamond P_1) &= \zeta_n(U_{11} \diamond W_{12} \diamond P_1) + \zeta_n(U_{12} \diamond W_{12} \diamond P_1) = \zeta_n(U_{11}) \diamond W_{12} \diamond P_1 + \\
&+ U_{11} \diamond \zeta_n(W_{12}) \diamond P_1 + U_{11} \diamond W_{12} \diamond \zeta_n(P_1) + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \zeta_p(U_{11}) \diamond \zeta_q(W_{12}) \diamond \zeta_r(P_1) + \\
&+ \zeta_n(U_{12}) \diamond W_{12} \diamond P_1 + U_{12} \diamond \zeta_n(W_{12}) \diamond P_1 + U_{12} \diamond W_{12} \diamond \zeta_n(P_1) + \\
&+ \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \zeta_p(U_{12}) \diamond \zeta_q(W_{12}) \diamond \zeta_r(P_1) = (\zeta_n(U_{11}) + \zeta_n(U_{12})) \diamond W_{12} \diamond P_1 + \\
&+ (U_{11}+U_{12}) \diamond \zeta_n(W_{12}) \diamond P_1 + (U_{11}+U_{12}) \diamond W_{12} \diamond \zeta_n(P_1) + \\
&+ \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} (\zeta_p(U_{11}) + \zeta_p(U_{12})) \diamond \zeta_q(W_{12}) \diamond \zeta_r(P_1).
\end{aligned}$$

On the other hand, by induction hypothesis, we have

$$\begin{aligned}
\zeta_n((U_{11}+U_{12}) \diamond W_{12} \diamond P_1) &= \zeta_n(U_{11}+U_{12}) \diamond W_{12} \diamond P_1 + (U_{11}+U_{12}) \diamond \zeta_n(W_{12}) \diamond P_1 + \\
&+ (U_{11}+U_{12}) \diamond W_{12} \diamond \zeta_n(P_1) + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \zeta_p(U_{11}+U_{12}) \diamond \zeta_q(W_{12}) \diamond \zeta_r(P_1) = \\
&= \zeta_n(U_{11}+U_{12}) \diamond W_{12} \diamond P_1 + (U_{11}+U_{12}) \diamond \zeta_n(W_{12}) \diamond P_1 + \\
&+ (U_{11}+U_{12}) \diamond W_{12} \diamond \zeta_n(P_1) + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} (\zeta_p(U_{11}) + \zeta_p(U_{12})) \diamond \zeta_q(W_{12}) \diamond \zeta_r(P_1).
\end{aligned}$$

By analyzing both equations, we conclude that $T \diamond W_{12} \diamond P_1 = 0$, which leads to the conclusion that $T_{11} = 0$. We now show that $T_{12} = 0$. Note that $U_{11} \diamond P_1 \diamond P_2 = 0$. We have

$$\begin{aligned}
\zeta_n((U_{11}+U_{12}) \diamond P_1 \diamond P_2) &= \zeta_n(U_{11} \diamond P_1 \diamond P_2) + \zeta_n(U_{12} \diamond P_1 \diamond P_2) = \\
&= \zeta_n(U_{11}) \diamond P_1 \diamond P_2 + U_{11} \diamond \zeta_n(P_1) \diamond P_2 + U_{11} \diamond P_1 \diamond \zeta_n(P_2) + \\
&+ \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \zeta_p(U_{11}) \diamond \zeta_q(P_1) \diamond \zeta_r(P_2) + \zeta_n(U_{12}) \diamond P_1 \diamond P_2 + U_{11} \diamond \zeta_n(P_1) \diamond P_2 + \\
&+ U_{11} \diamond P_1 \diamond \zeta_n(P_2) + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \zeta_p(U_{12}) \diamond \zeta_q(P_1) \diamond \zeta_r(P_2) \\
&= (\zeta_n(U_{11}) + \zeta_n(U_{12})) \diamond P_1 \diamond P_2 + (U_{11}+U_{12}) \diamond \zeta_n(P_1) \diamond P_2 \\
&+ (U_{11}+U_{12}) \diamond P_1 \diamond \zeta_n(P_2) + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} (\zeta_p(U_{11}) + \zeta_p(U_{12})) \diamond \zeta_q(P_1) \diamond \zeta_r(P_2).
\end{aligned}$$

