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STUDY OF HOMODERIVATIONS ON QUOTIENT NEAR RINGS SATISFYING CERTAIN ALGEBRAIC CONSTRAINTS

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Let N be a near ring and P, a prime ideal of N. In this paper, we define the notion of P-homoderivation in near rings. Also, we investigate the structure of quotient near ring N/P via P-homoderivation satisfying some specific identities. Moreover, we provide examples to justify the existence of the results.

1. Introduction. A left near ring N is a group that is additive and has a binary operation "$\cdot$" such that (i) $(N,\cdot)$ is a semigroup (ii) $u\cdot(v+w)=u\cdot v+u\cdot w$ for all $u,v,w\in N$. A left near ring N is considered as a right near ring if right distributive law holds instead of (ii). Remember that a near ring N is considered to be zero-symmetric if $0u=0$ for all $u\in N$ (left distributive law gives that $u0=0$). Throughout the paper, N is used to represent a zero-symmetric left near ring, $Z(N)=\{u\in N\mid uv=vu \text{ for all } v\in N\}$ is the multiplicative center of N and $Z(N/P)=\{\bar{u}\in N/P\mid \bar{u}\bar{v}=\bar{v}\bar{u} \text{ for all } \bar{v}\in N/P\}$ is the multiplicative center of the quotient near ring N/P, where P denotes the prime ideal of N. Typically, for every $u,v\in N$, $(u\circ v)$ and $[u,v]$ represent the well known Jordan product $uv+vu$ and Lie product $uv-vu$ respectively. Typically, a near ring N is said to as prime if for every $u,v\in N$, $uNv=\{0\}$ indicates that either $u=0$ or $v=0$. A normal subgroup P of $(N,+)$ is said to be a right (or resp. left) ideal of N if $(u+p)v-uv\in P$ (resp. $PN\subseteq P$) for all $u,v\in N$ and P is said to be an ideal of N if P is right as well as left ideal of N. A normal subgroup P of $(N,+)$ is said to be a symmetric ideal of N if P is an ideal of N and $PN\subseteq P$. In [9], Groenewald defined prime ideal in near rings as follows: An ideal P is called a prime ideal if for any $u,v\in N$, $uNv\subseteq P$ implies that either $u\in P$ or $v\in P$.

Bell and Mason [5] established the conviction of derivation in near rings as follows: A derivation on N is defined as an additive mapping $d\colon N\to N$ such that $d(uv)=d(u)v+ud(v)$ for all $u,v\in N$ or alternatively, as stated in [18], $d(uv)=ud(v)+d(u)v$ for all $u,v\in N$. A mapping $d\colon N\to N$ is called P-additive if $d(u+v)-(d(u)+d(v))\in P$ for all $u,v\in N$ and $d\colon N\to N$ is called P-trivial if $d(N)\subseteq P$. A large number of numerous results have been investigated by a various authors to obtain the commutativity of prime and semiprime rings and near rings considering specific constraints on derivations, generalized derivations, semiderivations, generalized semiderivations, homoderivations and (σ,τ)-derivations on some appropriate subsets of near ring N (for reference see [2], [3], [4], [6], [7], [10], [11], [12], [13]).

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In [17], El. Sofy investigated the notion of homoderivations in rings as follows: An additive mapping $d\colon R \to R$ is called a homoderivation on R if $d(uv) = d(u)d(v) + d(u)v + ud(v)$ for all $u, v \in R$. Also, El. Sofy [17] proved the following result: Let R be a prime ring and I, a nonzero two sided ideal of R. If R admits a homoderivation d such that $d([u, v]) = \pm[u, v]$ for all $u, v \in I$, then R is commutative. Koç has discussed similar conditions regarding derivations in the prime near ring in [14]. Koç and Gölbaşi studied this conditions for generalized derivations on semiprime near-ring in [15]. Further many authors investigated the commutativity on different subsets of prime ring and $*$-prime ring admitting a homoderivation. In [1], A. Al-Kenani et al. extended these results for nonzero $*$ ideal I of a $*$-prime rings via homoderivation d which commutes with $*$ and satisfying the following assertions: (i) $d([u, v]) = 0$, (ii) $d(u \circ v) = 0$, (iii) $d([u, v]) = [u, v]$ and (iv) $d(u \circ v) = (u \circ v)$ for all $u, v \in I$. Very recently, S. Mouhssine and A. Boua [16] established a distinctive concept of P-derivation in quotient near ring. Inspired by this terminology, we define the notion of a homoderivation $\widetilde{d}$ on quotient near ring N/P defined by $\widetilde{d}(\bar{u}) = \overline{d(u)}$ for all $u \in N$. Further, we establish the notion of P-homoderivation in near rings.

Definition 1. Let N be a near ring and P, an additive subgroup of N. A P-additive mapping $d\colon N \to N$ is said to be a P-homoderivation on N, if $d(uv) - (d(u)d(v) + d(u)v + ud(v)) \in P$ for all $u, v \in N$.

Definition 2. Let N be a near ring and P, an additive subgroup of N. A P-additive mapping $d\colon N \to N$ is said to be a P^+-homoderivation on N, if d is a P-homoderivation such that $d(d(uv) - (d(u)d(v) + d(u)v + ud(v))) \in P$ for all $u, v \in N$.

Every homoderivation on N is a P-homoderivation on N but the converse is not true in general. The following examples verify the existence of P-homoderivation which is not a homoderivation.

Example 1. Let S be a zero-symmetric left near ring. Define N, P by

$$N = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & w & 0 \end{pmatrix} \,\middle|\, 0, u, v, w \in S \right\}; \quad P = \left\{ \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, 0, u \in S \right\}.$$

It is easy to check that N is a left near ring and P is an ideal of N. Define $d\colon N \to N$ by

$$d \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & w & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be easily showed that d is a P-homoderivation but not a homoderivation on N.

Example 2. Let S be a zero-symmetric left near ring. Define N, P by

$$N = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, 0, u, v, w \in S \right\}; \quad P = \left\{ \begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, 0, v \in S \right\}.$$

It can be easily verified that N is a left near ring and P be an ideal of N. Let us define $d\colon N \to N$ by

$$d \begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to show that d is a P-homoderivation but not a homoderivation on N.

In this line of investigation, it is naturally more interesting to look forward this concept into homoderivation on quotient near rings. To continue this work, here we will generalize and extend the aforementioned results and also, some existing results will be obtained approaching in quotient near ring in the setting of homoderivations.

2. Commutativity of quotient near ring via P-homoderivation. We begin with the following lemma, which is essential to develop the proof of our findings.

Lemma 1 ([16], Lemma 2)**.** *Let N be a near ring and P, a prime ideal of N.*
(i) If $\overline{u} \in Z(N/P) \setminus \{\overline{0}\}$ and $\overline{v} \in N/P$ such that $\overline{u}\overline{v} \in Z(N/P)$, then $\overline{v} \in Z(N/P)$.
(ii) If $\overline{u} \in Z(N/P) \setminus \{\overline{0}\}$, then $\overline{u}$ is not a zero-divisor.
(iii) If $Z(N/P)$ contains a nonzero element $\overline{u}$ for which $\overline{u} + \overline{u} \in Z(N/P)$, then $(N/P, +)$ is abelian.

