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ON THE NUMBER OF g-DIMONOIDS OF SMALL ORDER

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We study g-dimonoids, that is, algebraic structures endowed with two associative binary operations satisfying a specified system of axioms. The paper investigates duality and isomorphisms of g-dimonoids and provides a complete characterization of all rectangular commutative g-dimonoids. Several new examples are constructed, including commutative iso-dual g-dimonoids that are not abelian, and noncommutative nonabelian rectangular g-dimonoids that are not dimonoids. A complete classification of all two-element g-dimonoids is obtained. Finally, we present the results of computer computations determining the numbers of all pairwise nonisomorphic g-dimonoids of orders up to 5, as well as all pairwise nonisomorphic commutative, abelian, and rectangular g-dimonoids of orders up to 6, obtained using `GAP`, `Python`, and `C++`.

Introduction. The concept of a dialgebra was introduced by Jean-Louis Loday [16] in his search for a class of (linear) algebras that generate Leibniz algebras in a manner analogous to how associative algebras give rise to Lie algebras. Recall that a Leibniz algebra is a linear algebra over a field whose bracket operation $[,]$ satisfies the Leibniz derivation identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]],$$

without necessarily being anticommutative.

Loday's idea was to "separate" the left and right multiplications, treating them as distinct associative operations, denoted $\vdash$ and $\dashv$. He observed that if these operations satisfy the following three axioms:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \qquad (D_1)$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \qquad (D_2)$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \qquad (D_3)$$

then the bracket defined by $[x, y] = x \dashv y - y \vdash x$ satisfies the Leibniz identity.

Accordingly, Loday defined a *dimonoid* [16] as an algebraic structure $(D, \dashv, \vdash)$ consisting of a set D equipped with two associative binary operations $\dashv$ and $\vdash$ satisfying axioms (D_1), (D_2), and (D_3). Since dialgebras are the linear analogues of dimonoids, many results on dimonoids have direct applications in the theory of dialgebras [2,4,16,17,25]. T. Pirashvili [21] introduced the concept of a duplex, a generalization of dimonoids, and constructed the free duplex. The properties of free dimonoids were employed in [16] to characterize free dialgebras and to study their cohomologies. In [15], the notion of a dimonoid was used to define and investigate one-sided dirings. Furthermore, dimonoids are closely related to restrictive bisemigroups [24] and doppelsemigroups [5, 8, 9, 27, 29, 33].

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In [1,5–11,13], several classes of dimonoids, doppelsemigroups, and extensions of doppelsemigroups were constructed and studied. Among numerous results, these works provide isomorphism criteria for the corresponding classes, structural classifications of objects of small order, descriptions of their automorphism groups, and enumerations of pairwise nonisomorphic structures of small order.

In [18], the notion of a *generalized dimonoid* (or *g-dimonoid*) was introduced as an algebraic structure $(D, \dashv, \vdash)$ consisting of a set D endowed with two associative binary operations $\dashv$ and $\vdash$ that satisfy the axioms (D_1) and (D_3). The construction of the free g-dimonoid was also described therein. Every semigroup $(D, \dashv)$ can naturally be regarded as a g-dimonoid $(D, \dashv, \dashv)$, referred to as the *trivial g-dimonoid*. In this sense, g-dimonoids constitute a genuine generalization of semigroups.

An *associative* 0*-dialgebra* [22], that is, a linear space over a field equipped with two associative binary operations $\dashv$ and $\vdash$ satisfying the same axioms (D_1) and (D_3), can be viewed as a linear analogue of a g-dimonoid. Given the central role of dimonoids and g-dimonoids in the study of Leibniz algebras, dialgebras, and associative 0-dialgebras, their investigation by semigroup-theoretic methods constitutes a natural and promising line of research.

In [31], a free n-nilpotent g-dimonoid was constructed, the least n-nilpotent congruence on a free g-dimonoid was examined, and a characterization of a free g-dimonoid was obtained. The construction of the free commutative g-dimonoid together with the description of the least commutative congruence on a free g-dimonoid was provided in [26]. Furthermore, Yu. Zhuchok [32] determined all isomorphisms between the endomorphism semigroups of free commutative g-dimonoids and proved that every automorphism of such an endomorphism semigroup is quasi-inner. In [19], Cayley-type theorems for g-dimonoids were proved using left and right actions of sets and the concept of a dialgebra. More recently, A. Zhuchok [30] described the least dimonoid congruence and the least semigroup congruence on the free (commutative, n-nilpotent) g-dimonoid. In [12], a construction of g-dimonoids based on inflations of semigroups was introduced. In addition, several classes of 2-dinilpotent commutative g-dimonoids were established, isomorphism criteria for these classes were obtained, and the dimonoids among them were characterized.