On the other hand, by the induction hypothesis, we have

$$\begin{aligned}
\zeta_n((U_{11}+U_{12}) \diamond P_1 \diamond P_2) &= \zeta_n(U_{11}+U_{12}) \diamond P_1 \diamond P_2 + (U_{11}+U_{12}) \diamond \zeta_n(P_1) \diamond P_2 + \\
&+ (U_{11}+U_{12}) \diamond P_1 \diamond \zeta_n(P_2) + \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \zeta_p(U_{11}+U_{12}) \diamond \zeta_q(P_1) \diamond \zeta_r(P_2) = \\
&= \zeta_n(U_{11}+U_{12}) \diamond P_1 \diamond P_2 + (U_{11}+U_{12}) \diamond \zeta_n(P_1) \diamond P_2 + (U_{11}+U_{12}) \diamond P_1 \diamond \zeta_n(P_2) + \\
&+ \sum_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} (\zeta_p(U_{11}) + \zeta_p(U_{12})) \diamond \zeta_q(P_1) \diamond \zeta_r(P_2).
\end{aligned}$$

Comparing the above two relations, we get $T \diamond P_1 \diamond P_2 = 0$. This implies that $T_{12} = 0$. Hence $T = 0$. Similarly, we can show that $\zeta_n(U_{21} + U_{22}) = \zeta_n(U_{21}) + \zeta_n(U_{22})$. □

Lemma 5. *For any* $U_{11} \in \mathfrak{E}_{11}, U_{12} \in \mathfrak{E}_{12}, U_{21} \in \mathfrak{E}_{21}$ *and* $U_{22} \in \mathfrak{E}_{22}$, *we have:*
(*i*) $\zeta_n(U_{11} + U_{12} + U_{21}) = \zeta_n(U_{11}) + \zeta_n(U_{12}) + \zeta_n(U_{21})$;
(*ii*) $\zeta_n(U_{12} + U_{21} + U_{22}) = \zeta_n(U_{12}) + \zeta_n(U_{21}) + \zeta_n(U_{22})$.

Proof. (i) By Claim 6 in [11], the result holds true for $n = 1$. Assume that the result holds for $k < n$, i.e., $\zeta_k(U_{11} + U_{12} + U_{21}) = \zeta_k(U_{11}) + \zeta_k(U_{12}) + \zeta_k(U_{21})$.
Let $T = \zeta_n(U_{11} + U_{12} + U_{21}) - \zeta_n(U_{11}) - \zeta_n(U_{12}) - \zeta_n(U_{21})$. We now show that $T = 0$. For any $W_{21} \in \mathfrak{E}_{21}$, using the fact $U_{11} \diamond W_{21} \diamond I = U_{12} \diamond W_{21} \diamond I = 0$ and Lemma 1, we have

$$\begin{gathered}
\zeta_n((U_{11} + U_{12} + U_{21}) \diamond W_{21} \diamond I) = \zeta_n(U_{11} \diamond W_{21} \diamond I) + \zeta_n(U_{12} \diamond W_{21} \diamond I) + \\
+ \zeta_n(U_{21} \diamond W_{21} \diamond I) = \zeta_n(U_{11}) \diamond W_{21} \diamond I + U_{11} \diamond \zeta_n(W_{21}) \diamond I + U_{11} \diamond W_{21} \diamond \zeta_n(I) + \\
+ \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} \zeta_p(U_{11}) \diamond \zeta_q(W_{21}) \diamond \zeta_r(I) + \zeta_n(U_{12}) \diamond W_{21} \diamond I + U_{12} \diamond \zeta_n(W_{21}) \diamond I + \\
+ U_{12} \diamond W_{21} \diamond \zeta_n(I) + \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} \zeta_p(U_{12}) \diamond \zeta_q(W_{21}) \diamond \zeta_r(I) + \\
+ \zeta_n(U_{21}) \diamond W_{21} \diamond I + U_{21} \diamond \zeta_n(W_{21}) \diamond I + U_{21} \diamond W_{21} \diamond \zeta_n(I) + \\
+ \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} \zeta_p(U_{21}) \diamond \zeta_q(W_{21}) \diamond \zeta_r(I) = (\zeta_n(U_{11}) + \zeta_n(U_{12}) + \\
+ \zeta_n(U_{21})) \diamond W_{21} \diamond I + (U_{11} + U_{12} + U_{21}) \diamond \zeta_n(W_{21}) \diamond I + (U_{11} + U_{12} + U_{21}) \diamond W_{21} \diamond \zeta_n(I) + \\
+ \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} (\zeta_p(U_{11}) + \zeta_p(U_{12}) + \zeta_p(U_{21})) \diamond \zeta_q(W_{21}) \diamond \zeta_r(I).
\end{gathered}$$