Theorem 1. *Let N be a near ring and P, an ideal of N. If $d\colon N \to N$ is a P-homoderivation on N which preserves P, then the mapping $\tilde{d}\colon N/P \to N/P$ defined by $\tilde{d}(\overline{u}) = \overline{d(u)}$ is a homoderivation on N/P.*

Proof. To verify $\tilde{d}$ is well-defined, it is sufficient to show that the images of two equal elements in the quotient, under $\tilde{d}$ are also equal. Assume that $\overline{u_1}, \overline{u_2} \in N/P$ such that $\overline{u_1} = \overline{u_2}$, it follows that $u_1 + p_1 = u_2 + p_2$ for some $p_1, p_2 \in P$, implying $-u_2 + u_1 = p_2 - p_1$. Let $p_2 - p_1 = p' \in P$, so $-u_2 + u_1 = p'$. Applying d to both sides, it gives $d(-u_2 + u_1) = d(p') \in P$ as d is P-preserving. Since d is P-additive, then $d(-u_2 + u_1) \equiv d(-u_2) + d(u_1) \pmod P$ and $d(-u_2) \equiv -d(u_2) \pmod P$. Therefore, we have $-d(u_2) + d(u_1) \in P$, which is equivalent to $\overline{d(u_1)} = \overline{d(u_2)}$. Hence, $\tilde{d}(\overline{u_1}) = \tilde{d}(\overline{u_2})$.

Next, for all $\overline{u}, \overline{v} \in N/P$, we have $\tilde{d}(\overline{u} + \overline{v}) = \tilde{d}(\overline{u+v}) = \overline{d(u+v)}$. By P-additivity of d, we have $d(u+v) \equiv d(u) + d(v) \pmod P$, equivalent to

$$\overline{d(u+v)} = \overline{d(u) + d(v)} = \overline{d(u)} + \overline{d(v)} = \tilde{d}(\overline{u}) + \tilde{d}(\overline{v}).$$

Thus, $\tilde{d}(\overline{u} + \overline{v}) = \tilde{d}(\overline{u}) + \tilde{d}(\overline{v})$ for all $\overline{u}, \overline{v} \in N/P$. Moreover, $\tilde{d}(\overline{u}.\overline{v}) = \tilde{d}(\overline{uv}) = \overline{d(uv)}$. Since d is a P-homoderivation, so $d(uv) \equiv d(u)d(v) + d(u)v + ud(v) \pmod P$. Equivalently,

$$\begin{aligned}\overline{d(uv)} &= \overline{d(u)d(v) + d(u)v + ud(v)} = \overline{d(u)d(v)} + \overline{d(u)v} + \overline{ud(v)} = \\ &= \overline{d(u)}\,\overline{d(v)} + \overline{d(u)}\overline{v} + \overline{u}\overline{d(v)} = \tilde{d}(\overline{u})\tilde{d}(\overline{v}) + \tilde{d}(\overline{u})\overline{v} + \overline{u}\tilde{d}(\overline{v}).\end{aligned}$$

Therefore, $\tilde{d}(\overline{u}.\overline{v}) = \tilde{d}(\overline{u})\tilde{d}(\overline{v}) + \tilde{d}(\overline{u})\overline{v} + \overline{u}\tilde{d}(\overline{v})$ for all $\overline{u}, \overline{v} \in N/P$. Hence, $\tilde{d}$ is a homoderivation on N/P. □

Theorem 2. *Let N be a near ring and P, an ideal of N. If $d\colon N \to N$ is a P-homoderivation on N which preserves P, then the following partial distributive law holds for N/P:*

$$\overline{d(uv)}(\overline{d(w)} + \overline{w}) = \overline{d(u)d(v)}(\overline{d(w)} + \overline{w}) + \overline{d(u)}\overline{v}(\overline{d(w)} + \overline{w}) + \overline{u}\overline{d(v)}(\overline{d(w)} + \overline{w})$$

for all $\overline{u}, \overline{v}, \overline{w} \in N/P$.

Proof. Since d is a P-homoderivation on N, then an induced map $\widetilde{d}\colon N/P \to N/P$ defined by $\widetilde{d}(\bar{u}) = \overline{d(u)}$ is a homoderivation by Theorem 1. For all $\bar{u}, \bar{v}, \bar{w} \in N/P$, we have

$$\widetilde{d}((\bar{u}\bar{v})\bar{w}) = \widetilde{d}(\bar{u}\bar{v})\widetilde{d}(\bar{w}) + \widetilde{d}(\bar{u}\bar{v})\bar{w} + \bar{u}\bar{v}\widetilde{d}(\bar{w}) = \widetilde{d}(\bar{u}\bar{v})(\widetilde{d}(\bar{w}) + \bar{w}) + \bar{u}\bar{v}\widetilde{d}(\bar{w}).$$

Alternatively, $\widetilde{d}(\bar{u}(\bar{v}\bar{w})) = \widetilde{d}(\bar{u})\widetilde{d}(\bar{v}\bar{w}) + \widetilde{d}(\bar{u})\bar{v}\bar{w} + \bar{u}\widetilde{d}(\bar{v}\bar{w})$. Applying the homoderivation property of $\widetilde{d}$ with the left distributivity, yielding

$$\widetilde{d}(\bar{u}(\bar{v}\bar{w})) =$$
$$= \widetilde{d}(\bar{u})\widetilde{d}(\bar{v})\widetilde{d}(\bar{w}) + \widetilde{d}(\bar{u})\widetilde{d}(\bar{v})\bar{w} + \widetilde{d}(\bar{u})\bar{v}\widetilde{d}(\bar{w}) + \widetilde{d}(\bar{u})\bar{v}\bar{w} + \bar{u}\widetilde{d}(\bar{v})\widetilde{d}(\bar{w}) + \bar{u}\widetilde{d}(\bar{v})\bar{w} + \bar{u}\bar{v}\widetilde{d}(\bar{w}).$$

This implies that

$$\widetilde{d}(\bar{u}(\bar{v}\bar{w})) = \widetilde{d}(\bar{u})\widetilde{d}(\bar{v})(\widetilde{d}(\bar{w}) + \bar{w}) + \widetilde{d}(\bar{u})\bar{v}(\widetilde{d}(\bar{w}) + \bar{w}) + \bar{u}\widetilde{d}(\bar{v})(\widetilde{d}(\bar{w}) + \bar{w}) + \bar{u}\bar{v}\widetilde{d}(\bar{w})$$

Since $(uv)w \equiv u(vw) \pmod{P}$, it follows that $(\bar{u}\bar{v})\bar{w} = \bar{u}(\bar{v}\bar{w})$. Comparing the expressions $\widetilde{d}((\bar{u}\bar{v})\bar{w})$ and $\widetilde{d}(\bar{u}(\bar{v}\bar{w}))$, we conclude that

$$\widetilde{d}(\overline{uv})(\widetilde{d}(\bar{w}) + \bar{w}) = \widetilde{d}(\bar{u})\widetilde{d}(\bar{v})(\widetilde{d}(\bar{w}) + \bar{w}) + \widetilde{d}(\bar{u})\bar{v}(\widetilde{d}(\bar{w}) + \bar{w}) + \bar{u}\widetilde{d}(\bar{v})(\widetilde{d}(\bar{w}) + \bar{w}),$$

which can rewrite as

$$\overline{d(uv)}(\overline{d(w)} + \bar{w}) = \overline{d(u)}\ \overline{d(v)}(\overline{d(w)} + \bar{w}) + \overline{d(u)}\bar{v}(\overline{d(w)} + \bar{w}) + \bar{u}\overline{d(v)}(\overline{d(w)} + \bar{w})$$

for all $\bar{u}, \bar{v}, \bar{w} \in N/P$. $\square$

Theorem 3. *Let N be a near ring and P, a prime ideal of N. If N admits a non-P trivial P-homoderivation $d\colon N \to N$ such that $\bar{u}\overline{d(N)} = \{\bar{0}\}$, then $\bar{u} = \bar{0}$.*