The present paper investigates duality and isomorphisms of g-dimonoids. We introduce several new classes of g-dimonoids and determine their automorphism groups and halos. Based on these constructions, a complete classification, up to isomorphism, of all two-element g-dimonoids is obtained. Moreover, all rectangular commutative g-dimonoids are fully characterized. Furthermore, we construct examples of commutative iso-dual g-dimonoids that are nonabelian, as well as noncommutative nonabelian rectangular g-dimonoids that are not dimonoids. Finally, computer-assisted calculations yield the numbers of all pairwise nonisomorphic g-dimonoids of orders up to 5, and all pairwise nonisomorphic commutative, abelian, and rectangular g-dimonoids of orders up to 6.

1. Preliminaries on semigroups. An element e of a semigroup $(S, *)$ is called *a left identity* (resp. *a right identity*) in S if $e * a = a$ (resp. $a * e = a$) for any $a \in S$. An element 1 is an *identity* if it is both a left and a right identity. An element e of a semigroup $(S, *)$ is called *a middle identity* in S if $\ell * e * r = \ell * r$ for all $\ell, r \in S$. It is clear that each left (right) identity is a middle identity.

An element e of a semigroup $(S, *)$ is called an *idempotent* if $e * e = e$. The semigroup is a *band*, if all its elements are idempotents. Commutative bands are called *semilattices*. By

L_n we denote the *linear semilattice* $\{0, 1, \dots, n-1\}$ of order n, endowed with the operation of minimum.

An element z of a semigroup S is called *a left zero* (resp. *a right zero*) in S if $z * a = z$ (resp. $a * z = z$) for any $a \in S$. An element 0 is called a *zero* if it is both a left and a right zero.

Let $(S, *)$ be a semigroup and $0 \notin S$. The binary operation $*$ defined on S can be extended to $S \cup \{0\}$ putting $0 * s = s * 0 = 0$ for all $s \in S \cup \{0\}$. The notation $(S, *)^{+0}$ denotes a semigroup $(S \cup \{0\}, *)$ obtained from $(S, *)$ by adjoining the extra zero 0 (regardless of whether $(S, *)$ has a zero).

A semigroup $(S, *)$ is called a *null semigroup* if there exists an element $0 \in S$ such that $x*y = 0$ for all $x, y \in S$. In this case 0 is a zero of S. By O_{S^0} we denote a null semigroup with zero 0 on a set S. The null semigroups O_{S^0} and O_{T^z} are isomorphic if and only if $|S| = |T|$. If S is a set of cardinality $|S| = n$, we use the notation O_n for a representative of the class of semigroups isomorphic to O_{S^0}.

If $(S, *)$ is a semigroup, then the semigroup $(S, *^d)$ with operation $x *^d y = y * x$ is called *dual* to $(S, *)$, denoted $(S, *)^d$. It follows that $(S, *)^d = (S, *)$ if and only if $(S, *)$ is a commutative semigroup, and $\mathrm{Aut}\,(S, *)^d = \mathrm{Aut}\,(S, *)$.

A semigroup $(S, *)$ is said to be a *left* (resp. *right*) *zero semigroup* if $a * b = a$ (resp. $a * b = b$) for any $a, b \in S$. By LO_S and RO_S we denote a left zero and a right zero semigroup on a set S, respectively. If S is a set of cardinality $|S| = n$, we use the notations LO_n and RO_n for representatives of the classes of semigroups isomorphic to LO_S and RO_S, respectively.

A semigroup $(S, *)$ is called *rectangular* [10,28] if every element of S is a middle identity, that is, $x * y * z = x * z$ for all $x, y, z \in S$. In other words, the product in S depends only on the first and the last factors. A nontrivial null semigroup is an example of a commutative rectangular semigroup that has no identity element. The semigroups LO_S and RO_S are dual rectangular bands. In [20], it was proved that a semigroup $(S, *)$ is rectangular if and only if the factor semigroup S/θ is a right zero semigroup, where $\theta = \{(a, b) \in S \times S : x * a = x * b \text{ for all } x \in S\}$. The authors also described a method for constructing all rectangular semigroups. In [10], the study focused on rectangular semigroups, including rectangular ideal extensions of left (right) zero semigroups by null quotients, and provided new combinatorial results for the numbers of pairwise nonisomorphic instances.

Let S be a nonempty set, $A \subseteq S$ a nonempty subset, and $a \in A$. Define an associative binary operation $*$ on S as follows:

$$x * y = \begin{cases} x & \text{if } x \in A; \\ a & \text{if } x \notin A. \end{cases}$$

We denote the semigroup $(S, *)$ by $LO_{A_a \leftarrow S}$. It follows that all elements $z \in A$ are left zeros of $LO_{A_a \leftarrow S}$. If $A = \{a\}$, then $LO_{A_a \leftarrow S}$ coincides with a null semigroup O_{S^a} with zero a. If $A = S$, then $LO_{A_a \leftarrow S}$ coincides with a left zero semigroup LO_S. The semigroups $LO_{A_a \leftarrow S}$ and $LO_{B_b \leftarrow T}$ are isomorphic if and only if $|S| = |T|$ and $|A| = |B|$. If S is a finite set of cardinality $|S| = n$ and $|A| = m$, we use the notation $LO_{m \leftarrow n}$ for a representative of the class of semigroups isomorphic to $LO_{A_a \leftarrow S}$.

