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LINEAR GEOMETRY: COMPLETIONS AND PROJECTIVIZATIONS

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Linear Geometry describes geometric properties that depend on the fundamental notion of a line. In this paper we survey basic notions and results related to completions and free projectivizations of liners. The paper is the second survey in the series of surveys that describe the contents of the monograph "Linear Geometry and Algebra" [1]. The first survey [2] concentrated at properties of liners that depend on flat hulls (flats, ranks, regularity, parallelity). In this paper we survey basic notions and results related to completions and free projectivizations of liners. The material covers Chapters 7, 8, 9 of the book [1].

Introduction

This paper is the second survey in the series of surveys that describe the contents of the monograph "Linear Geometry and Algebra" [1]. The first survey [2] concentrated at properties of liners that depend on flat hulls (flats, ranks, regularity, parallelity). In this survey we consider completions and projectivizations of liners. The material covers Chapters 7, 8, 9 of the book [1].

1 Preliminaries

In this section we recall the basic notions of Linear Geometry, needed for understanding the main results of this survey.

A *liner* is a set X of *points*, endowed with a family $\mathcal{L}$ of subsets of X called *lines* such that the following two axioms are satisfied:

- any two distinct points belong to a unique line;
- any line contains at least two distinct points.

A liner X is *κ-long* for a cardinal κ if every line L in X has cardinality $|L| \ge \kappa$.

Let $(X, \mathcal{L})$ be a liner. For two distinct points $x, y \in X$, we denote by $\overline{x\,y}$ the unique line containing those points. If $x = y$, then we put $\overline{x\,y} := \{x\} = \{y\}$.

A subset $A \subseteq X$ is called *flat* if $\overline{x\,y} \subseteq A$ for any points $x, y \in A$. For a subset $A \subseteq X$ its *flat hull* $\overline{A}$ is the smallest flat set in X that contains the set A. The *rank* $\|A\|$ of a set $A \subseteq X$ is the smallest cardinality of a subset $B \subseteq X$ such that $A \subseteq \overline{B}$. Flat subsets of rank 3 in a liner are called *planes*. Two lines in a liner are *coplanar* if they are contained in some plane. Two lines are *concurrent* if their intersection is a singleton. A proper flat H in a liner X is called a *hyperplane* if every flat B in X with $H \subseteq B \subseteq X$ is equal to H or X.

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A liner X is *ranked* (resp. κ*-ranked* for a cardinal number κ) if for any distinct flats $A \subset B$ of finite rank (with $\|B\| \le \kappa$) we have $\|A\| < \|B\|$.

For two flats A, B in the liner X, we say that A is *subparallel* to B and write $A \,\|\, B$ if $A \subseteq \overline{B \cup \{a\}}$ for all $a \in A$. We say that flats $A, B \subseteq X$ are *parallel* and write $A \parallel B$ if $A \,\|\, B$ and $B \,\|\, A$. It is easy to see that two lines L, Λ in a 3-ranked liner are parallel if and only if either $L = \Lambda$ or $L \cap \Lambda = \varnothing$ and $\overline{L \cup \Lambda}$ is a plane.

A bijective map $F\colon X \to Y$ between two liners $(X, \mathcal{L}_X)$ and $(Y, \mathcal{L}_Y)$ is called a *liner isomorphism* if $\mathcal{L}_Y = \{F[L] \colon L \in \mathcal{L}_X\}$. Each subset A of a liner $(X, \mathcal{L})$ carries the line structure $\mathcal{L}_A := \{L \cap A \colon L \in \mathcal{L} \ \wedge\ |L \cap A| \ge 2\}$ turning A into a liner, called a *subliner* of the liner $(X, \mathcal{L})$.

Definition 1.1. A liner X is called

- *strongly regular* if for every nonempty flat $A \subseteq X$ and point $b \in X \setminus A$, we have $\overline{A \cup \{b\}} = \bigcup_{a \in A} \overline{a\,b}$;
- *regular* if for every flat $A \subseteq X$ and points $o \in A$, $b \in X \setminus A$ we have $\overline{A \cup \{b\}} = \bigcup_{a \in A} \bigcup_{y \in \overline{o\,b}} \overline{a\,y}$;
- *weakly regular* if for every flat $A \subseteq X$ and points $o \in A$, $b \in X \setminus A$ we have $\overline{A \cup \{b\}} = \bigcup_{a \in A} \overline{\{o, a, b\}}$.

Next, we recall some Parallelity Postulates and Axioms.

Definition 1.2. A liner X is defined to be

- *Proclus* if for every plane $P \subseteq X$, line $L \subseteq P$ and point $x \in P \setminus L$ there exists at most one line Λ in X such that $x \in \Lambda \subseteq P \setminus L$;
- *Bolyai* if for every plane $P \subseteq X$, line $L \subseteq P$ and point $x \in P \setminus L$ there exists a line Λ in X such that $x \in \Lambda \subseteq P \setminus L$;
- *Playfair* if for every plane $P \subseteq X$, line $L \subseteq P$ and point $x \in P \setminus L$ there exists a unique line Λ in X such that $x \in \Lambda \subseteq P \setminus L$;
- *para-Playfair* if for every plane $P \subseteq X$, disjoint lines $L, \Lambda \subseteq P$ and point $x \in P \setminus L$ there exists a unique line L_x in X such that $x \in L_x \subseteq P \setminus L$.