Proof. Assume that $\bar{u}\overline{d(N)} = \{\bar{0}\}$, then for all $v, w \in N$, we have

$$\bar{u}\overline{d(v.w)} = \bar{0}. \tag{1}$$

By P-homoderivation identity, we have $d(v.w) - (d(v)d(w) + d(v)w + vd(w)) \in P$, which is equivalent to

$$\overline{d(v.w)} = \overline{d(v)d(w) + d(v)w + vd(w)} = \overline{d(v)d(w)} + \overline{d(v)w} + \overline{vd(w)}.$$

Thus, (1) with the left distributivity in N/P simplifies to

$$\bar{u}\overline{d(v)}\ \overline{d(w)} + \bar{u}\overline{d(v)}\bar{w} + \bar{u}\bar{v}\overline{d(w)} = \bar{0}, \quad \forall\, \bar{u}, \bar{v}, \bar{w} \in N/P.$$

Applying the hypothesis, this reduces to $\bar{u}\bar{v}\overline{d(w)} = \bar{0}$ for all $\bar{u}, \bar{v}, \bar{w} \in N/P$. Since N is a near ring and P is a prime ideal of N, so N/P is a prime near ring. Thus, the primeness of N/P together with the fact that $d(N) \nsubseteq P$ gives that $\bar{u} = \bar{0}$. $\square$

Theorem 4. *Let N be a near ring and P, a symmetric prime ideal of N such that N/P is 2-torsion-free. If N admits a P^{+}-homoderivation $d\colon N \to N$ such that $d^2(N) \subseteq P$, then $d(N) \subseteq P$.*

Proof. Suppose that $d^2(N) \subseteq P$, it follows that $\overline{d^2(u)} = \bar{0}$ for all $u \in N$. Applying it to the product uv, we get

$$\overline{d^2(uv)} = \bar{0}, \quad \forall\, u, v \in N. \tag{2}$$

Since d is a P^+-homoderivation, then $d^2(uv) \equiv d(d(u)d(v) + d(u)v + ud(v) \pmod P$, which is equivalent to

$$\overline{d^2(uv)} = \overline{d(d(u)d(v) + d(u)v + ud(v))}.$$

Thus, (2) becomes

$$\overline{d(d(u)d(v) + d(u)v + ud(v))} = \bar{0}.$$

P-additivity of d, yielding

$$\overline{d(d(u)d(v)} + \overline{d(d(u)v)} + \overline{d(ud(v))} = \bar{0}.$$

By P-homoderivation identity, we have $d(uv) \equiv d(u)d(v) + d(u)v + ud(v) \pmod P$. Applying this identity to each term of last expression, ensuring

$$\overline{d^2(u)}\ \overline{d^2(v)} + \overline{d^2(u)}\ \overline{d(v)} + \overline{d(u)}\ \overline{d^2(v)} + \overline{d^2(u)}\ \overline{d(v)} + \overline{d^2(u)}\bar{v} + \overline{d(u)}\ \overline{d(v)} + \overline{d(u)}\ \overline{d^2(v)}$$
$$+ \overline{d(u)}\ \overline{d(v)} + \bar{u}\overline{d^2(v)} = \bar{0}.$$

Using the hypothesis, we get $2\overline{d(u)}\ \overline{d(v)} = \bar{0}$ for all $u, v \in N$. By 2-torsion-freeness of N/P, we obtain $\overline{d(u)}\ \overline{d(v)} = \bar{0}$ for all $u, v \in N$. Substituting vw for v in the preceding relation, we find that $\overline{d(u)}\bar{v}\overline{d(w)} = \bar{0}$ for all $u, v, w \in N$. By the primeness of N/P, we have $\overline{d(u)} = \bar{0}$ or $\overline{d(w)} = \bar{0}$ for all $u, w \in N$. Consequently, $\overline{d(N)} = \{\bar{0}\}$, equivalent to $d(N) \subseteq P$. □

Theorem 5. *Let N be a near ring and P, a prime ideal of N. If N admits a non P-trivial P-homoderivation d which preserves P, and satisfies $\overline{d(N)} \subseteq Z(N/P)$, then $(N/P, +)$ is abelian. Moreover, if $d^2(N) \nsubseteq P$, then N/P is a commutative ring.*

Proof. Since $d(N) \nsubseteq P$, then $d(w) \notin P$ for some $w \in N$. Thus, $\overline{d(w)} \neq \bar{0}$ in N/P, and hypothesis yields $\overline{d(w)} \in Z(N/P) \setminus \{\bar{0}\}$. By P-additivity, $d(w + w) - (d(w) + d(w)) \in P$, which implies that

$$\overline{d(w + w)} = \overline{d(w) + d(w)} = \overline{d(w)} + \overline{d(w)}.$$

Therefore, $\overline{d(w)} + \overline{d(w)} \in Z(N/P)$ as $\overline{d(w + w)} \in Z(N/P)$ by hypothesis. Now, for arbitrary $\bar{u}, \bar{v} \in N/P$,

$$(\overline{d(w)} + \overline{d(w)})(\bar{u} + \bar{v}) = (\bar{u} + \bar{v})(\overline{d(w)} + \overline{d(w)}).$$

Applying the left distributivity in N/P, we find

$$(\overline{d(w)} + \overline{d(w)})\bar{u} + (\overline{d(w)} + \overline{d(w)})\bar{v} = (\bar{u} + \bar{v})\overline{d(w)} + (\bar{u} + \bar{v})\overline{d(w)}$$
$$\bar{u}\overline{d(w)} + \bar{u}\overline{d(w)} + \bar{v}\overline{d(w)} + \bar{v}\overline{d(w)} = \bar{u}\overline{d(w)} + \bar{v}\overline{d(w)} + \bar{u}\overline{d(w)} + \bar{v}\overline{d(w)}.$$

The last expression simplifies to

$$\bar{u}\overline{d(w)} + \bar{v}\overline{d(w)} = \bar{v}\overline{d(w)} + \bar{u}\overline{d(w)},$$

which implies that $\overline{d(w)}(\bar{u}, \bar{v}) = \bar{0}$ for all $\bar{u}, \bar{v} \in N/P$. Since N/P is a prime near ring and $\overline{d(w)} \in Z(N/P) \setminus \{\bar{0}\}$, we obtain $(\bar{u}, \bar{v}) = \bar{0}$ for all $\bar{u}, \bar{v} \in N/P$. Hence, the additive group $(N/P, +)$ is abelian. Moreover, the hypothesis yielding

$$\overline{d(uv)}(\overline{d(w)} + \bar{w}) = (\overline{d(w)} + \bar{w})\overline{d(uv)}.$$

Appealing to Theorem 2 and P-homoderivation property, we derive

$$\begin{aligned}&\overline{d(u)}\,\overline{d(v)}(\overline{d(w)}+\bar{w})+\overline{d(u)}\bar{v}(\overline{d(w)}+\bar{w})+\bar{u}\overline{d(v)}(\overline{d(w)}+\bar{w})=\\&=(\overline{d(w)}+\bar{w})\overline{d(u)}\,\overline{d(v)}+(\overline{d(w)}+\bar{w})\overline{d(u)}\bar{v}+(\overline{d(w)}+\bar{w})\bar{u}\overline{d(v)},\end{aligned}$$

leading to

$$\overline{d(u)}\bar{v}(\overline{d(w)}+\bar{w})+\bar{u}\overline{d(v)}(\overline{d(w)}+\bar{w})=(\overline{d(w)}+\bar{w})\overline{d(u)}\bar{v}+(\overline{d(w)}+\bar{w})\bar{u}\overline{d(v)}.$$