By $RO_{A_a \leftarrow S}$ we denote a dual semigroup of $LO_{A_a \leftarrow S}$ and use the notation $RO_{m \leftarrow n}$ accordingly. Both $LO_{A_a \leftarrow S}$ and $RO_{A_a \leftarrow S}$ are rectangular semigroups. Each of them is a rectangular band precisely when $A = S$.

Following the algebraic tradition, we take for a model of the class of cyclic groups of order n the multiplicative group $C_n = \{z \in \mathbb{C} : z^n = 1\}$ of n-th roots of 1. For a set X, we denote by S_X the group of all bijections of X.

2. Some definitions and basic properties of g-dimonoids. In this section, we recall several useful results on dimonoids that will be used in the subsequent investigations.

An element e of a g-dimonoid $(D, \dashv, \vdash)$ is called a *bar-unit* if $e \vdash d = d = d \dashv e$ for all $d \in D$. In contrast to monoids a g-dimonoid may have many bar-units. The set of all bar-units of a g-dimonoid $(D, \dashv, \vdash)$ is called the *halo* of $(D, \dashv, \vdash)$, denoted $\mathrm{Halo}\,(D, \dashv, \vdash)$. A nonempty subset $B \subset D$ is called a *g-subdimonoid* [31] of a g-dimonoid $(D, \dashv, \vdash)$ if $a \dashv b, a \vdash b \in B$ for all $a, b \in B$. If the halo of $(D, \dashv, \vdash)$ is nonempty, then it is a g-subdimonoid of $(D, \dashv, \vdash)$.

An element $0 \in D$ is called a *zero of a g-dimonoid* $(D, \dashv, \vdash)$ [31] if 0 is a zero of $(D, \dashv)$ and a zero of $(D, \vdash)$. Let $(D, \dashv, \vdash)$ be a g-dimonoid and $0 \notin D$. The binary operations defined on D can be extended to $D \cup \{0\}$ putting $0 \dashv d = d \dashv 0 = 0 = 0 \vdash d = d \vdash 0$ for all $d \in D \cup \{0\}$. The notation $(D, \dashv, \vdash)^{+0}$ denotes a g-dimonoid $D \cup \{0\}$ obtained from D by adjoining the extra zero 0. It follows that $\mathrm{Halo}\,((D, \dashv, \vdash)^{+0}) = \mathrm{Halo}\,(D, \dashv, \vdash)$.

The axioms of a g-dimonoid imply the following proposition.

Proposition 1. *Let $(D, \dashv, \vdash)$ be a g-dimonoid. If the semigroup $(D, \dashv)$ contains a left identity or the semigroup $(D, \vdash)$ contains a right identity, then the operations of a g-dimonoid $(D, \dashv, \vdash)$ coincide.*

Let $(D, \dashv, \vdash)$ be a g-dimonoid. Define new operations $\dashv^d$ and $\vdash^d$ on D by

$$x \dashv^d y = y \vdash x \quad \text{and} \quad x \vdash^d y = y \dashv x.$$

It is immediate to check that $(D, \dashv^d, \vdash^d)$ is a new g-dimonoid, called the *dual g-dimonoid of* $(D, \dashv, \vdash)$, which we denote by $(D, \dashv, \vdash)^d$. It follows that the unary duality operation $d\colon (D, \dashv, \vdash) \mapsto (D, \dashv, \vdash)^d$ is involutive in the sense that $((D, \dashv, \vdash)^d)^d = (D, \dashv, \vdash)$. In fact, $(D, \dashv, \vdash)^d$ is a g-dimonoid if and only if $(D, \dashv, \vdash)$ is a g-dimonoid. As usual, a g-dimonoid $(D, \dashv, \vdash)$ is said to be *self-dual* if $(D, \dashv, \vdash)^d = (D, \dashv, \vdash)$.

Remark 1. Observe that if we put $x \dashv' y = y \dashv x$ and $x \vdash' y = y \vdash x$, then, in general, the structure $(D, \dashv', \vdash')$ is not a g-dimonoid. Indeed, consider the left-zero and right-zero dimonoid (and hence a g-dimonoid) $(D, \dashv, \vdash)$, where the operations are given by $x \dashv y = x$ and $x \vdash y = y$ (see [16]). Taking into account that $x \dashv' y = y \dashv x = y$ and $x \vdash' y = y \vdash x = x$, we conclude that $(x \dashv' y) \dashv' z = z$ while $x \dashv' (y \vdash' z) = x \dashv' y = y$ for all $x, y, z \in D$. Hence, the axiom (D_1) fails, and consequently $(D, \dashv', \vdash')$ can not be a g-dimonoid.