The Parallel Postulates are tightly related to the following Parallelity Axioms.

Definition 1.3. A liner X is defined to be

- *projective* if $\forall o, x, y \in X \ \ \forall p \in \overline{x\,y} \ \ \forall v \in \overline{o\,y} \setminus \{p\} \ \ (\overline{v\,p} \cap \overline{o\,x} \ne \varnothing)$;
- *proaffine* if $\forall o, x, y \in X \ \forall p \in \overline{x\,y} \setminus \overline{o\,x} \ \exists u \in \overline{o\,y} \ \forall v \in \overline{o\,y} \setminus \{u\} \ \ (\overline{v\,p} \cap \overline{o\,x} \ne \varnothing)$;
- *affine* if $\forall o, x, y \in X \ \forall p \in \overline{x\,y} \setminus \overline{o\,x} \ \exists u \in \overline{o\,y} \ \forall v \in \overline{o\,y} \ \ (u = v \ \Leftrightarrow\ \overline{v\,p} \cap \overline{o\,x} = \varnothing)$.

It is easy to show that for every liner the following implications hold:

$$\begin{array}{ccccccc}
\text{strongly regular} & \Longrightarrow & \text{para-Playfair} & \Longleftarrow & & & \text{Playfair} \\
\Updownarrow & & \Downarrow & & & & \Downarrow \\
\text{projective} & \Longrightarrow & \text{Proclus} & \Longrightarrow & \text{proaffine} & \Longleftarrow & \text{affine}
\end{array}$$

Theorem 1.4 ([1, 7.1.3])**.** *Every proaffine regular liner X of rank $\|X\| \ne 3$ is para-Playfair.*

On the other hand, we have

Example 1.5 ([1, 7.1.5])**.** There exists a countable ω-long Proclus plane, which is not para-Playfair.

2 Bolyai lines in liners

Definition 2.1. A line L in a liner X is defined to be *Bolyai* if for every $x \in X \setminus L$ there exists a line Λ in X such that $x \in \Lambda \subseteq \overline{L \cup \{x\}} \setminus L$.

It is easy to see that a line L in a liner X is Bolyai if and only if for every plane $P \subseteq X$ that contains the line L and every point $x \in P \setminus L$, there exists a line $\Lambda \subset P$ such that $x \in \Lambda$ and $\Lambda \cap L = \varnothing$.

Observe that a liner X is Bolyai if and only if every line in X is Bolyai. This fact implies the following characterization of Playfair liners.

Proposition 2.2 ([1, 7.2.3])**.** *For a liner X the following conditions are equivalent:* (1) *X is Playfair;* (2) *X is Proclus and Bolyai;* (3) *X is Proclus and every line in X is Bolyai.*

Proposition 2.3 ([1, 7.2.4])**.** *A line L in a para-Playfair regular liner X of rank $\|X\| \geq 3$ is Bolyai if and only if there exists a line L' such that $L \cap L' = \varnothing$ and the lines L, L' are coplanar.*

In fact, the regularity of the para-Playfair liner in Proposition 2.3 can be replaced by the 3-long property of the liner.

Proposition 2.4 ([1, 7.2.5])**.** *A line L in a 3-long para-Playfair liner X of rank $\|X\| \geq 3$ is Bolyai if and only if there exists a line L' such that $L \cap L' = \varnothing$ and the lines L, L' are coplanar.*

Definition 2.5. A liner X is called *bi-Bolyai* if for every concurrent Bolyai lines L, Λ in X, any line in the plane $\overline{L \cup \Lambda}$ is Bolyai in X.

Definitions 1.2 and 2.5 imply that for every liner the following implications hold:

$$\text{Playfair} \Rightarrow \text{Bolyai} \Rightarrow \text{bi-Bolyai}.$$

Theorem 2.6 ([1, 7.3.2])**.** *Every 4-long para-Playfair bi-Bolyai liner X is regular.*

Remark 2.7. By Theorem 10.5, every *finite* 4-long para-Playfair liner is bi-Bolyai and regular. On the other hand, there exists a ω-long para-Playfair liner which is not bi-Bolyai, see Example 8.4.3 in [1].

3 Spreading lines in liners

Definition 3.1. Let X be a liner and $\mathcal{L}$ be the family of lines in X. A subfamily $\mathcal{S} \subseteq \mathcal{L}$ is called a *spread of lines* in X if every point $x \in X$ belongs to a unique line $L \in \mathcal{S}$.

A line $L \in \mathcal{L}$ is called *spreading* if the family

$$L_{\parallel} := \{\Lambda \in \mathcal{L} \colon \Lambda \parallel L\}$$

is a spread of lines in X such that $L_{\parallel} = \Lambda_{\parallel}$ for every line $\Lambda \in L_{\parallel}$.