On the other hand, by induction hypothesis, we have

$$\begin{gathered}
\zeta_n((U_{11} + U_{12} + U_{21}) \diamond W_{21} \diamond I) = \zeta_n(U_{11} + U_{12} + U_{21}) \diamond W_{21} \diamond I + \\
+ (U_{11} + U_{12} + U_{21}) \diamond \zeta_n(W_{21}) \diamond I + (U_{11} + U_{12} + U_{21}) \diamond W_{21} \diamond \zeta_n(I) + \\
+ \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} \zeta_p(U_{11} + U_{12} + U_{21}) \diamond \zeta_q(W_{21}) \diamond \zeta_r(I) = \zeta_n(U_{11} + U_{12} + U_{21}) \diamond W_{21} \diamond I + \\
+ (U_{11} + U_{12} + U_{21}) \diamond \zeta_n(W_{21}) \diamond I + (U_{11} + U_{12} + U_{21}) \diamond W_{21} \diamond \zeta_n(I) + \\
+ \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} (\zeta_p(U_{11}) + \zeta_p(U_{12}) + \zeta_p(U_{21})) \diamond \zeta_q(W_{21}) \diamond \zeta_r(I).
\end{gathered}$$

The above two relations give that $T \diamond W_{21} \diamond I = 0$, which implies that

$$W_{21}^* T - T^* W_{21} = 0. \tag{12}$$

Following the same procedure as above and replacing W_{21} by iW_{21}, we obtain

$$W_{21}^* T + T^* W_{21} = 0. \tag{13}$$

From (12) and (13) and using the primeness of $\mathfrak{E}$, we get $T_{21} = T_{22} = 0$.
In view of $U_{21} \diamond P_1 \diamond I = 0$. It follow the Lemmas 1 and 4(i), we obtain

$$\begin{gathered}
\zeta_n((U_{11} + U_{12} + U_{21}) \diamond P_1 \diamond I) = \zeta_n((U_{11} + U_{12}) \diamond W_{12} \diamond I) + \zeta_n(U_{21} \diamond P_1 \diamond I) = \\
= \zeta_n(U_{11} + U_{12}) \diamond P_1 \diamond I + (U_{11} + U_{12}) \diamond \zeta_n(P_1) \diamond I + (U_{11} + U_{12}) \diamond P_1 \diamond \zeta_n(I) + \\
+ \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} \zeta_p(U_{11} + U_{12}) \diamond \zeta_q(P_1) \diamond \zeta_r(I) + \zeta_n(U_{21}) \diamond P_1 \diamond I + U_{21} \diamond \zeta_n(P_1) \diamond I + \\
+ U_{21} \diamond P_1 \diamond \zeta_n(I) + \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} \zeta_p(U_{21}) \diamond \zeta_q(P_1) \diamond \zeta_r(I) =
\end{gathered}$$

$$= (\zeta_n(U_{11}) + \zeta_n(U_{12}) + \zeta_n(U_{21})) \diamond P_1 \diamond I + (U_{11} + U_{12} + U_{21}) \diamond \zeta_n(P_1) \diamond I +$$
$$+ (U_{11} + U_{12} + U_{21}) \diamond P_1 \diamond \zeta_n(I) +$$
$$+ \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} (\zeta_p(U_{11}) + \zeta_p(U_{12}) + \zeta_p(U_{21})) \diamond \zeta_q(P_1) \diamond \zeta_r(I).$$

Alternatively, based on the induction hypothesis, we can conclude that

$$\zeta_n((U_{11} + U_{12} + U_{21}) \diamond P_1 \diamond I) = \zeta_n(U_{11} + U_{12} + U_{21}) \diamond P_1 \diamond I + (U_{11} +$$
$$+ U_{12} + U_{21}) \diamond \zeta_n(P_1) \diamond I + (U_{11} + U_{12} + U_{21}) \diamond P_1 \diamond \zeta_n(I) +$$
$$+ \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} \zeta_p(U_{11} + U_{12} + U_{21}) \diamond \zeta_q(P_1) \diamond \zeta_r(I) = \zeta_n(U_{11} + U_{12} + U_{21}) \diamond P_1 \diamond I +$$
$$+ (U_{11} + U_{12} + U_{21}) \diamond \zeta_n(P_1) \diamond I + (U_{11} + U_{12} + U_{21}) \diamond P_1 \diamond \zeta_n(I) +$$
$$+ \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} (\zeta_p(U_{11}) + \zeta_p(U_{12}) + \zeta_p(U_{21})) \diamond \zeta_q(P_1) \diamond \zeta_r(I).$$

Comparing the above two relations, we get $T \diamond P_1 \diamond I = 0$. This implies that $T_{12} = 0$ and

$$T_{11} - T_{11}^* = 0. \tag{14}$$

By applying the same procedure for iP_1 instead of P_1 in above two relations, we obtain

$$T_{11} + T_{11}^* = 0. \tag{15}$$

Adding (14) and (15), we get $T_{11} = 0$. Thus $T = 0$.
Similarly, one can easily obtain $\zeta_n(U_{12} + U_{21} + U_{22}) = \zeta_n(U_{12}) + \zeta_n(U_{21}) + \zeta_n(U_{22})$. □