Substituting $d(u)$ for u in this expression, we deduce

$$\overline{d^2(u)}\bar{v}(\overline{d(w)}+\bar{w})+\overline{d(u)}\,\overline{d(v)}(\overline{d(w)}+\bar{w})=(\overline{d(w)}+\bar{w})\overline{d^2(u)}\bar{v}+(\overline{d(w)}+\bar{w})\overline{d(u)}\,\overline{d(v)}.$$

Invoking the hypothesis, resulting in

$$\overline{d^2(u)}\bar{v}(\overline{d(w)}+\bar{w})=(\overline{d(w)}+\bar{w})\overline{d^2(u)}\bar{v}. \tag{3}$$

Replacing $\bar{v}$ by $\bar{v}\bar{s}$ in (3) and using (3), it gives $\overline{d^2(u)}\bar{v}[\overline{d(w)}+\bar{w},\bar{s}]=\bar{0}$ for all $\bar{s},\bar{u},\bar{v},\bar{w}\in N/P$, i.e., $\overline{d^2(u)}N/P[\overline{d(w)}+\bar{w},\bar{s}]=\{\bar{0}\}$ for all $\bar{s},\bar{u},\bar{w}\in N/P$. By the primeness of N/P and $d^2(N)\nsubseteq P$, we get $\overline{d(w)}+\bar{w}\in Z(N/P)$. Thus, for each $\bar{r},\bar{w}\in N/P$, we have

$$(\overline{d(w)}+\bar{w})(\overline{d(r)}+\bar{r})=(\overline{d(r)}+\bar{r})(\overline{d(w)}+\bar{w}),\quad\forall\,\bar{w},\bar{r}\in N/P.$$

Expanding with left distributivity, we obtain

$$\begin{aligned}(\overline{d(w)}+\bar{w})\overline{d(r)}+(\overline{d(w)}+\bar{w})\bar{r}&=(\overline{d(r)}+\bar{r})\overline{d(w)}+(\overline{d(r)}+\bar{r})\bar{w},\\ \overline{d(r)}(\overline{d(w)}+\bar{w})+\bar{r}(\overline{d(w)}+\bar{w})&=\overline{d(w)}(\overline{d(r)}+\bar{r})+\bar{w}(\overline{d(r)}+\bar{r}),\\ \overline{d(r)}\,\overline{d(w)}+\overline{d(r)}\bar{w}+\bar{r}\overline{d(w)}+\bar{r}\bar{w}&=\overline{d(w)}\,\overline{d(r)}+\overline{d(w)}\bar{r}+\bar{w}\overline{d(r)}+\bar{w}\bar{r},\end{aligned}$$

reduces to

$$\overline{d(r)}\bar{w}+\bar{r}\overline{d(w)}+\bar{r}\bar{w}=\overline{d(w)}\bar{r}+\bar{w}\overline{d(r)}+\bar{w}\bar{r},\quad\forall\,\bar{w},\bar{r}\in N/P.$$

Since $(N/P,+)$ is abelian and $\overline{d(N)}\subseteq Z(N/P)$, we conclude $[\bar{w},\bar{r}]=\bar{0}$ for all $\bar{r},\bar{w}\in N/P$. Hence, N/P is a commutative ring. □

Theorem 6. *Let N be a near ring and P, a prime ideal of N, where N/P is 2-torsion-free. If N admits a non P-trivial P-homoderivation d which preserves P and satisfies $d([u,v])\in P$ for all $u,v\in N$, then N/P is a commutative ring.*

Proof. By hypothesis, $d([u,v])\in P$ for all $u,v\in N$, i.e., $d([u,v])\equiv 0 \pmod P$. This can rewrite as $\overline{d([u,v])}=\bar{0}$ for all $u,v\in N$. Since d is a P-homoderivation, then there exists an induced map $\tilde{d}\colon N/P\to N/P$ defined by $\tilde{d}(\bar{u})=\overline{d(u)}$ is a homoderivation by Theorem 1. Therefore, our hypothesis becomes

$$\tilde{d}\left(\left[\bar{u},\bar{v}\right]\right)=\bar{0},\quad\forall\,\bar{u},\bar{v}\in N/P. \tag{4}$$

Replacing $\bar{v}$ by $\bar{u}\bar{v}$ in (4) and using $\left[\bar{u},\bar{u}\bar{v}\right]=\bar{u}\left[\bar{u},\bar{v}\right]$, we obtain

$$\tilde{d}\left(\bar{u}\right)\tilde{d}(\left[\bar{u},\bar{v}\right])+\tilde{d}\left(\bar{u}\right)\left[\bar{u},\bar{v}\right]+\bar{u}\tilde{d}\left(\left[\bar{u},\bar{v}\right]\right)=\bar{0},\quad\forall\,\bar{u},\bar{v}\in N/P.$$

In view of (4), this yields

$$\widetilde{d}\left(\overline{u}\right)\left[\overline{u},\overline{v}\right]=\overline{0},\quad \forall\, \overline{u},\overline{v}\in N/P.$$

This can expressed as

$$\widetilde{d}\left(\overline{u}\right)\overline{u}\overline{v}=\widetilde{d}\left(\overline{u}\right)\overline{v}\overline{u},\quad \forall\, \overline{u},\overline{v}\in N/P. \tag{5}$$

Taking $\overline{v}\overline{w}$ instead of $\overline{v}$, where $\overline{w}\in N/P$ in (5) and using (5), simplifies to

$$\widetilde{d}\left(\overline{u}\right)\overline{v}\left[\overline{u},\overline{w}\right]=\overline{0},\quad \forall\, \overline{u},\overline{v},\overline{w}\in N/P.$$

By primeness of N/P, we get either $\left[\overline{u},\overline{w}\right]=\overline{0}$ or $\widetilde{d}\left(\overline{u}\right)=\overline{0}$ for each $\overline{u}\in N/P$. Thus, we conclude that

$$\overline{u}\in Z(N/P)\ \text{ or }\ \widetilde{d}\left(\overline{u}\right)=\overline{0},\quad \forall\, \overline{u}\in N/P. \tag{6}$$

Suppose there exists an element $\overline{u_0}\in N/P$ such that $\widetilde{d}\left(\overline{u_0}\right)=\overline{0}$, then replacing $\overline{u}$ by $\overline{u_0}$ in (4), we have $\widetilde{d}(\overline{u_0}\overline{v})=\widetilde{d}(\overline{v}\overline{u_0})$ for all $\overline{v}\in N/P$. Applying the homoderivation property, it gives $\overline{u_0}\widetilde{d}(\overline{v})=\widetilde{d}(\overline{v})\overline{u_0}$ for all $\overline{v}\in N/P$. This follows that $(\widetilde{d}(\overline{u_0})+\overline{u_0})\widetilde{d}(\overline{v})=\widetilde{d}(\overline{v})(\widetilde{d}(\overline{u_0})+\overline{u_0})$ for all $\overline{v}\in N/P$. Replacing $\overline{v}$ by $\widetilde{d}(\overline{v})\overline{w}$ in the previous relation, and expand with Theorem 2, we see that

$$\overline{u_0}\widetilde{d}^2(\overline{v})\widetilde{d}(\overline{w})+\overline{u_0}\widetilde{d}^2(\overline{v})\overline{w}+\overline{u_0}\widetilde{d}(\overline{v})\widetilde{d}(\overline{w})=\widetilde{d}^2(\overline{v})\widetilde{d}(\overline{w})\overline{u_0}+\widetilde{d}^2(\overline{v})\overline{w}\overline{u_0}+\widetilde{d}(\overline{v})\widetilde{d}(\overline{w})\overline{u_0}.$$