It can be shown that a family of lines $\mathcal{S}$ in a liner X is a spread of lines if and only if $\bigcup \mathcal{S} = X$ and any two distinct lines in $\mathcal{S}$ are disjoint.

Theorem 3.2 ([1, 7.4.3])**.** *For a line L in a proaffine regular liner X, the following conditions are equivalent:* (1) *L is spreading;* (2) *L is Bolyai;* (3) $\bigcup L_{\parallel} = X$.

Theorem 3.3 ([1, 7.4.4])**.** *A liner X is regular and Playfair if and only if X is 3-ranked and every line in X is speading.*

Remark 3.4. Let A, B be two spreading lines A, B in a liner X. If the spreads $A_\|$ and $B_\|$ contain some common line L, then $A_\| = L_\| = B_\|$. Therefore, for two spreading lines A, B, the spreads $A_\|$ and $B_\|$ either coincide or else are disjoint.

Proposition 3.5 ([1, 7.4.6]). *If L is a spreading line in a 3-ranked liner X, then every line $\Lambda \notin L_\|$ in X is 3-long.*

4 The spread completion of a liner

Let X be a liner, $\mathcal{L}$ be the family of all lines in X, and $\mathcal{S}$ be the family of all spreading lines in X. The set

$$\partial X := \{L_\| \colon L \in \mathcal{S}\}$$

of spreads of spreading lines in X is called *the boundary* of X. Since $\bigcup(\partial X) = \mathcal{S}$, the family $\mathcal{S}$ can be recovered from the boundary of X. Elements $L_\| \in \partial X$ of the boundary will be called *directions* in the liner X.

We claim that $X \cap \partial X = \varnothing$. Indeed, assuming that some point $x \in X$ is equal to the spread $L_\|$ for some speading line $L \in \mathcal{S}$, we can find a line L_x such that $x \in L_x \in L_\| = x$, which contradicts the Axiom of Foundation in Set Theory[1].

For a plane $\Pi \subseteq X$ its boundary $\partial\Pi$ in X is the set $\{L_\| \in \partial X \colon L \in \mathcal{L} \ \wedge\ L \subseteq \Pi\}$.

Attaching to the liner X its boundary ∂X, we obtain the set

$$\overline{X} := X \cup \partial X,$$

which is the underlying set of the spread completion of X. A subset $\Lambda \subseteq \overline{X}$ is defined to be a line in $\overline{X}$ if either $\Lambda = L \cup L_\|$ for some spreading line L in X or else $\Lambda = \partial\Pi$ for some plane $\Pi \subseteq X$ with $|\partial\Pi| \geq 2$. If the liner X is 3-ranked, then $\overline{X} = X \cup \partial X$ is a liner, called the *spread completion* of X.

Proposition 4.1 ([1, 7.5.4]). *For every isomorphism $A \colon X \to Y$ between 3-ranked liners X, Y, there exists a unique isomorphism $\bar{A} \colon \overline{X} \to \overline{Y}$ of the spead completions of the liners X, Y such that $A = \bar{A}{\restriction}_X$.*

Proposition 4.1 justifies the following definition.

Definition 4.2. For an isomorphism $A \colon X \to Y$ between 3-ranked liners, its *spread completion* is a unique isomorphism $\bar{A} \colon \overline{X} \to \overline{Y}$ such that $A = \bar{A}{\restriction}_X$.

5 Completely regular liners

Definition 5.1. A liner X is called *completely regular* if X is 3-ranked and its spread completion $\overline{X}$ is strongly regular (= projective).

Theorem 5.2 ([1, 7.6.2]). *A liner X is completely regular if and only if X is regular, para-Playfair, and bi-Bolyai.*

Therefore, for every liner we have the implications:

projective ⇔ strongly regular ⇒ completely regular ⇒ regular ⇒ weakly regular.

Theorems 5.2, 2.6 imply the following characterization of 4-long completely regular liners.

Corollary 5.3 ([1, 7.6.8]). *A 4-long liner is completely regular if and only if it is para-Playfair and bi-Bolyai.*

The Axiom of Foundation says that every nonempty set x contains an element $y \in x$ such that $y \cap x = \varnothing$.

Theorem 5.4 ([1, 7.6.9]). *Every proaffine regular liner X of rank $\|X\| \neq 3$ is completely regular.*

Corollary 5.5 ([1, 7.6.10]). *Every affine regular liner X is completely regular.*

Proposition 5.6 ([1, 7.6.11]). *If a liner X is completely regular and not projective, then the spread completion $\overline{X}$ of X is projective and 3-long.*

Proposition 5.6 implies the following corollary.

Corollary 5.7 ([1, 7.6.12]). *The spread completion of any 3-long completely regular liner X is projective and 3-long.*

6 Complete properties of liners

In this section we develop some terminology related to properties of liners. In the sequel we shall strive to define properties of liners which are complete in the following sense.