Lemma 6. *For any $U_{11} \in \mathfrak{E}_{11}, U_{12} \in \mathfrak{E}_{12}, U_{21} \in \mathfrak{E}_{21}$ and $U_{22} \in \mathfrak{E}_{22}$, we have*

$$\zeta_n(U_{11} + U_{12} + U_{21} + U_{22}) = \zeta_n(U_{11}) + \zeta_n(U_{12}) + \zeta_n(U_{21}) + \zeta_n(U_{22}).$$

Proof. By Claim 7 in [11], the result holds true for $n = 1$. Assume that the result holds for $k < n$, i.e., $\zeta_k(U_{11} + U_{12} + U_{21} + U_{22}) = \zeta_k(U_{11}) + \zeta_k(U_{12}) + \zeta_k(U_{21}) + \zeta_k(U_{22})$.
Let $T = \zeta_n(U_{11} + U_{12} + U_{21} + U_{22}) - \zeta_n(U_{11}) - \zeta_n(U_{12}) - \zeta_n(U_{21}) - \zeta_n(U_{22})$.
We now show that $T = 0$. For any $W_{12} \in \mathfrak{E}_{12}$, using Lemmas 1 and 4($i$) and the fact $U_{21} \diamond W_{12} \diamond I = U_{22} \diamond W_{12} \diamond I = 0$, we have

$$\zeta_n((U_{11} + U_{12} + U_{21} + U_{22}) \diamond W_{12} \diamond I) = \zeta_n((U_{11} + U_{12}) \diamond W_{12} \diamond I) + \zeta_n(U_{21} \diamond W_{12} \diamond I) +$$
$$+ \zeta_n(U_{22} \diamond W_{12} \diamond I) = \zeta_n(U_{11} + U_{12}) \diamond W_{12} \diamond I + (U_{11} + U_{12}) \diamond \zeta_n(W_{12}) \diamond I +$$
$$+ (U_{11} + U_{12}) \diamond W_{12} \diamond \zeta_n(I) + \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} \zeta_p(U_{11} + U_{12}) \diamond \zeta_q(W_{12}) \diamond \zeta_r(I) +$$
$$+ \zeta_n(U_{21}) \diamond W_{12} \diamond I + U_{21} \diamond \zeta_n(W_{12}) \diamond I + U_{21} \diamond W_{12} \diamond \zeta_n(I) +$$
$$+ \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} \zeta_p(U_{21}) \diamond \zeta_q(W_{12}) \diamond \zeta_r(I) + \zeta_n(U_{22}) \diamond W_{12} \diamond I + U_{22} \diamond \zeta_n(W_{12}) \diamond I +$$
$$+ U_{22} \diamond W_{12} \diamond \zeta_n(I) + \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} \zeta_p(U_{22}) \diamond \zeta_q(W_{12}) \diamond \zeta_r(I) =$$
$$= (\zeta_n(U_{11}) + \zeta_n(U_{12}) + \zeta_n(U_{21}) + \zeta_n(U_{22})) \diamond W_{12} \diamond I +$$
$$+ (U_{11} + U_{12} + U_{21} + U_{22}) \diamond \zeta_n(W_{12}) \diamond I + (U_{11} + U_{12} + U_{21} + U_{22}) \diamond W_{12} \diamond \zeta_n(I) +$$
$$+ \sum_{\substack{p+q+r=n \\ 0 \le p,q,r \le n-1}} (\zeta_p(U_{11}) + \zeta_p(U_{12}) + \zeta_p(U_{21}) + \zeta_p(U_{22})) \diamond \zeta_q(W_{12}) \diamond \zeta_r(I).$$

Alternatively, based on the induction hypothesis, we can conclude that

$$\zeta_n((U_{11} + U_{12} + U_{21} + U_{22}) \diamond W_{12} \diamond I) = \zeta_n(U_{11} + U_{12} + U_{21} + U_{22}) \diamond W_{12} \diamond I +$$
$$+ (U_{11} + U_{12} + U_{21} + U_{22}) \diamond \zeta_n(W_{12}) \diamond I + (U_{11} + U_{12} + U_{21} + U_{22}) \diamond W_{12} \diamond \zeta_n(I) +$$

$$+\sum_{\substack{p+q+r=n\\0\le p,q,r\le n-1}}\zeta_p(U_{11}+U_{12}+U_{21}+U_{22})\diamond\zeta_q(W_{12})\diamond\zeta_r(I)=$$
$$=\zeta_n(U_{11}+U_{12}+U_{21}+U_{22})\diamond W_{12}\diamond I+(U_{11}+U_{12}+U_{21}+U_{22})\diamond\zeta_n(W_{12})\diamond I+$$
$$+(U_{11}+U_{12}+U_{21}+U_{22})\diamond W_{12}\diamond\zeta_n(I)+$$
$$+\sum_{\substack{p+q+r=n\\0\le p,q,r\le n-1}}(\zeta_p(U_{11})+\zeta_p(U_{12})+\zeta_p(U_{21})+\zeta_p(U_{22}))\diamond\zeta_q(W_{12})\diamond\zeta_r(I).$$