Using the relation that $\overline{u_0}\widetilde{d}(\overline{v})=\widetilde{d}(\overline{v})\overline{u_0}$ for all $\overline{v}\in N/P$ in the last expression, we deduce that $\overline{u_0}\widetilde{d}^2(\overline{v})\overline{w}=\widetilde{d}^2(\overline{v})\overline{w}\overline{u_0}$ for all $\overline{v},\overline{w}\in N/P$. Substituting $\overline{w}\overline{s}$ for $\overline{w}$, where $\overline{s}\in N/P$ in this relation and using it again, we get $\widetilde{d}^2(\overline{v})N/P[\overline{u_0},\overline{s}]=\{\overline{0}\}$ for all $\overline{s},\overline{v}\in N/P$. Owing to primeness of N/P, we get $\overline{u_0}\in Z(N/P)$ or $\widetilde{d}^2(\overline{v})=\overline{0}$ for all $\overline{v}\in N/P$. If the second alternative occurs, i,e., $\widetilde{d}^2(\overline{v})=\overline{0}$ for all $\overline{v}\in N/P$, leads to a contradiction by Theorem 4. Thus, only the first case holds, i.e., $\overline{u_0}\in Z(N/P)$. Therefore, (6) implies that $\overline{u}\in Z(N/P)$ for all $\overline{u}\in N/P$, which shows that N/P is commutative. □

Theorem 7. *Let N be a near ring and P, a prime ideal of N, where N/P is 2-torsion-free. Then there is no non P-trivial P-homoderivation d which preserves P and satisfies $d(u\circ v)\in P$ for all $u,v\in N$.*

Proof. Assume that $d(u\circ v)\in P$ for all $\overline{u},\overline{v}\in N/P$, which follows that

$$\widetilde{d}(\overline{u}\circ\overline{v})=\overline{0},\quad \forall\, \overline{u},\overline{v}\in N/P. \tag{7}$$

Substituting $\overline{u}\overline{v}$ for $\overline{v}$ in (7) and applying the homoderivation property, we get

$$\widetilde{d}(\overline{u})\widetilde{d}(\overline{u}\circ\overline{v})+\widetilde{d}(\overline{u})(\overline{u}\circ\overline{v})+\overline{u}\widetilde{d}(\overline{u}\circ\overline{v})=\overline{0},\quad \forall\, \overline{u},\overline{v}\in N/P.$$

Using (7), this becomes $\widetilde{d}(\overline{u})(\overline{u}\circ\overline{v})=\overline{0}$ for all $\overline{u},\overline{v}\in N/P$, which can rewrite as

$$\widetilde{d}(\overline{u})\overline{u}\overline{v}=-\widetilde{d}(\overline{u})\overline{v}\overline{u},\quad \forall\, \overline{u},\overline{v}\in N/P. \tag{8}$$

Taking $\overline{v}\overline{w}$, $\overline{w} \in N/P$ instead of $\overline{v}$ in (8) and using (8), we derive

$$\widetilde{d}(\overline{u})\overline{u}\overline{v}\overline{w} = -\widetilde{d}(\overline{u})\overline{v}\overline{w}\overline{u} = (\widetilde{d}(\overline{u})\overline{v}\overline{w})(-\overline{u}) = (-\widetilde{d}(\overline{u})\overline{v}\overline{u})\overline{w} = ((\widetilde{d}(\overline{u})\overline{v})(-\overline{u}))\overline{w}.$$

Thus, $\widetilde{d}(\overline{u})\overline{v}[-\overline{u}, \overline{w}] = \overline{0}$ for all $\overline{u}, \overline{v}, \overline{w} \in N/P$. Putting $-\overline{u}$ for $\overline{u}$ in this relation, we see that $\widetilde{d}(-\overline{u})\overline{v}[\overline{u}, \overline{w}] = \overline{0}$ for all $\overline{u}, \overline{v}, \overline{w} \in N/P$. By primeness of N/P, we obtain $\widetilde{d}(-\overline{u}) = \overline{0}$ or $[\overline{u}, \overline{w}] = \overline{0}$ for all $\overline{u} \in N/P$. By additivity of $\widetilde{d}$ is additive, we arrive at

$$\overline{u} \in Z(N/P) \ \text{ or } \ \widetilde{d}(\overline{u}) = \overline{0}, \quad \forall\, \overline{u} \in N/P. \tag{9}$$

Suppose the first alternative occurs, $\overline{u_0} \in Z(N/P)$ for some $\overline{u_0} \in N/P$. Replacing $\overline{u}$ with $\overline{u_0}$ in (7) and invoking 2-torsion-freeness to develop

$$\widetilde{d}(\overline{u_0})\widetilde{d}(\overline{v}) + \widetilde{d}(\overline{u_0})\overline{v} + \overline{u_0}\widetilde{d}(\overline{v}) = \overline{0}, \quad \forall\, \overline{v} \in N/P.$$

Substitute $\overline{v}$ with $\overline{v} \circ \overline{w}$ in this expression and using (7), resulting in $\widetilde{d}(\overline{u_0})(\overline{v} \circ \overline{w}) = \overline{0}$ for all $\overline{v}, \overline{w} \in N/P$, which gives

$$\widetilde{d}(\overline{u_0})\overline{v}\overline{w} = -\widetilde{d}(\overline{u_0})\overline{w}\overline{v}, \quad \forall\, \overline{v}, \overline{w} \in N/P.$$

Replacing $\overline{w}$ with $\overline{w}\overline{s}$, and applying the same techniques as after (8), we obtain

$$\overline{v} \in Z(N/P) \ \text{ or } \ \widetilde{d}(\overline{u_0}) = \overline{0}, \quad \forall\, \overline{v} \in N/P.$$

Hence, (9) concludes that N/P is a commutative ring or $d(N) \subseteq P$. If N/P is commutative, then (7) with 2-torsion-freeness gives that $\widetilde{d}(\overline{u}\overline{v}) = \overline{0}$ for all $\overline{u}, \overline{v} \in N/P$. Putting $\overline{w}\overline{u}$ for $\overline{u}$ in last relation and developing it, we arrive at $\widetilde{d}(\overline{w})\overline{u}\overline{v} = \overline{0}$ for all $\overline{u}, \overline{v}, \overline{w} \in N/P$. In view of primeness of N/P, we get $\widetilde{d}(\overline{w}) = \overline{0}$ for all $\overline{w} \in N/P$. Hence, in each case, we get $d(N) \subseteq P$, a contradiction. □

Theorem 8. *Let N be a near ring and P, a prime ideal of N, where N/P is 2-torsion-free. If N admits a non P-trivial P-homoderivation d which preserves P and satisfies $d([u, v]) - [u, v] \in P$ for all $u, v \in N$, then N/P is a commutative ring.*

Proof. By hypothesis, $d([u, v]) - [u, v] \in P$, i.e., $d([u, v]) - [u, v] \equiv 0 \pmod P$ for all $u, v \in N$. Equivalently, $\overline{d([u, v]) - [u, v]} = \overline{0}$ in N/P, which implies that $\overline{d([u, v])} - \overline{[u, v]} = \overline{0}$ in N/P, thus by Theorem 1, we obtain

$$\widetilde{d}([\overline{u}, \overline{v}]) - [\overline{u}, \overline{v}] = \overline{0}, \quad \forall\, \overline{u}, \overline{v} \in N/P. \tag{10}$$

We divide the proof into two cases:

Case 1: Assume that $d(N) \subseteq P$, then $\overline{d(N)} = \{\overline{0}\}$ in N/P. Thus, $\widetilde{d}(N/P) = \{\overline{0}\}$ by Theorem 1. Therefore, (10) gives that $-[\overline{u}, \overline{v}] = \overline{0}$ for all $\overline{u}, \overline{v} \in N/P$, which forces that N/P is a commutative ring.