Definition 6.1. A property $\mathcal{P}$ of liners is called *complete* if a completely regular liner X has property $\mathcal{P}$ if and only if the spread completion $\overline{X}$ of X has property $\mathcal{P}$.

It can be shown that the properties of a liner to be completely regular, regular, proregular, weakly regular, ranked, Proclus, para-Playfair, proaffine, hyperbolic are complete. On the other hand, the properties of a liner to be strongly regular, modular, projective, affine, hyperaffine, injective, Playfair, Bolyai, Lobachevsky, Boolean are not complete.

There is a simple method of turning a property of projective liners to a complete property of arbitrary (not necessarily projective) liners.

Definition 6.2. Let $\mathcal{P}$ be a property of projective liners. A liner X is called *completely* $\mathcal{P}$ if X is completely regular and its spread completion $\overline{X}$ has property $\mathcal{P}$.

Proposition 6.3 ([1, 7.7.5]). *For every property $\mathcal{P}$ of projective liners, the property of liners to be completely $\mathcal{P}$ is complete.*

In the book [1] instances of the following general problem are often considered.

Problem 6.4. *Given a property $\mathcal{P}$ of projective liners, find an inner characterization of completely $\mathcal{P}$ liners.*

Remark 6.5. Such characterizations were found for the properties of projective liners to be Desarguesian, Pappian, Fano, Moufang, see Corollaries 10.9.7, 20.7.4, 40.3.6, 41.4.3 in [1].

Next, we consider two methods of turning a property of affine liners into a property of projective liners. By Proposition 9.12, for every hyperplane H in a projective liner Y, the subliner $X := Y \setminus H$ is affine and regular.

Definition 6.6. Let $\mathcal{P}$ be a property of affine liners. A projective liner X is defined to be

- *everywhere* $\mathcal{P}$ if for every hyperplane $H \subset X$, the affine liner $X \setminus H$ has property $\mathcal{P}$;
- *somewhere* $\mathcal{P}$ if for some hyperplane $H \subset X$, the affine liner $X \setminus H$ has property $\mathcal{P}$.

Definition 6.7. A property $\mathcal{P}$ of liners is called a *liner property* if for any isomorphic liners X, Y, the liner X has property $\mathcal{P}$ if and only if Y has property $\mathcal{P}$.

Theorem 6.8 ([1, 7.7.10]). *Let $\mathcal{P}$ be a complete liner property. For any 4-long projective liner X, the following conditions are equivalent:*

(1) *X has property $\mathcal{P}$;* (2) *X is everywhere $\mathcal{P}$;* (3) *X is somewhere $\mathcal{P}$.*

7 Projective and normal completions of liners

Definition 7.1. A *completion* of a liner X is a 3-long liner Y that contains X as a subliner such that $\overline{Y \setminus X} \neq Y$. If the liner Y is projective, then Y is called a *projective completion* of the liner X. If X is a line, then we shall additionally require that the remainder $Y \setminus X$ is not empty, in order to make the projective completion of a line unique.

Example 7.2. By Proposition 5.6, the spread completion $\overline{X}$ of any (completely regular) nonempty 3-long 3-ranked liner X is a (projective) completion of X.

Proposition 7.3 ([1, 8.1.5]). *Let Y be a completion of a liner X. For every line $L \subseteq Y$ with $L \not\subseteq \overline{Y \setminus X}$, the intersection $L \cap X$ is a line in the liner X.*

Example 7.4 ([1, 8.1.6]). Let Y be a Steiner projective plane, L be a line in Y and $X := Y \setminus L$. Then Y is a projective completion of X, and $P := Y$ is a plane such that $P \cap X = X$ is not a plane, because every line in X contains exactly two elements and so X has rank $\|X\| = |X| = 4 > 3$.

Proposition 7.3 and Example 7.4 suggest the following natural definition.

Definition 7.5. A completion Y of a liner X is called *normal* if for every plane $P \subseteq Y$ with $P \not\subseteq \overline{Y \setminus X}$, the intersection $P \cap X$ is a plane in X.

Proposition 7.6 ([1, 8.1.8]). *Any projective completion Y of any liner X of rank $\|X\| \leq 3$ is normal.*

Theorem 7.7 ([1, 8.1.10]). *Any projective completion Y of a 3-long liner X is normal.*

Proposition 7.8 ([1, 8.1.11]). *The spread completion $\overline{X}$ of any 3-ranked liner X is normal.*

Let us recall that a projective plane X is called *Steiner* if each line in X contains exactly three points. A Steiner projective plane contains exactly seven points. A Steiner projective plane contains 7 points, 7 lines and is unique up to an isomorphism.

Theorem 7.9 ([1, 8.1.12]). *If a Proclus plane X is not projective and not 3-long, then X has a projective completion and $|X| \leq 6$. More precisely, $X = P \setminus H$ for some Steiner projective plane P and some set $H \subseteq P$ of cardinality $|H| \in \{1, 2\}$. If X is regular or para-Playfair, then $|X| = 6$ and $|H| = 1$.*

Corollary 7.10 ([1, 8.1.13]). *Any projective completion Y of a non-projective Proclus plane X is normal.*

8 The uniqueness of projective completions

In this section we discuss the uniqueness of projective completions.