Comparing the above two equations, we arrive at $T\diamond W_{12}\diamond I=0$, which gives

$$W_{12}^*T-T^*W_{12}=0. \tag{16}$$

Following the same procedure as above and replacing W_{12} by iW_{12}, we obtain

$$W_{12}^*T+T^*W_{12}=0. \tag{17}$$

From (16) and (17) and using the primeness of $\mathfrak{E}$, we get $T_{11}=T_{12}=0$.

By substituting W_{21} for W_{12} in the latest calculation, we arrive at the result $T_{21}=T_{22}=0$. Thus $T=0$. □

Lemma 7. *For each $U_{ij},V_{ij}\in\mathfrak{E}_{ij}$ such that $i\neq j$, we have*

$$\zeta_n(U_{ij}+V_{ij})=\zeta_n(U_{ij})+\zeta_n(V_{ij}).$$

Proof. It easy to show that $(P_i+U_{ij}^*)\diamond(P_j+V_{ij})\diamond\frac{I}{2}=-U_{ij}-V_{ij}+U_{ij}^*+V_{ij}^*$. It follows from Lemmas 2 and 6, that

$$\zeta_n(-U_{ij}-V_{ij})+\zeta_n(U_{ij}^*+V_{ij}^*)=\zeta_n\big((P_i+U_{ij}^*)\diamond(P_j+V_{ij})\diamond\frac{I}{2}\big)=$$
$$=\zeta_n(P_i+U_{ij}^*)\diamond(P_j+V_{ij})\diamond\frac{I}{2}+(P_i+U_{ij}^*)\diamond\zeta_n(P_j+V_{ij})\diamond\frac{I}{2}+$$
$$+(P_i+U_{ij}^*)\diamond(P_j+V_{ij})\diamond\zeta_n\Big(\frac{I}{2}\Big)+\sum_{\substack{p+q+r=n\\0\le p,q,r\le n-1}}\zeta_p(P_i+U_{ij}^*)\diamond\zeta_q(P_j+V_{ij})\diamond\zeta_r\Big(\frac{I}{2}\Big)=$$
$$=(\zeta_n(P_i)+\zeta_n(U_{ij}^*))\diamond(P_j+V_{ij})\diamond\frac{I}{2}+(P_i+U_{ij}^*)\diamond(\zeta_n(P_j)+\zeta_n(V_{ij}))\diamond\frac{I}{2}+$$
$$+\sum_{\substack{p+q+r=n\\0\le p,q,r\le n-1}}(\zeta_p(P_i)+\zeta_p(U_{ij}^*))\diamond(\zeta_q(P_j)+\zeta_q(V_{ij}))\diamond\zeta_r\Big(\frac{I}{2}\Big)=$$
$$=\zeta_n\big(P_i\diamond P_j\diamond\frac{I}{2}\big)+\zeta_n\big(P_i\diamond V_{ij}\diamond\frac{I}{2}\big)+\zeta_n\big(U_{ij}^*\diamond P_j\diamond\frac{I}{2}\big)+\zeta_n\Big(U_{ij}^*\diamond V_{ij}\diamond\frac{I}{2}\Big)=$$
$$=\zeta_n(-V_{ij})+\zeta_n(V_{ij}^*)+\zeta_n(-U_{ij})+\zeta_n(U_{ij}^*).$$

This implies that

$$\zeta_n(-U_{ij}-V_{ij})+\zeta_n(U_{ij}^*+V_{ij}^*)=\zeta_n(-U_{ij})+\zeta_n(-V_{ij})+\zeta_n(U_{ij}^*)+\zeta_n(V_{ij}^*).$$

By applying Lemma 3, the above equation yields

$$-\zeta_n(U_{ij}+V_{ij})+\zeta_n(U_{ij}^*+V_{ij}^*)=-\zeta_n(U_{ij})-\zeta_n(V_{ij})+\zeta_n(U_{ij}^*)+\zeta_n(V_{ij}^*) \tag{18}$$

A straightforward initial calculation shows that

$$(P_i+U_{ij}^*)\diamond(iP_j+iV_{ij})\diamond\frac{-I}{2}=iU_{ij}+iV_{ij}+iU_{ij}^*+iV_{ij}^*.$$