A g-dimonoid $(D, \dashv, \vdash)$ is called *abelian* if $x \dashv y = y \vdash x$ for all $x, y \in D$.

Proposition 2. *Let $(D, \dashv, \vdash)$ be a g-dimonoid. Then the following conditions are equivalent:*

1) $(D, \dashv)$ and $(D, \vdash)$ are dual semigroups;

2) $(D, \dashv, \vdash)$ is abelian;

3) $(D, \dashv, \vdash)$ is self-dual.

Proof. (1) $\Rightarrow$ (2) If $(D, \dashv)$ and $(D, \vdash)$ are dual semigroups, then $x \dashv y = y \vdash x$ for all $x, y \in D$, and hence $(D, \dashv, \vdash)$ is an abelian g-dimonoid.

(2) $\Rightarrow$ (3) Let $(D, \dashv, \vdash)$ be an abelian g-dimonoid. Taking into account that

$$x \dashv^d y = y \vdash x = x \dashv y \quad \text{and} \quad x \vdash^d y = y \dashv x = x \vdash y,$$

we conclude that $\dashv^d = \dashv$ and $\vdash^d = \vdash$, and hence $(D, \dashv, \vdash)$ is a self-dual g-dimonoid.

(3) $\Rightarrow$ (1) If $(D, \dashv, \vdash)$ is a self-dual g-dimonoid, then $x \dashv y = x \dashv^d y = y \vdash x$ for all $x, y \in D$, and hence $(D, \dashv)$ and $(D, \vdash)$ are dual semigroups. $\square$

Corollary 1. *The class of nonabelian g-dimonoids decomposes into pairs of dual g-dimonoids.*

The definition of the halo of a g-dimonoid and Proposition 2 imply the following corollary.

Corollary 2. *Let $(D, \dashv, \vdash)$ be an abelian g-dimonoid. If the halo of $(D, \dashv, \vdash)$ is nonempty, then it coincides with the subsemigroup of all right identities of a semigroup $(D, \dashv)$, and the subsemigroup of all left identities of a semigroup $(D, \vdash)$.*

A g-dimonoid $(D, \dashv, \vdash)$ is called *commutative* [26] if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative.

Since commutative semigroups $(D, \dashv)$ and $(D, \vdash)$ are dual if and only if their operations coincide, Proposition 2 implies the following corollaries.

Corollary 3. *Commutative nontrivial g-dimonoids are nonabelian.*

Corollary 4. *Abelian nontrivial g-dimonoids are noncommutative.*

A g-dimonoid $(D, \dashv, \vdash)$ is said to be *rectangular* if both $(D, \dashv)$ and $(D, \vdash)$ are rectangular semigroups.

The left-zero and right-zero dimonoid is an example of a noncommuative abelian rectangular g-dimonoid. Theorem 1 gives examples of nonabelian commuative rectangular g-dimonoids that are not dimonoids.

Observing that a semigroup $(S, *)$ is rectangular if and only if its dual is rectangular, we obtain the following proposition.

Proposition 3. *Let $(D, \dashv, \vdash)$ be a g-dimonoid. Then $(D, \dashv, \vdash)$ is rectangular if and only if $(D, \dashv, \vdash)^d$ is rectangular.*

For a g-dimonoid $(D, \dashv, \vdash)$, if $\mathbb{S}$ and $\mathbb{T}$ denote the semigroups $(D, \dashv)$ and $(D, \vdash)$, respectively, then $\mathbb{S} \int \mathbb{T}$ stands for the g-dimonoid $(D, \dashv, \vdash)$.

3. Isomorphisms of g-dimonoids. A map $\varphi\colon D_1 \to D_2$ is called a *homomorphism* from a g-dimonoid $(D_1, \dashv_1, \vdash_1)$ to a g-dimonoid $(D_2, \dashv_2, \vdash_2)$ [32] if
$$\varphi(a \dashv_1 b) = \varphi(a) \dashv_2 \varphi(b) \quad \text{and} \quad \varphi(a \vdash_1 b) = \varphi(a) \vdash_2 \varphi(b)$$
for all $a, b \in D_1$.