A function $F\colon X \to Y$ between the underlying sets of two liners $(X, \mathcal{L}_X)$ and $(Y, \mathcal{L}_Y)$ is called

- a *liner morphism* if $\mathcal{L}_X \subseteq \{F^{-1}[L] \colon L \in \mathcal{L}_Y\}$;
- a *liner embedding* if F is an injective map with $\mathcal{L}_X = \{F^{-1}[L] \colon L \in \mathcal{L}_Y\}$;
- a *liner isomorphism* if F and F^{-1} are liner embeddings.

Theorem 8.1 ([1, 8.2.1]). *Let Y be a normal completion of a 3-long liner X. For any projective completion Z of the liner X, there exists a unique injective liner morphism $F\colon Y \to Z$ such that $F(x) = x$ for all $x \in X$. If the liner Y is 3-ranked, then the map $F\colon Y \to Z$ is a liner embedding.*

Theorem 8.2 ([1, 8.2.9]). *For any projective completions Y, Z of a 3-long liner X, there exists a unique liner isomorphism $F\colon Y \to Z$ such that $F(x) = x$ for all $x \in X$.*

Theorems 5.2, 8.2 and Proposition 5.6 imply the following corollary.

Corollary 8.3 ([1, 8.2.10]). *Let X be a 3-long completely regular liner and $\overline{X}$ be the spread completion of X. For every projective completion Y of X, there exists a unique liner isomorphism $F\colon Y \to \overline{X}$ such that $F(x) = x$ for every $x \in X$.*

Theorem 8.4 ([1, 8.2.11]). *Let X, Y be two 3-long liners and $\widetilde{X}, \widetilde{Y}$ be projective completions of the liners X, Y, respectively. Every isomorphism $F\colon X \to Y$ uniquely extends to an isomorphism $\widetilde{F}\colon \widetilde{X} \to \widetilde{Y}$.*

9 Interplay between a liner and its horizon

For a 3-long liner X possessing a projective completion Y, the complement $Y \setminus X$ is called the *horizon* of X. Since a projective completion of X is unique (up to an isomorphism), the horizon of X is also uniquely determined. In this section we discuss the interplay betwen properties of a liner and its horizon.

We start with a duality between the proaffinity a liner and the proflat property of its horizon.

Definition 9.1. A subset A of a liner X is called *proflat* in X if $|\overline{x\,y} \setminus A| \le 1$ for every points $x, y \in A$. It is clear that every flat in a liner X is a proflat in X.

Example 9.2. For every flats $A \subseteq B$ in a liner X, the set $B \setminus A$ is proflat in X.

Example 9.3. For an increasing sequence $(A_n)_{n\in\mathbb{Z}}$ of flats in a liner X, the set $\bigcup_{n\in\omega} A_{2n+1} \setminus A_{2n}$ is proflat in X.

Proposition 9.4 ([1, 8.3.5]). *For every proflat set H in a projective liner Y, the subliner $X := Y \setminus H$ of Y is Proclus.*

Theorem 9.5 ([1, 8.3.6]). *For a projective completion Y of a liner X, the following conditions are equivalent: (1) the liner X is Proclus; (2) the liner X is proaffine;*
(3) the horizon $Y \setminus X$ of X is proflat in Y.

Theorem 9.6 ([1, 8.3.7]). *Let Y be a projective completion of a liner X. The liner X is para-Playfair if and only if the set $H := Y \setminus X$ is flat in Y.*

Proposition 9.7 ([1, 8.3.8]). *Let Y be a projective completion of a 3-long liner X. If the horizon $Y \setminus X$ of X is flat in Y, then: (1) for every set $A \subseteq X$ and its flat hull $\overline{A}$ in the liner Y, the intersection $\overline{A} \cap X$ coincides with the flat hull of A in the liner X; (2) the liner X is regular; (3) the liners X and Y have the same rank.*

Corollary 9.8 ([1, 8.3.9]). *The rank $\|X\|$ of a 3-long liner X coincides with the rank $\|Y\|$ of any projective completion Y of X.*

Example 9.9 ([1, 8.3.10]). *Every 2-element set H is a Steiner projective plane P is proflat in P but the liner $P \setminus H$ is not 3-regular.*

Proposition 9.10 ([1, 8.3.11]). *If a 3-long liner X of rank $\|X\| \le 3$ has a projective completion, then X is regular.*

Theorem 9.11 ([1, 8.3.12]). *Let Y be a projective completion of a 3-long proaffine liner X of rank $\|X\| \ge 4$. The liner X is regular if and only if X is weakly regular if and only if the horizon $Y \setminus X$ is flat in Y.*

Proposition 9.12 ([1, 8.3.13]). *For every hyperplane H in a projective liner Y, the subliner $X := Y \setminus H$ of X is affine and regular.*

Theorem 9.13 ([1, 8.3.14]). *Let Y be a projective completion of a liner X such that $|Y|_2 = 3$. The liner X is regular if and only if $|Y \setminus X| \le 1$ or $Y \setminus X$ is a hyperplane in Y.*

Theorem 9.14 ([1, 8.3.19]). *For a projective completion Y of a liner X, the following conditions are equivalent:* (1) *the liner X is affine and regular;* (2) *the liner X is affine;* (3) *$Y \setminus X$ is a hyperplane in Y.*

10 The (non)existence of projective completions

In this section we detect liners that have (or do not have) projective completions.