Therefore, we obtain

$$\zeta_n(iU_{ij}+iV_{ij})+\zeta_n(iU_{ij}^*+iV_{ij}^*)=\zeta_n\big((P_i+U_{ij}^*)\diamond(iP_j+iV_{ij})\diamond\frac{-I}{2}\big)=$$
$$=\zeta_n(P_i+U_{ij}^*)\diamond(iP_j+iV_{ij})\diamond\frac{-I}{2}+(P_i+U_{ij}^*)\diamond\zeta_n(iP_j+iV_{ij})\diamond\frac{-I}{2}+$$
$$+(P_i+U_{ij}^*)\diamond(iP_j+iV_{ij})\diamond\zeta_n\Big(\frac{-I}{2}\Big)+\sum_{\substack{p+q+r=n\\0\le p,q,r\le n-1}}\zeta_p(P_i+U_{ij}^*)\diamond\zeta_q(iP_j+iV_{ij})\diamond\zeta_r\Big(\frac{-I}{2}\Big)$$

$$= (\zeta_n(P_i) + \zeta_n(U_{ij}^*)) \diamond (iP_j + iV_{ij}) \diamond \frac{-I}{2} + (P_i + U_{ij}^*) \diamond (\zeta_n(iP_j) + \zeta_n(iV_{ij})) \diamond \frac{-I}{2} +$$
$$+ \sum\nolimits_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} (\zeta_p(P_i) + \zeta_p(U_{ij}^*)) \diamond (\zeta_q(iP_j) + \zeta_q(iV_{ij})) \diamond \zeta_r\Big(\frac{-I}{2}\Big) =$$
$$= \zeta_n\big(P_i \diamond iP_j \diamond \frac{-I}{2}\big) + \zeta_n\big(P_i \diamond iV_{ij} \diamond \frac{-I}{2}\big) + \zeta_n\big(U_{ij}^* \diamond iP_j \diamond \frac{-I}{2}\big) + \zeta_n\Big(U_{ij}^* \diamond iV_{ij} \diamond \frac{-I}{2}\Big) =$$
$$= \zeta_n(iV_{ij}) + \zeta_n(iV_{ij}^*) + \zeta_n(iU_{ij}) + \zeta_n(iU_{ij}^*).$$

Therefore, we establish the following equality

$$\zeta_n(iU_{ij} + iV_{ij}) + \zeta_n(iU_{ij}^* + iV_{ij}^*) = \zeta_n(iU_{ij}) + \zeta_n(iV_{ij}) + \zeta_n(iU_{ij}^*) + \zeta_n(iV_{ij}^*).$$

Using Lemma 3 in above equation yields

$$\zeta_n(U_{ij} + V_{ij}) + \zeta_n(U_{ij}^* + V_{ij}^*) = \zeta_n(U_{ij}) + \zeta_n(V_{ij}) + \zeta_n(U_{ij}^*) + \zeta_n(V_{ij}^*) \tag{19}$$

By collecting (18) and (19), we obtain $\zeta_n(U_{ij} + V_{ij}) = \zeta_n(U_{ij}) + \zeta_n(V_{ij})$. □

Lemma 8. *For each $U_{ii}, V_{ii} \in \mathfrak{E}_{ii}$ such that $1 \le i \le 2$, we have*

$$\zeta_n(U_{ii} + V_{ii}) = \zeta_n(U_{ii}) + \zeta_n(V_{ii}).$$

Proof. According to Claim 9 in [11], the statement is valid for $n = 1$. Suppose that for all $k < n$, the statement holds, that is,

$$\zeta_k(U_{ii} + V_{ii}) = \zeta_k(U_{ii}) + \zeta_k(V_{ii}).$$

Let $T = \zeta_n(U_{ii} + V_{ii}) - \zeta_n(U_{ii}) - \zeta_n(V_{ii})$. We now demonstrate that $T = 0$.
Note that for $i \ne j$, $U_{ii} \diamond P_j \diamond I = V_{ii} \diamond P_j \diamond I = 0$. It follows from Lemma 1, that

$$\zeta_n((U_{ii} + V_{ii}) \diamond P_j \diamond I) = \zeta_n(U_{ii} \diamond P_j \diamond I) + \zeta_n(V_{ii} \diamond P_j \diamond I) = \zeta_n(U_{ii}) \diamond P_j \diamond I +$$
$$+ U_{ii} \diamond \zeta_n(P_j) \diamond I + U_{ii} \diamond P_j \diamond \zeta_n(I) + \sum\nolimits_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \zeta_p(U_{ii}) \diamond \zeta_q(P_j) \diamond \zeta_r(I) +$$
$$+ \zeta_n(V_{ii}) \diamond P_j \diamond I + V_{ii} \diamond \zeta_n(P_j) \diamond I + V_{ii} \diamond P_j \diamond \zeta_n(I) + \sum\nolimits_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \zeta_p(V_{ii}) \diamond \zeta_q(P_j) \diamond \zeta_r(I) =$$
$$= (\zeta_n(U_{ii}) + \zeta_n(V_{ii})) \diamond P_j \diamond I + (U_{ii} + V_{ii}) \diamond \zeta_n(P_j) \diamond I + (U_{ii} + V_{ii}) \diamond P_j \diamond \zeta_n(I) +$$
$$+ \sum\nolimits_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} (\zeta_p(U_{ii}) + \zeta_p(V_{ii})) \diamond \zeta_q(P_j) \diamond \zeta_r(I).$$