A bijective homomorphism $\psi\colon D_1 \to D_2$ is called an *isomorphism* from a g-dimonoid $(D_1, \dashv_1, \vdash_1)$ to a g-dimonoid $(D_2, \dashv_2, \vdash_2)$. If there exists an isomorphism from a g-dimonoid $(D_1, \dashv_1, \vdash_1)$ to a g-dimonoid $(D_2, \dashv_2, \vdash_2)$, then $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are said to be *isomorphic*, denoted $(D_1, \dashv_1, \vdash_1) \cong (D_2, \dashv_2, \vdash_2)$. An isomorphism $\psi\colon D \to D$ is called an *automorphism* of a g-dimonoid $(D, \dashv, \vdash)$. By $\mathrm{Aut}\,(D, \dashv, \vdash)$ we denote the automorphism group of a g-dimonoid $(D, \dashv, \vdash)$. It follows that $\mathrm{Aut}\,((D, \dashv, \vdash)^{+0}) \cong \mathrm{Aut}\,(D, \dashv, \vdash)$.

Proposition 4. *Let $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ be g-dimonoids. Then for an arbitrary map $\varphi\colon D_1 \to D_2$ the following conditions are equivalent:*

1) φ is a homomorphism from $(D_1, \dashv_1, \vdash_1)$ to $(D_2, \dashv_2, \vdash_2)$;

2) φ is a homomorphism from $(D_1, \dashv_1, \vdash_1)^d$ to $(D_2, \dashv_2, \vdash_2)^d$.

Proof. (1) $\Rightarrow$ (2) Let φ be a homomorphism from $(D_1, \dashv_1, \vdash_1)$ to $(D_2, \dashv_2, \vdash_2)$. Observing that

$$\varphi(x \dashv_1^d y) = \varphi(y \vdash_1 x) = \varphi(y) \vdash_2 \varphi(x) = \varphi(x) \dashv_2^d \varphi(y) \text{ and}$$
$$\varphi(x \vdash_1^d y) = \varphi(y \dashv_1 x) = \varphi(y) \dashv_2 \varphi(x) = \varphi(x) \vdash_2^d \varphi(y),$$

for all $x, y \in D_1$, we deduce that φ is a homomorphism from $(D_1, \dashv_1, \vdash_1)^d$ to $(D_2, \dashv_2, \vdash_2)^d$.

(2) $\Rightarrow$ (1) Let φ be a homomorphism from $(D_1, \dashv_1, \vdash_1)^d$ to $(D_2, \dashv_2, \vdash_2)^d$. Noting that

$$\varphi(x \dashv_1 y) = \varphi(y \vdash_1^d x) = \varphi(y) \vdash_2^d \varphi(x) = \varphi(x) \dashv_2 \varphi(y) \text{ and}$$
$$\varphi(x \vdash_1 y) = \varphi(y \dashv_1^d x) = \varphi(y) \dashv_2^d \varphi(x) = \varphi(x) \vdash_2 \varphi(y),$$

for all $x, y \in D_1$, we conclude that φ is a homomorphism from $(D_1, \dashv_1, \vdash_1)$ to $(D_2, \dashv_2, \vdash_2)$. □

Corollary 5. *Let $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ be g-dimonoids. Then $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic if and only if $(D_1, \dashv_1, \vdash_1)^d$ and $(D_2, \dashv_2, \vdash_2)^d$ are isomorphic.*

Corollary 6. *Let $(D, \dashv, \vdash)$ be a g-dimonoid. Then* $\mathrm{Aut}\,(D, \dashv, \vdash)^d = \mathrm{Aut}\,(D, \dashv, \vdash)$.

Proposition 5. *Let $(D, \dashv, \vdash)$ be a g-dimonoid. Then* $\mathrm{Halo}\,(D, \dashv, \vdash)^d = \mathrm{Halo}\,(D, \dashv, \vdash)$. *Moreover, $(D, \dashv, \vdash)$ is commutative if and only if $(D, \dashv, \vdash)^d$ is commutative.*

Proof. Let e be a bar-unit of the g-dimonoid $(D, \dashv, \vdash)$. Then $e \vdash d = d = d \dashv e$ for all $d \in D$. Since $e \vdash^d d = d \dashv e = d = e \vdash d = d \dashv^d e$ for all $d \in D$, it follows that e is a bar-unit of the g-dimonoid $(D, \dashv, \vdash)^d$ as well. Using involutivity of the unary duality operation, we conclude that $\mathrm{Halo}\,(D, \dashv, \vdash)^d = \mathrm{Halo}\,(D, \dashv, \vdash)$.