Theorem 10.1 ([1, 8.4.1]). *A non-projective 3-long 3-ranked liner X has a projective completion if and only if X is a normal completion of some 3-long affine regular liner $A \subseteq X$.*

Example 10.2 ([1, 8.4.3]). There exists an ω-long para-Playfair plane having no projective completions.

In contrast to Example 10.2, finite proaffine regular liners do have projective completions.

Theorem 10.3 ([1, 8.4.6]). *Every non-projective finite proaffine regular liner X has a projective completion.*

The following classification of finite Proclus planes is due to Kuiper and Dembowski [4].

Theorem 10.4 (Kuiper–Dembowski, 1962; [1, 8.5.1]). *Every finite Proclus plane X is equal to the complement $P \setminus H$ of a proflat set H of rank $\|H\| \le 2$ in some projective plane P. The proflat set H is one of the following:*

- *the empty set, in which case X is a projective plane;*
- *a singleton, in which case X is a punctured projective plane;*
- *a line, in which case X is an affine plane;*
- *a line with a removed point, in which case X is an affine liner with attached point at infinity.*

Finally, we present a characterization of completely regular (finite) liners, extending the characterization given in Theorem 5.2.

Theorem 10.5 ([1, 8.6.1]). *For a non-projective liner X, the following are equivalent:*

(1) *X is completely regular;*
(2) *X is a regular para-Playfair liner possessing a projective completion;*
(3) *X is regular and has a projective completion Y with flat horizon $Y \setminus X$;*

If the liner X is finite, then the conditions (1)–(3) *are equivalent to*

(4) *X is regular and para-Playfair.*

If the liner X is finite and 4-long, then the conditions (1)–(4) *are equivalent to*

(5) *X is para-Playfair.*

Remark 10.6. Example 4.1.11 [1] of a non-regular Hall liner shows that in Theorem 10.5 the condition (5) is not equivalent to the conditions (1)–(4) for 3-long Playfair liners. We recall [1, §4.4] that a liner X is *Hall* if it is Steiner and Playfair.

11 The free closure of a liner

In this section we consider a functorial embedding of a liner into its free closure, which is defined as follows.

Let $(X, \mathcal{L})$ be a liner, and $\mathcal{D} := \big\{\{A, B\}\colon A, B \in \mathcal{L} \;\wedge\; A \cap B = \varnothing\big\}$ be the family of pairs of disjoint lines in X. For any pair $\{A, B\} \in \mathcal{D}$, consider the "point" $p_{A,B} := \{X, A, B\}$ and observe that $p_{A,B} \notin X$. Indeed, assuming that $p_{A,B} \in X$, we obtain the contradiction $X \in \{X, A, B\} = p_{A,B} \in X$ with the Axiom of Foundation.

The *free closure* of the liner X is the set of points

$$\hat{X} := X \cup \{p_{A,B}\colon \{A, B\} \in \mathcal{D}\},$$

endowed with the family of lines

$$\mathcal{L}_{\hat{X}} := \{L \cup p_{L,\Lambda}\colon \Lambda \in \mathcal{L} \;\wedge\; \Lambda \cap L = \varnothing\} \cup \big\{\{x, y\}\colon x, y \in \hat{X} \setminus X \;\wedge\; x \neq y\big\}$$
$$\cup \big\{\{p_{A,B}, x\}\colon \{A, B\} \in \mathcal{D} \;\wedge\; x \in X \setminus (A \cup B)\big\}.$$

It is easy to see that the free closure of a liner X coincides with X if and only if X contains no disjoint lines if and only if X is either empty, a singleton, a line or a projective plane.

Definition 11.1. A subliner X of a liner Y is defined to be *preclosed in* Y if for any lines $A, B \subseteq X$ their flat hulls in Y have a common point $y \in \overline{A} \cap \overline{B} \subseteq Y$.

The definition of the free closure implies that every liner is preclosed in its free closure. Moreover, the free closures have the following universality property.

Proposition 11.2 ([1, 9.1.2])**.** *Let X be a liner, $Y := \hat{X}$ be its free closure and $Z := \hat{Y}$ be the free closure of Y. If $X \neq Y$, then:* (1) *$Y \neq Z$;* (2) *no line in X remains a line in Z.*

Proposition 11.3 ([1, 9.1.3])**.** *Let $\Phi\colon X \to Y$ be a liner embedding. If the set $\Phi[X]$ is preclosed in Y, then there exists a unique liner morphism $\hat{\Phi}\colon \hat{X} \to Y$ such that $\hat{\Phi}(x) = \Phi(x)$ for all $x \in X$.*

Proposition 11.3 implies the following corollary.