On the other hand, by the induction hypothesis, we have

$$\zeta_n((U_{ii} + V_{ii}) \diamond P_j \diamond I) = \zeta_n(U_{ii} + V_{ii}) \diamond P_j \diamond I + (U_{ii} + V_{ii}) \diamond \zeta_n(P_j) \diamond I +$$
$$+ (U_{ii} + V_{ii}) \diamond P_j \diamond \zeta_n(I) + \sum\nolimits_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \zeta_p(U_{ii} + V_{ii}) \diamond \zeta_q(P_j) \diamond \zeta_r(I) =$$
$$= \zeta_n(U_{ii} + V_{ii}) \diamond P_j \diamond I + (U_{ii} + V_{ii}) \diamond \zeta_n(P_j) I + (U_{ii} + V_{ii}) \diamond P_j \diamond \zeta_n(I) +$$
$$+ \sum\nolimits_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} (\zeta_p(U_{ii}) + \zeta_p(V_{ii})) \diamond \zeta_q(P_j) \diamond \zeta_r(I).$$

Comparing the above two equations, we get $T \diamond P_j \diamond I = 0$. This implies that $T_{ji} = 0$ and

$$T_{jj} - T_{jj}^* = 0. \tag{20}$$

By applying the same procedure for iP_j instead of P_j in above two relations, we obtain

$$T_{jj} + T_{jj}^* = 0. \tag{21}$$

Adding (20) and (21), we get $T_{jj} = 0$.
For any $W_{ij} \in \mathfrak{E}_{ij}$ and using Lemmas 6 and 7, we obtain

$$\zeta_n((U_{ii} + V_{ii}) \diamond W_{ij} \diamond I) = \zeta_n(U_{ii} \diamond W_{ij} \diamond I) + \zeta_n(V_{ii} \diamond W_{ij} \diamond I) = \zeta_n(U_{ii}) \diamond W_{ij} \diamond I +$$
$$+ U_{ii} \diamond \zeta_n(W_{ij}) \diamond I + U_{ii} \diamond W_{ij} \diamond \zeta_n(I) + \sum\nolimits_{\substack{p+q+r=n \\ 0\le p,q,r\le n-1}} \zeta_p(U_{ii}) \diamond \zeta_q(W_{ij}) \diamond \zeta_r(I) +$$

$$+\zeta_n(V_{ii})\diamond W_{ij}\diamond I+V_{ii}\diamond\zeta_n(W_{ij})\diamond I+V_{ii}\diamond W_{ij}\diamond\zeta_n(I)+\sum_{\substack{p+q+r=n\\0\le p,q,r\le n-1}}\zeta_p(V_{ii})\diamond\zeta_q(W_{ij})\diamond\zeta_r(I)=$$
$$=(\zeta_n(U_{ii})+\zeta_n(V_{ii}))\diamond W_{ij}\diamond I+(U_{ii}+V_{ii})\diamond\zeta_n(W_{ij})\diamond I+(U_{ii}+V_{ii})\diamond W_{ij}\diamond\zeta_n(I)+$$
$$+\sum_{\substack{p+q+r=n\\0\le p,q,r\le n-1}}(\zeta_p(U_{ii})+\zeta_p(V_{ii}))\diamond\zeta_q(W_{ij})\diamond\zeta_r(I).$$

On the other hand, by the induction hypothesis, we obtain

$$\zeta_n((U_{ii}+V_{ii})\diamond W_{ij}\diamond I)=$$
$$=\zeta_n(U_{ii}+V_{ii})\diamond W_{ij}\diamond I+(U_{ii}+V_{ii})\diamond\zeta_n(W_{ij})\diamond I+(U_{ii}+V_{ii})\diamond W_{ij}\diamond\zeta_n(I)+$$
$$+\sum_{\substack{p+q+r=n\\0\le p,q,r\le n-1}}\zeta_p(U_{ii}+V_{ii})\diamond\zeta_q(W_{ij})\diamond\zeta_r(I)=\zeta_n(U_{ii}+V_{ii})\diamond W_{ij}\diamond I+(U_{ii}+V_{ii})\diamond\zeta_n(W_{ij})\diamond I+$$
$$+(U_{ii}+V_{ii})\diamond W_{ij}\diamond\zeta_n(I)+\sum_{\substack{p+q+r=n\\0\le p,q,r\le n-1}}(\zeta_p(U_{ii})+\zeta_p(V_{ii}))\diamond\zeta_q(W_{ij})\diamond\zeta_r(I).$$

Comparing the above two relations, we get $T\diamond W_{ij}\diamond I=0$. This yields that

$$W_{ij}^*T-T^*W_{ij}=0. \tag{22}$$

Following the same procedure as above and replacing W_{ij} by iW_{ij}, we obtain

$$W_{ij}^*T+T^*W_{ij}=0. \tag{23}$$

From (22) and (23) and using the primeness of $\mathfrak{E}$, we get $T_{ii}=T_{ij}=0$. Hence $T=0$. □

In view of Lemmas 6–8, we can easily prove the following lemma.