Since commutativity of an operation $\dashv$ is equivalent to commutativity of an operation $\vdash^d$ and commutativity of an operation $\vdash$ is equivalent to commutativity of an operation $\dashv^d$, we conclude that $(D, \dashv, \vdash)$ is commutative if and only if $(D, \dashv, \vdash)^d$ is commutative. □

Proposition 6. *Let $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ be g-dimonoids such that and $(D_1, \dashv_1)$ and $(D_2, \dashv_2)$ are left zero semigroups. The g-dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic if and only if the semigroups $(D_1, \vdash_1)$ and $(D_2, \vdash_2)$ are isomorphic.*

Proof. It is immediate to observe that if the g-dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic, then the semigroups $(D_1, \vdash_1)$ and $(D_2, \vdash_2)$ are also isomorphic. Conversely, let $\psi\colon D_1 \to D_2$ be an isomorphism from the semigroup $(D_1, \vdash_1)$ to the semigroup $(D_2, \vdash_2)$. Then, necessarily, $|D_1| = |D_2|$. Taking into account that any bijective map is an isomorphism from the left zero semigroup $(D_1, \dashv_1)$ to the left zero semigroup $(D_2, \dashv_2)$, it follows that ψ is also an isomorphism from the left zero semigroup $(D_1, \dashv_1)$ to the left zero semigroup $(D_2, \dashv_2)$. Therefore, ψ is an isomorphism from the g-dimonoid $(D_1, \dashv_1, \vdash_1)$ to the g-dimonoid $(D_2, \dashv_2, \vdash_2)$. □

Corollary 7. *Let $(D, \dashv, \vdash)$ be a g-dimonoid such that and $(D, \dashv)$ is a left zero semigroup. Then* $\mathrm{Aut}\,(D, \dashv, \vdash) = \mathrm{Aut}\,(D, \vdash)$.

Dually, one can prove the following proposition.

Proposition 7. *Let $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ be g-dimonoids such that and $(D_1, \vdash_1)$ and $(D_2, \vdash_2)$ are right zero semigroups. The g-dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic if and only if the semigroups $(D_1, \dashv_1)$ and $(D_2, \dashv_2)$ are isomorphic.*

Proposition 8. *Let $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ be abelian g-dimonoids. The g-dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic if and only if the semigroups $(D_1, \dashv_1)$ and $(D_2, \dashv_2)$ are isomorphic.*

Proof. It is immediate to show that if the g-dimonoids $(D_1, \dashv_1, \vdash_1)$ and $(D_2, \dashv_2, \vdash_2)$ are isomorphic, then the semigroups $(D_1, \dashv_1)$ and $(D_2, \dashv_2)$ are isomorphic as well. Conversely, let $\psi\colon D_1 \to D_2$ be an isomorphism from the semigroup $(D_1, \dashv_1)$ to the semigroup $(D_2, \dashv_2)$. Since $\psi(x \vdash_1 y) = \psi(y \dashv_1 x) = \psi(y) \dashv_2 \psi(x) = \psi(x) \vdash_2 \psi(y)$ for all $x, y \in D_1$, it follows that $\psi\colon D_1 \to D_2$ is an isomorphism from the g-dimonoid $(D_1, \dashv_1, \vdash_1)$ to the g-dimonoid $(D_2, \dashv_2, \vdash_2)$. $\square$

Corollary 8. *Let $(D, \dashv, \vdash)$ be an abelian g-dimonoid. Then* $\mathrm{Aut}\,(D, \dashv, \vdash) = \mathrm{Aut}\,(D, \dashv) = \mathrm{Aut}\,(D, \vdash)$.

4. Commutative rectangular g-dimonoids. In this section, we describe, up to isomorphism, all commutative rectangular g-dimonoids and calculate their halos and automorphism groups.

The following proposition is an immediate consequence of the definitions.

Proposition 9. *Each commutative rectangular semigroup S is a null semigroup.*

We define a g-dimonoid $(D, \dashv, \vdash)$ to be *iso-dual* if it is isomorphic to its dual g-dimonoid $(D, \dashv, \vdash)^d$. It is evident that every abelian (self-dual) g-dimonoid is iso-dual. Theorem 1 provides an example of an iso-dual g-dimonoid that is not abelian.

Theorem 1. *Let D be a set, and $0 \in D$, $a \in D$. An algebraic structure $O_{D^0} \int O_{D^a} = (D, \dashv, \vdash)$, where $(D, \dashv)$ and $(D, \vdash)$ are null semigroups with zeros 0 and a, respectively, is an iso-dual commutative rectangular g-dimonoid with* $\mathrm{Aut}\,(O_{D^0} \int O_{D^a}) \cong S_{D\setminus\{0,a\}}$. *Moreover, if $0 \neq a$, the g-dimonoid $O_{D^0} \int O_{D^a}$ is neither abelian nor a dimonoid, and if $|D| > 1$, it has an empty halo.*

Proof. It follows directly from the definition of the g-dimonoid operations that, for all $x, y, z \in D$, $(x \dashv y) \dashv z = 0 = x \dashv (y \vdash z)$ and $(x \dashv y) \vdash z = a = x \vdash (y \vdash z)$. Therefore, $(D, \dashv, \vdash)$ is a g-dimonoid.

On the other hand, if $0 \neq a$, then

$$0 \dashv a = 0 \neq a = a \vdash 0 \quad \text{and} \quad (x \vdash y) \dashv z = 0 \neq a = x \vdash (y \dashv z),$$

and hence the g-dimonoid $(D, \dashv, \vdash)$ is neither abelian nor a dimonoid.