Corollary 11.4 ([1, 9.1.4])**.** *For every isomorphism $\Phi\colon X \to Y$ between liners, there exists a unique isomorphism $\hat{\Phi}\colon \hat{X} \to \hat{Y}$ such that $\hat{\Phi}{\restriction}_X = \Phi$.*

Definition 11.5. A liner X is defined to be 3-*wide* if for every point $x \in X$ there exist three distinct lines A, B, C in X such that $x \in A \cap B \cap C$ and $\min\{|A|, |B|, |C|\} \geq 3$.

Exercise 11.6. Show that every 3-long liner X of rank $\|X\| \geq 3$ is 3-wide.

Proposition 11.7 ([1, 9.1.7])**.** *Let X be a liner and $\hat{X}$ be its free closure. Any 3-wide subliner S of $\hat{X}$ is contained in X.*

Corollary 11.8 ([1, 9.1.8])**.** *For every isomorphism $\Phi\colon \hat{X} \to \hat{Y}$ between the free closures of 3-wide liners X, Y, the restriction $\Phi{\restriction}_X\colon X \to Y$ is a well-defined isomorphism between the liners X, Y.*

Corollary 11.4 implies that the construction of free closure is a functor in the category Liners of liners and their isomorphisms. This functor assigns to each liner X its free closure $\hat{X}$, and to each isomorphism $\Phi\colon X \to Y$ between liners the unique liner isomorphism $\hat{\Phi}\colon \hat{X} \to \hat{Y}$ such that $\hat{\Phi}{\restriction}_X = \Phi$. For a liner X, the monoid of morphisms from X to X in the category Liners coinsides with the automorphism group $\mathrm{Aut}(X)$ of X.

Corollaries 11.4 and 11.8 imply the following theorem.

Theorem 11.9 ([1, 9.1.9]). *For every* 3*-wide liner* X*, the map* $\mathrm{Aut}(X) \to \mathrm{Aut}(\hat{X})$*,* $\Phi \mapsto \hat{\Phi}$*, assigning to each automorphism* $\Phi\colon X \to X$ *its unique exension* $\hat{\Phi}\colon \hat{X} \to \hat{Y}$ *is a well-defined isomorphism between the automorphism groups* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(\hat{X})$*. Its inverse is the restriction operator* $\mathrm{Aut}(\hat{X}) \to \mathrm{Aut}(X)$*,* $\Phi \mapsto \Phi{\restriction}_X$*.*

A liner X is *rigid* if its automorphism group $\mathrm{Aut}(X)$ is trivial. Theorem 11.9 implies the following corollary.

Corollary 11.10 ([1, 9.1.10]). *A* 3*-wide liner is rigid iff its free closure is a rigid liner.*

12 The free projectivization of a liner

Given any liner X, consider the increasing sequence of liners $(X_n)_{n\in\omega}$ where $X_0 := X$ and for every $n \in \omega$, the liner X_{n+1} is the free closure of the liner X_n. For every $n \in \omega$, let $\mathcal{L}_n$ be the family of lines in the liner X_n. The *free projectivization* of the liner X is the liner $\widehat{X} := \bigcup_{n\in\omega} X_n$ endowed with the family of lines

$$\mathcal{L}_{\widehat{X}} := \Big\{ \bigcup_{n\in\omega} L_n \colon (L_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{L}_n \ \wedge\ \forall n \in \omega\ (L_n \subseteq L_{n+1}) \Big\}.$$

The free projectivization was introduced by Marshall Hall [5] in 1943.

Theorem 12.1 ([1, 9.2.1]). *Let* X *be a liner and let* $\widehat{X}$ *be its free projectivization.*

(1) *The liner* $\widehat{X}$ *has cardinality* $|\widehat{X}| = \max\{|X|, \omega\}$*.*
(2) *If the liner* X *contains no disjoint lines, then* $\widehat{X} = X = \hat{X}$*.*
(3) *If* X *contains disjoint lines, then its free projectivization* $\widehat{X}$ *is an* ω*-long projective plane.*
(4) *The liner* $\hat{X}$ *has rank* $\|\widehat{X}\| = \min\{\|X\|, 3\}$*.*
(5) *Every finite* 3*-wide subliner of* $\widehat{X}$ *is contained in* X*.*

Theorem 12.2 ([1, 9.2.2]). *For any isomorphism* $\Phi\colon X \to Y$ *between liners* X, Y*, there exists a unique isomorphism* $\widehat{\Phi}\colon \widehat{X} \to \widehat{Y}$ *such that* $\widehat{\Phi}{\restriction}_X = \Phi$*.*

Remark 12.3. The construction of free projectivization is a functor in the category of liners and their isomorphisms. This functor assigns to each liner X is free projectivization $\widehat{X}$ and to each liner isomorpism $\Phi\colon X \to Y$ the unique isomorphism $\widehat{\Phi}\colon \widehat{X} \to \widehat{Y}$ such that $\widehat{\Phi}{\restriction}_X = \Phi$. The isomorphism $\widehat{\Phi}$ is well-defined, by Theorem 12.2.