Case 2: Suppose that $d(N) \nsubseteq P$. Replacing $\overline{v}$ by $\overline{u}\overline{v}$ in (10) and applying $[\overline{u}, \overline{u}\overline{v}] = \overline{u}[\overline{u}, \overline{v}]$, we get $\widetilde{d}(\overline{u}[\overline{u}, \overline{v}]) = \overline{u}[\overline{u}, \overline{v}]$ for all $\overline{u}, \overline{v} \in N/P$. Applying the homoderivation identity, we see that $\widetilde{d}(\bar{u})\widetilde{d}([\bar{u},\bar{v}])+\widetilde{d}(\bar{u})[\bar{u},\bar{v}]+\bar{u}\widetilde{d}([\bar{u},\bar{v}])=\bar{u}[\bar{u},\bar{v}]$ for all $\bar{u},\bar{v}\in N/P$. In light of (10), this simplifies to

$$\widetilde{d}(\bar{u})\widetilde{d}([\bar{u},\bar{v}])+\widetilde{d}(\bar{u})[\bar{u},\bar{v}]=\bar{0},\quad \forall\, \bar{u},\bar{v}\in N/P. \tag{11}$$

Substitute $\bar{u}$ with $[\bar{r},\bar{s}]$ in (11) and using (10), we get $2([\bar{r},\bar{s}][[\bar{r},\bar{s}],\bar{v}])=\bar{0}$ for all $\bar{r},\bar{s},\bar{v}\in N/P$. By 2-torsion-freeness of N/P, we obtain $[\bar{r},\bar{s}][[\bar{r},\bar{s}],\bar{v}]=\bar{0}$ for all $\bar{r},\bar{s},\bar{v}\in N/P$. Applying left distributivity, we wrie

$$[\bar{r},\bar{s}][\bar{r},\bar{s}]\bar{v}=[\bar{r},\bar{s}]\bar{v}[\bar{r},\bar{s}],\quad \forall\, \bar{r},\bar{s},\bar{v}\in N/P. \tag{12}$$

Replace $\bar{v}$ with $\bar{v}\bar{w}$ in (12) and using (12), we find $[\bar{r},\bar{s}]\bar{v}[[\bar{r},\bar{s}],\bar{w}]=\bar{0}$ for all $\bar{r},\bar{s},\bar{v},\bar{w}\in N/P$, i.e., $[\bar{r},\bar{s}]N/P[[\bar{r},\bar{s}],\bar{w}]=\{\bar{0}\}$ for all $\bar{r},\bar{s},\bar{w}\in N/P$. By primeness of N/P, we conclude that $[\bar{r},\bar{s}]=\bar{0}$ or $[\bar{r},\bar{s}]\in Z(N/P)$ for all $\bar{r},\bar{s}\in N/P$, which forces that $[\bar{r},\bar{s}]\in Z(N/P)$ for all $\bar{r},\bar{s}\in N/P$. Putting $\bar{r}\bar{s}$ instead of $\bar{s}$ in the last expression, we obtain $\bar{r}[\bar{r},\bar{s}]\in Z(N/P)$ for all $\bar{r},\bar{s}\in N/P$. Appealing to Lemma 1(i), we get $[\bar{r},\bar{s}]=\bar{0}$ or $\bar{r}\in Z(N/P)$, which forces that $\bar{r}\in Z(N/P)$ for all $\bar{r}\in N/P$. Hence, N/P is a commutative ring. □

Theorem 9. *Let N be a near ring and P, a prime ideal of N, where N/P is 2-torsion-free. Then there is no non-P-trivial P-homoderivation d which preserves P and satisfies $d(u\circ v)-(u\circ v)\in P$ for all $u,v\in N$.*

Proof. By assumption, $d(u\circ v)-(u\circ v)\in P$ for all $u,v\in N$, it follows that

$$\widetilde{d}(\bar{u}\circ\bar{v})-(\bar{u}\circ\bar{v})=\bar{0},\quad \forall\, \bar{u},\bar{v}\in N/P. \tag{13}$$

Replacing $\bar{v}$ by $\bar{u}\bar{v}$ in (13) and using $(\bar{u}\circ\bar{u}\bar{v})=\bar{u}(\bar{u}\circ\bar{v})$, we have $\widetilde{d}(\bar{u}(\bar{u}\circ\bar{v}))=\bar{u}(\bar{u}\circ\bar{v})$ for all $\bar{u},\bar{v}\in N/P$. By homoderivation identity, this yields

$$\widetilde{d}(\bar{u})\widetilde{d}(\bar{u}\circ\bar{v})+\widetilde{d}(\bar{u})(\bar{u}\circ\bar{v})+\bar{u}\widetilde{d}(\bar{u}\circ\bar{v})=\bar{u}(\bar{u}\circ\bar{v}),\quad \forall\, \bar{u},\bar{v}\in N/P.$$

Applying (13), simplifies to

$$\widetilde{d}(\bar{u})\widetilde{d}(\bar{u}\circ\bar{v})+\widetilde{d}(\bar{u})(\bar{u}\circ\bar{v})=\bar{0},\quad \forall\, \bar{u},\bar{v}\in N/P. \tag{14}$$

Substituting $(\bar{r}\circ\bar{s})$ for $\bar{u}$ in (14) and using (13), we obtain $2(\bar{r}\circ\bar{s})((\bar{r}\circ\bar{s})\circ\bar{v})=\bar{0}$ for all $\bar{r},\bar{s},\bar{v}\in N/P$. By 2-torsion-freeness of N/P, we get $(\bar{r}\circ\bar{s})((\bar{r}\circ\bar{s})\circ\bar{v})=\bar{0}$ for all $\bar{r},\bar{s},\bar{v}\in N/P$. This implies that

$$(\bar{r}\circ\bar{s})(\bar{r}\circ\bar{s})\bar{v}=-(\bar{r}\circ\bar{s})\bar{v}(\bar{r}\circ\bar{s}),\quad \forall\, \bar{r},\bar{s},\bar{v}\in N/P. \tag{15}$$

Replacing $\bar{v}$ by $\bar{v}\bar{w}$ on (15) and using (15), we deduce that

$$(\bar{r}\circ\bar{s})(\bar{r}\circ\bar{s})\bar{v}\bar{w}=-(\bar{r}\circ\bar{s})\bar{v}\bar{w}(\bar{r}\circ\bar{s})=((\bar{r}\circ\bar{s})\bar{v}\bar{w})(-(\bar{r}\circ\bar{s})),$$

while,

$$(\bar{r}\circ\bar{s})(\bar{r}\circ\bar{s})\bar{v}\bar{w}=(-(\bar{r}\circ\bar{s})\bar{v}(\bar{r}\circ\bar{s}))\bar{w}=((\bar{r}\circ\bar{s})\bar{v}(-(\bar{r}\circ\bar{s})))\bar{w}.$$

Therefore, we have $(\bar{r}\circ\bar{s})\bar{v}((-\bar{w})(-(\bar{r}\circ\bar{s}))+(-(\bar{r}\circ\bar{s}))\bar{w})=\bar{0}$ for all $\bar{r},\bar{s},\bar{v},\bar{w}\in N/P$, i.e., $(\bar{r}\circ\bar{s})N/P((-\bar{w})(-(\bar{r}\circ\bar{s}))+(-(\bar{r}\circ\bar{s}))\bar{w})=\{\bar{0}\}$ for all $\bar{r},\bar{s},\bar{w}\in N/P$. By the primeness of N/P, we get $(\overline{r} \circ \overline{s}) = \overline{0}$ or $-(\overline{r} \circ \overline{s}) \in Z(N/P)$ for all $\overline{r}, \overline{s} \in N/P$, which forces that $-(\overline{r} \circ \overline{s}) \in Z(N/P)$ for all $\overline{r}, \overline{s} \in N/P$. Taking $\overline{r}\,\overline{s}$ in place of $\overline{s}$ in the previous expression, we obtain $-(\overline{r} \circ \overline{r}\,\overline{s}) = -(\overline{r}(\overline{r} \circ \overline{s})) = \overline{r}(-(\overline{r} \circ \overline{s})) \in Z(N/P)$ for all $\overline{r}, \overline{s} \in N/P$. An application of Lemma 1(i), we get $(\overline{r} \circ \overline{s}) = \overline{0}$ or $\overline{r} \in Z(N/P)$ for all $\overline{r}, \overline{s} \in N/P$.