Since both $(D, \dashv)$ and $(D, \vdash)$ are commutative (rectangular) semigroups, $(D, \dashv, \vdash)$ is a commutative (rectangular) g-dimonoid as well.

Consider its dual g-dimonoid $O_{D^a} \int O_{D^0} = (D, \dashv^d, \vdash^d)$, where $(D, \dashv^d)$ is a null semigroup with zero a and $(D, \vdash^d)$ is a null semigroup with zero 0. Then the bijective map $\psi : D \to D$ defined by $\psi(a) = 0$, $\psi(0) = a$, and $\psi(x) = x$ for all $D \setminus \{a, 0\}$, is isomorphism from $(D, \dashv, \vdash)$ to $(D, \dashv^d, \vdash^d)$. Indeed, for all $x, y \in D$,

$$\psi(x \dashv y) = \psi(0) = a = \psi(x) \dashv^d \psi(y) \quad \text{and} \quad \psi(x \vdash y) = \psi(a) = 0 = \psi(x) \vdash^d \psi(y).$$

Since for $|D| > 1$ the semigroup $(D, \dashv)$ contains no right identities, the halo of the g-dimonoid $O_{D^0} \int O_{D^a}$ is empty.

Let ψ be an arbitrary automorphism of the g-dimonoid $(D, \dashv, \vdash)$. Then ψ is an automorphism of the semigroup $(D, \dashv)$ and ψ is an automorphism of the semigroup $(D, \vdash)$. Since automorphisms preserves zeros and 0 is a zero $(D, \dashv)$, a is a zero $(D, \vdash)$, it follows that $\psi(0) = 0$ and $\psi(a) = a$.

On the other hand, let f be any bijection of D such that $f(0) = 0$ and $f(a) = a$. Then for all $x, y \in D$,

$$f(x \dashv y) = f(0) = 0 = f(x) \dashv f(y) \quad \text{and} \quad f(x \vdash y) = f(a) = a = f(x) \vdash f(y).$$

It follows that any bijection of D that preserves 0 and a generates an automorphism of the g-dimonoid $O_{D^0} \int O_{D^a}$. Therefore, $\operatorname{Aut}(O_{D^0} \int O_{D^a}) \cong S_{D\setminus\{0,a\}}$. □

Proposition 10. *Let D be a set, and let $0 \in D$. If $\{a, b\} \subset D \setminus \{0\}$, then the g-dimonoids $O_{D^0} \int O_{D^a}$ and $O_{D^0} \int O_{D^b}$ are isomorphic while the g-dimonoids $O_{D^0} \int O_{D^a}$ and $O_{D^0} \int O_{D^0} = O_{D^0}$ are not isomorphic, where for $p \in \{0, a, b\}$ the semigroup O_{D^p} is a null semigroup with zero p.*

Proof. One may verify that the dimonoids $O_{D^0} \int O_{D^a}$ and $O_{D^0} \int O_{D^b}$ are isomorphic via the bijection $\psi \colon D \to D$ defined by $\psi(a) = b$, $\psi(b) = a$, and $\psi(x) = x$ for all $x \notin \{a, b\}$.

To prove that the dimonoids $O_{D^0} \int O_{D^0}$ and $O_{D^0} \int O_{D^a}$ are not isomorphic, suppose, to the contrary, that $\phi \colon O_{D^0} \int O_{D^0} \to O_{D^0} \int O_{D^a}$ is an isomorphism. Then ϕ is an automorphism of the semigroup O_{D^0} and ϕ is an isomorphism from the semigroup O_{D^0} to the semigroup O_{D^a}. Since automorphisms preserve zeros, and 0 is zero of O_{D^0}, we obtain $\phi(0) = 0$. However, as isomorphisms preserve zeros and 0 and a are the zeros of O_{D^0} and O_{D^a}, respectively, we would have $\phi(0) = a$, leading to a contradiction. □

Taking into account that the null semigroups O_{S^0} and O_{T^z} are isomorphic if and only if $|S| = |T|$, together with Propositions 9 and 10 and Theorem 1, we obtain the following characterization theorem.

Theorem 2. *Let $\kappa > 1$ be a cardinal, and let D be a set of cardinality $|D| = \kappa$ with distinct elements $0, z \in D$. Up to isomorphism, there exist exactly two commutative rectangular g-dimonoids of order κ: the trivial one O_{D^0} with $\operatorname{Aut}(O_{D^0}) \cong S_{D\setminus\{0\}}$, and the nonabelian iso-dual nontrivial g-dimonoid $O_{D^0} \int O_{D^z}$ with $\operatorname{Aut}(O_{D^0} \int O_{D^z}) \cong S_{D\setminus\{0,z\}}$, which is not a dimonoid.*

5. Some classes of noncommutative rectangular g-dimonoids. This section is devoted to constructing noncommutative rectangular dimonoids.