Theorem 12.4 ([1, 9.2.4]). *Let* X, Y *be liners and* $\Phi\colon \widehat{X} \to \widehat{Y}$ *be an isomorphism between their free projectivizations. If the liners* X, Y *are finite and* 3*-wide, then* $\Phi[X] = Y$ *and hence the restriction* $\Psi = \Phi{\restriction}_X$ *is an isomorphism of the liners* X, Y*. Moreover,* $\Phi = \widehat{\Psi}$*.*

Theorems 12.2 and 12.4 imply the following corollary.

Corollary 12.5 ([1, 9.2.5]). *Two finite* 3*-wide liners are isomorphic if and only if their free projectivizations are isomorphic.*

Corollaries 11.4 and 11.8 imply the following theorem.

Theorem 12.6 ([1, 9.2.6]). *For every finite* 3*-wide liner* X*, the map* $\mathrm{Aut}(X) \to \mathrm{Aut}(\widehat{X})$*,* $\Phi \mapsto \widehat{\Phi}$*, assigning to each automorphism* $\Phi\colon X \to X$ *its unique exension* $\widehat{\Phi}\colon \widehat{X} \to \widehat{Y}$ *is a well-defined isomorphism between the automorphism groups* $\mathrm{Aut}(X)$ *and* $\mathrm{Aut}(\widehat{X})$*. Its inverse is the restriction operator* $\mathrm{Aut}(\widehat{X}) \to \mathrm{Aut}(X)$*,* $\Phi \mapsto \Phi{\restriction}_X$*.*

Theorems 11.9 and 12.6 imply the following corollary.

Theorem 12.7 ([1, 9.2.7]). *For a finite* 3*-wide liner* X*, the following conditions are equivalent:* (1) *the liner* X *is rigid;* (2) *the free closure* $\hat{X}$ *of* X *is rigid;* (3) *the free projectivization* $\widehat{X}$ *of* X *is rigid.*

Example 12.8. There exists at least 28 rigid affine planes of order 25.

By Example 12.8, there exists a 3-wide rigid liner of cardinality 625.

Example 12.9. By [3, II.1.28], there exist exactly 80 nonisomorphic Steiner liners X of cardinality $|X| = 15$. All of them are 3-wide (because every point of a Steiner liner of cardinality 15 is contained in 7 lines). By [3, II.1.29], exactly 36 of those 80 Steiner liners are rigid.

Problem 12.10. *What is the smallest cardinality of a* 3*-wide rigid liner?*

Theorem 12.11 ([1, 9.2.11]). *There exists a rigid countable* ω*-long projective plane.*

Proof. By Example 12.8, there exists a rigid finite 3-wide liner X. By Theorems 12.7 and 12.1, its free projectivization $\widehat{X}$ is a rigid countable ω-long projective plane. □

Problem 12.12. *Is there a rigid finite projective plane?*

Corollary 12.13 ([1, 9.2.13]). *Every liner* X *is a subliner of an* ω*-long projective plane* Y *such that every finite* 3*-wide subliner* $S \subseteq Y$ *is contained in* X.

Corollary 12.13 suggests the following known open problem (probably due to Marshall Hall).

Problem 12.14. *Is every finite liner a subliner of a finite projective plane?*

Remark 12.15. Rigid projective planes were also constructed by Kegel and Schleiermacher ([6]).

REFERENCES

1. T. Banakh, *Linear Geometry and Algebra*, preprint, 2025. https://arxiv.org/abs/2506.14060 doi:10.48550/arXiv.2506.14060
2. T. Banakh, I. Hetman, A. Ravsky, V. Pshyk, *Linear Geomery: Flats, Ranks, Regularity, Parallelity*, Mat. Stud. **65** (1) (2026), 74–96. doi:10.30970/ms.65.1.74-96; https://arxiv.org/abs/2511.19455
3. C.J. Colbourn, J.H. Dinitz (eds.), *Handbook of Combinatorial Designs*, Discrete Mathematics and its Applications, Chapman & Hall/CRC, Boca Raton, FL, 2007. xxii+984 pp. doi:10.1201/9781003040897
4. P. Dembowski, *Semiaffine Ebenen*, Arch. Math., **13** (1962), 120–131. doi:10.1007/BF01650055
5. M. Hall, *Projective Planes*, Trans. Amer. Math. Soc., **54** (1943), 229–277. doi:10.1090/S0002-9947-1943-0008892-4
6. O. Kegel, A. Schleiermacher, *Amalgams and Embeddings of Projective Planes*, Geometriae Dedicata, **2** (1973), 379–395. doi:10.1007/BF00181482

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