If there exists $\overline{r_0} \in N/P$ such that $(\overline{r_0} \circ \overline{s}) = \overline{0}$, then $\overline{r_0}\,\overline{s} = -\overline{s}\,\overline{r_0}$ for all $\overline{s} \in N/P$. Substitute $\overline{s}$ with $\overline{s}\,\overline{t}$, we derive $\overline{r_0}\,\overline{s}\,\overline{t} = -\overline{s}\,\overline{t}\,\overline{r_0} = \overline{s}\,\overline{t}(-\overline{r_0}) = (-\overline{s}\,\overline{r_0})\overline{t} = (\overline{s}(-\overline{r_0}))\overline{t}$, which implies that $\overline{s}[-\overline{r_0}, \overline{t}] = \overline{0}$ for all $\overline{s}, \overline{t} \in N/P$. Owing to primeness of N/P, we deduce that $\overline{r_0} \in Z(N/P)$. Therefore, $\overline{r} \in Z(N/P)$ for all $\overline{r} \in N/P$ in each case. Hence, N/P is a commutative ring. In this case, (13) together with the 2-torsion-freeness of N/P yields that $\widetilde{d}(\overline{u}\,\overline{v}) = \overline{u}\,\overline{v}$ for all $\overline{u}, \overline{v} \in N/P$. Replacing $\overline{u}$ by $\overline{u}\,\overline{w}$ in the previous relation and using it again, we arrive at $\overline{u}N/P\widetilde{d}(\overline{v}) = \{\overline{0}\}$ for all $\overline{u}, \overline{v} \in N/P$. By the primeness of N/P, we get $\widetilde{d}(\overline{v}) = \overline{0}$ for all $\overline{v} \in N/P$. Thus, $\overline{d(v)} = \overline{0}$ in N/P, equivalent to $d(N) \subseteq P$, a contradiction. □

Theorem 10. *Let N be a near ring, P a prime ideal of N, where N/P is 2-torsion-free. If d is a non-P-trivial P-homoderivation which preserves P and, is zero power valued on N satisfying any one of the following assertions:*

(i) $[\overline{d(u) + u}, \overline{v}] \in Z(N/P)$ *for all* $\overline{u}, \overline{v} \in N/P$,

(ii) $(\overline{d(u) + u}) \circ \overline{v} \in Z(N/P)$ *for all* $\overline{u}, \overline{v} \in N/P$,

(iii) $\overline{d(u \circ v)} + \overline{(u \circ v)} \in Z(N/P)$ *for all* $\overline{u}, \overline{v} \in N/P$,

(iv) $\overline{d(uv)} + \overline{uv} \in Z(N/P)$ *for all* $\overline{u}, \overline{v} \in N/P$,

then N/P is a commutative ring.

Proof. (i) By hypothesis,

$$[\overline{d(u) + u}, \overline{v}] \in Z(N/P), \quad \forall\ \overline{u}, \overline{v} \in N/P. \tag{16}$$

Substituting $(\overline{d(u) + u})\overline{v}$ for $\overline{v}$ in (16), we obtain

$$[\overline{d(u) + u}, (\overline{d(u) + u})\overline{v}] = (\overline{d(u) + u})[\overline{d(u) + u}, \overline{v}] \in Z(N/P).$$

An application of Lemma 1 gives that

$$[\overline{d(u) + u}, \overline{v}] = \overline{0} \ \text{ or } \ (\overline{d(u) + u}) \in Z(N/P), \quad \forall\ \overline{u}, \overline{v} \in N/P. \tag{17}$$

Suppose that there exists an element $\overline{u_0} \in N/P$ such that $[\overline{d(u_0) + u_0}, \overline{v}] = \overline{0}$, which implies that $(\overline{d(u_0) + u_0})\overline{v} = \overline{v}(\overline{d(u_0) + u_0})$ for all $\overline{v} \in N/P$. Putting $\overline{v}\,\overline{w}$ for $\overline{v}$ in the last expression and using it again, we have $N/P[\overline{d(u_0) + u_0}, \overline{w}] = \{\overline{0}\}$ for all $\overline{w} \in N/P$. In view of primeness of N/P, we get $(\overline{d(u_0) + u_0}) \in Z(N/P)$. Therefore, (17) yields that

$$(\overline{d(u) + u}) \in Z(N/P), \quad \forall\ \overline{u} \in N/P. \tag{18}$$

Assume that $d(u) \notin P$, i.e., $\overline{d(u)} \neq \overline{0}$ for all $u \in N \setminus \{0\}$, then by recurrence, we have $\overline{d^l(u)} \neq \overline{0}$ for all $u \in N \setminus \{0\}$, $l \in \mathbb{N}^*$. Since d is zero power valued on N, then there exists an integer $l(u) > 1$ such that $\overline{d^{l(u)}(u)} = \overline{0}$ for all $u \in N$. This implies that $\overline{r} = \overline{d^{l(u)-1}(u)} \neq \overline{0}$ for $\overline{r} \in N/P$, thus $\overline{d(r)} = \overline{d^{l(u)}(u)} = \overline{0}$, a contradiction as d is non-P-trivial on N. Therefore,

there must exists an element $u' \in N \setminus \{0\}$ with $\overline{d(u')} = \bar{0}$. Hence, $\bar{u'} = \overline{d(u')} + \bar{u'} \in Z(N/P)$ and $\bar{u'} + \bar{u'} = \overline{d(u' + u')} + \bar{u'} + \bar{u'} \in Z(N/P)$, which forces that N/P is abelian by Lemma 1. Therefore, Putting $\bar{u} - \overline{d(u)} + \overline{d^2(u)} + \ldots + (-1)^{l(u)-1}\overline{d^{l(u)-1}(u)}$ instead of $\bar{u}$ in (18), we arrive at $\bar{u} \in Z(N/P)$ for all $u \in N$, which completes the proof.
(ii) By assumption,

$$(\overline{d(u) + u}) \circ \bar{v} \in Z(N/P), \quad \forall \ \bar{u}, \bar{v} \in N/P. \tag{19}$$

Substituting $(\overline{d(u) + u})\bar{v}$ for $\bar{v}$ in (19), we have

$$(\overline{d(u) + u}) \circ (\overline{d(u) + u})\bar{v} = (\overline{d(u) + u})((\overline{d(u) + u}) \circ \bar{v}) \in Z(N/P).$$

Applying Lemma 1, we get

$$(\overline{d(u) + u}) \circ \bar{v} = \bar{0} \ \text{ or } \ (\overline{d(u) + u}) \in Z(N/P), \quad \forall \ \bar{u} \in N/P. \tag{20}$$