Lemma 9. *ζ_n is additive on $\mathfrak{E}$.*

Lemma 10. *For all $U\in\mathfrak{E}$, we have $\zeta_n(U^*)=\zeta_n(U)^*$.*

Proof. By applying Lemma 2, we have $\zeta_n\Big(U\diamond\frac{I}{2}\diamond\frac{I}{2}\Big)=\zeta_n(U)\diamond\frac{I}{2}\diamond\frac{I}{2}$. This implies that $\zeta_n(U-U^*)=\zeta_n(U)-\zeta_n(U)^*$. Hence $\zeta_n(U^*)=\zeta_n(U)^*$. □

Lemma 11. *ζ_n is an additive $*$- higher derivation on $\mathfrak{E}$.*

Proof. Based on the fact that $U^*\diamond V\diamond\frac{I}{2}=-UV+V^*U^*$ for any $U,V\in\mathfrak{E}$ and using Lemmas 2 and 10, we have

$$\zeta_n(-UV+V^*U^*)=\zeta_n\big(U^*\diamond V\diamond\tfrac{I}{2}\big)=\zeta_n(U^*)\diamond V\diamond\tfrac{I}{2}+U^*\diamond\zeta_n(V)\diamond\tfrac{I}{2}+$$
$$+\sum_{\substack{p+q+r=n\\0\le p,q,r\le n-1}}\zeta_p(U^*)\diamond\zeta_q(V)\diamond\zeta_r\Big(\tfrac{I}{2}\Big).$$

This implies that

$$\zeta_n(-UV+V^*U^*)=V^*\zeta_n(U)^*+\zeta_n(V)^*U^*-\zeta_n(U)V-U\zeta_n(V)+$$
$$+\sum_{\substack{p+q+r=n\\0\le p,q,r\le n-1}}\zeta_p(U)^*\diamond\zeta_q(V)\diamond\zeta_r\Big(\frac{I}{2}\Big)=V^*\zeta_n(U)^*+\zeta_n(V)^*U^*-\zeta_n(U)V-U\zeta_n(V)+$$
$$+\sum_{\substack{p+q=n\\0\le p,q\le n-1}}\zeta_p(U)^*\diamond\zeta_q(V)\diamond\frac{I}{2}=V^*\zeta_n(U)^*+\zeta_n(V)^*U^*-\zeta_n(U)V-U\zeta_n(V)+$$
$$+\sum_{\substack{p+q=n\\0\le p,q\le n-1}}(\zeta_q(V)^*\zeta_p(U)^*-\zeta_p(U)\zeta_q(V)). \tag{24}$$

Replacing U by iU in above equation, and using the Lemmas 3 and 10, we obtain

$$i\zeta_n(-UV-V^*U^*)=-iV^*\zeta_n(U)^*-i\zeta_n(V)^*U^*-i\zeta_n(U)V-iU\zeta_n(V)+$$
$$+i\sum_{\substack{p+q=n\\0\le p,q\le n-1}}(-\zeta_q(V)^*\zeta_p(U)^*-\zeta_p(U)\zeta_q(V)).$$

This yields that

$$\zeta_n(-UV-V^*U^*)=-V^*\zeta_n(U)^*-\zeta_n(V)^*U^*-\zeta_n(U)V-U\zeta_n(V)+$$

$$+\sum_{\substack{p+q=n\\0\leq p,q\leq n-1}}(-\zeta_q(V)^*\zeta_p(U)^*-\zeta_p(U)\zeta_q(V)). \qquad (25)$$

Adding (24) and (25), we have

$$\zeta_n(UV)=\zeta_n(U)V+U\zeta_n(V)+\sum_{\substack{p+q=n\\0\leq p,q\leq n-1}}\zeta_p(U)\zeta_q(V)=\sum_{\substack{p+q=n\\0\leq p,q\leq n}}\zeta_p(U)\zeta_q(V).$$

□

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Department of Mathematics, Faculty of Science, Aligarh Muslim University
Aligarh, India

asma_ali2@rediffmail.com
shakivaliamu@gmail.com
tasleemh59@gmail.com

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