Assume that there exists $\bar{u_0} \in N/P$ such that $(\overline{d(u_0) + u_0}) \circ \bar{v} = \bar{0}$ for all $\bar{v} \in N/P$. This implies that $\bar{v}(\overline{d(u_0) + u_0}) = -(\overline{d(u_0) + u_0})\bar{v}$ for all $\bar{v} \in N/P$. Replacing $\bar{v}$ by $\bar{v}\bar{s}$ in previous expression, we find that

$$\bar{v}\bar{s}(\overline{d(u_0) + u_0}) = -((\overline{d(u_0) + u_0})\bar{v}\bar{s}) = ((\overline{d(u_0) + u_0})\bar{v})(-\bar{s}),$$

while

$$((\overline{d(u_0) + u_0})\bar{v})(-\bar{s}) = (-\bar{v}(\overline{d(u_0) + u_0}))(-\bar{s}) = \bar{v}(-(\overline{d(u_0) + u_0}))(-\bar{s}).$$

Thus, $\bar{v}[-(\overline{d(u_0) + u_0}), \bar{s}] = \bar{0}$ for all $\bar{v}, \bar{s} \in N/P$, i.e., $N/P[-(\overline{d(u_0) + u_0}), \bar{s}] = \{\bar{0}\}$ for all $\bar{s} \in N/P$. By primeness of N/P, we obtain $-(\overline{d(u_0) + u_0}) \in Z(N/P)$, which means that $-\bar{u}_0 + \overline{d(-u_0)} \in Z(N/P)$. Proceeding the same techniques as we have done after (18) in (i), we can conclude the result. Now for the later case, suppose that $(\overline{d(u) + u}) \in Z(N/P) \setminus \{\bar{0}\}$ for all $u \in N$, then (19) becomes $(\overline{d(u) + u})(\bar{v} + \bar{v}) \in Z(N/P)$ for all $\bar{u}, \bar{v} \in N/P$. In view of Lemma 1(i), we get $(\bar{v} + \bar{v}) \in Z(N/P)$ for all $\bar{v} \in N/P$. Therefore, replacing $\bar{v}$ by $\bar{s}\bar{v}$ in the last relation and using it with the Lemma 1(i), we conclude that N/P is a commutative ring.
(iii) Assume that

$$\overline{d(u \circ v)} + \overline{(u \circ v)} \in Z(N/P), \quad \forall \ u, v \in N. \tag{21}$$

Substituting uv for v in (21), we have

$$\begin{aligned}\overline{d(u(u \circ v))} + \overline{u(u \circ v)} &= \overline{d(u)}\ \overline{d(u \circ v)} + \overline{d(u)}\overline{(u \circ v)} + \bar{u}\overline{d(u \circ v)} + \bar{u}\overline{(u \circ v)} = \\ &= \overline{d(u)}(\overline{d(u \circ v)} + \overline{(u \circ v)}) + \bar{u}(\overline{d(u \circ v)} + \overline{(u \circ v)}) \in Z(N/P).\end{aligned}$$

Applying (21) in above expression, we get $(\overline{d(u \circ v)} + \overline{(u \circ v)})(\overline{d(u) + u}) \in Z(N/P)$ for all $u, v \in N$. By Lemma 1, we deduce that

$$\overline{d(u \circ v)} + \overline{(u \circ v)} = \bar{0} \ \text{ or } \ \overline{d(u) + u} \in Z(N/P), \quad \forall \ u \in N. \tag{22}$$

Assume that there exists an element $u_0 \in N$ such that $\overline{d(u_0 \circ v)} + \overline{(u_0 \circ v)} = \bar{0}$ for all $v \in N$. Therefore, by recurrence, we can find

$$\overline{d^l(u_0 \circ v)} + (-1)^{l+1}\overline{(u_0 \circ v)} = \bar{0}, \quad \forall \ v \in N, \ l \in \mathbb{N}^*. \tag{23}$$

Since d is zero power valued on N, then there exists an integer $l(u \circ v) > 1$ such that $\overline{d^{l(u_0 \circ v)}(u \circ v)} = \bar{0}$ for all $v \in N$. Putting $l(u \circ v)$ in place of l in (23), we arrive at $\overline{u_0 \circ v} = \bar{0}$ for all $v \in N$, which implies that $\bar{v}\bar{u_0} = -\bar{u_0}\bar{v}$ for all $\bar{v} \in N/P$. Replacing $\bar{v}$ by $\bar{v}\bar{r}$ in the last relation, we obtain $\bar{v}\bar{r}\bar{u_0} = -\bar{u_0}\bar{v}\bar{r} = (\bar{u_0}\bar{v})(-\bar{r}) = (-\bar{v}\bar{u_0})(-\bar{r}) = (\bar{v}(-\bar{u_0}))(-\bar{r})$ for all $\bar{r}, \bar{v} \in N/P$, which implies that $N/P(-\bar{r}(-\bar{u_0}) + (-\bar{u_0})\bar{r}) = \{\bar{0}\}$. By the primeness of N/P, we get $-\bar{u_0} \in Z(N/P)$, thus $\bar{u_0} \in Z(N/P)$, which shows that N/P is a commutative ring. Now, considering the later from (22) and arguing in the similar manner as we have done after (18), we can conclude the result.

(iv) By hypothesis

$$\overline{d(uv)} + \overline{uv} \in Z(N/P), \quad \forall\, \bar{u}, \bar{v} \in N/P. \tag{24}$$

Replacing $\bar{v}$ by $\bar{u}\bar{v}$ in (24) and developing the proof with the Lemma 1, we arrive at

$$\overline{d(uv)} + \overline{uv} = \bar{0} \ \text{ or } \ \overline{d(u) + u} \in Z(N/P), \quad \forall\, \bar{u}, \bar{v} \in N/P. \tag{25}$$

Assume that there exists an element $\bar{u_0} \in N/P$ such that $\overline{d(u_0 v)} + \overline{u_0 v} = \bar{0}$ for all $\bar{v} \in N/P$. By recurrence, we can find $\overline{d^l(u_0 v)} + (-1)^{l+1}\overline{u_0 v} = \bar{0}$ for all $\bar{v} \in N/P$. Since d is zero power valued on N, then there must exists an integer $l(u_0 v) > 1$ such that $\overline{d^{l(u_0 v)}(u_0 v)} = \bar{0}$ for all $v \in N$. Putting $l(u_0 v)$ in place of l in the preceding relation, we deduce that $\bar{u_0}\bar{v} = \bar{0}$, i.e., $\bar{u_0} N/P \bar{u_0} = \{\bar{0}\}$. The primeness of P yields that $\bar{u_0} = \bar{0}$, a contradiction. Therefore, (25) leads to $\overline{d(u) + u} \in Z(N/P)$ for all $u \in N$, which forces that N/P is commutative ring. $\square$

The following example demonstrates that the primeness hypothesis is essential for our results.

Example 3. Let S be a zero-symmetric left near ring. Define N, P by

$$N = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & w & 0 \end{pmatrix} \,\middle|\, 0, u, v, w \in S \right\}; \quad P = \left\{ \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, 0, u \in S \right\}.$$

It is easy to check that P is an ideal of near ring N which is not prime. Define $d\colon N \to N$ by

$$d \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & w & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that d is a P-homoderivation on N satisfying the hypothesis of the theorems but N/P is not commutative.

3. Conclusion. Finally, we conclude our work by posing two crucial queries: (i) One pertinent direction is to explore whether analogous conclusions can be derived by substituting homoderivation with generalized homoderivation or multiplicative generalized homoderivation or (α, β) homoderivations? (ii) A significant interest is to examine the validity of our principal findings persists when the underlying structure extend for semiprime quotient near rings.

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