Proposition 11. *Let $(D, \dashv)$ be a left zero semigroup and $(D, \vdash)$ be an arbitrary semigroup. For an algebraic structure $(D, \dashv, \vdash)$ the following conditions are equivalent:*

1) *$(D, \dashv, \vdash)$ is a g-dimonoid;*
2) *$(D, \dashv, \vdash)$ is a dimonoid;*
3) *$(D, \vdash)$ is a rectangular semigroup.*

Proof. (1) $\Rightarrow$ (2) Since $(D, \dashv)$ is a left zero semigroup, axiom (D_2) holds:

$$(x \vdash y) \dashv z = x \vdash y = x \vdash (y \dashv z),$$

and therefore $(D, \dashv, \vdash)$ is a dimonoid.

(2) $\Rightarrow$ (3) Given that $(D, \dashv)$ is a left zero semigroup, axiom (D_3) yields

$$x \vdash (y \vdash z) = (x \dashv y) \vdash z = x \vdash z,$$

for all $x, y, z \in D$. Consequently, $(D, \vdash)$ is a rectangular semigroup.
(3) $\Rightarrow$ (1) Since $(D, \dashv)$ is a left zero semigroup,

$$x \dashv (y \dashv z) = x = x \dashv (y \vdash z)$$

for all $x, y, z \in D$, so axiom (D_1) holds.

Moreover, taking into account that $(D, \vdash)$ is a rectangular semigroup, we conclude that

$$(x \dashv y) \vdash z = x \vdash z = x \vdash (y \vdash z)$$

for any $x, y, z \in D$, and hence axiom (D_3) is also satisfied. Consequently, $(D, \dashv, \vdash)$ is a g-dimonoid. $\square$

Dually, one can prove the following proposition.

Proposition 12. *Let $(D, \dashv)$ be an arbitrary semigroup and $(D, \vdash)$ be a right zero semigroup. For an algebraic structure $(D, \dashv, \vdash)$ the following conditions are equivalent:*

1) $(D, \dashv, \vdash)$ is a g-dimonoid;

2) $(D, \dashv, \vdash)$ is a dimonoid;

3) $(D, \dashv)$ is a rectangular semigroup.

Since the semigroups $LO_{A_a \leftarrow S}$ and $RO_{A_a \leftarrow S}$ are rectangular, Propositions 11 and 12 imply that the rectangular g-dimonoids $LO_D \int LO_{A_a \leftarrow D}$, $LO_D \int RO_{A_a \leftarrow D}$, $LO_{A_a \leftarrow D} \int RO_D$, and $RO_{A_a \leftarrow D} \int RO_D$ are also rectangular dimonoids. These g-dimonoids were introduced and studied in [6].

Proposition 13. *Let $(D, \dashv)$ be a semigroup and $(D, \vdash)$ be a null semigroup with zero 0. An algebraic structure $(D, \dashv, \vdash)$ is a g-dimonoid if and only if $x \dashv y \dashv z = x \dashv 0$ for all $x, y, z \in D$.*

Proof. Since $(D, \vdash)$ is a null semigroup, the axiom (D_3) holds:

$$(x \dashv y) \vdash z = 0 = x \vdash (y \vdash z).$$

Taking into account that $x \dashv (y \vdash z) = x \dashv 0$ for any $x, y, z \in D$, we conclude that axiom (D_1) holds if and only if $x \dashv y \dashv z = x \dashv 0$ for any $x, y, z \in D$. $\square$

The following theorem provides an example of noncommutative rectangular g-dimonoid which is not a dimonoid.

Theorem 3. *Let A be a nonempty proper subset of a set D such that $a \in A$ and $0 \in D \setminus A$. Then $LO_{A_a \leftarrow D} \int O_{D^0} = (D, \dashv, \vdash)$, where $(D, \vdash)$ is a null semigroup with zero 0 and*

$$x \dashv y = \begin{cases} x & \text{if } x \in A; \\ a & \text{if } x \in D \setminus A, \end{cases}$$

is a nonabelian rectangular g-dimonoid which is not a dimonoid, and $\mathrm{Aut}\,(LO_{A_a \leftarrow D} \int O_{D^0}) \cong S_{A\setminus\{a\}} \times S_{D\setminus(A\cup\{0\})}$, $\mathrm{Halo}\,(LO_{A_a \leftarrow D} \int O_{D^0}) = \varnothing$*. Moreover, if* $|A| > 1$*, then* $LO_{A_a \leftarrow D} \int O_D$ *is a noncommutative g-dimonoid.*

Proof. Using the definition of the semigroup $LO_{A_a \leftarrow D}$, it is immediate to check that $x \dashv y \dashv z = x \dashv 0$ for all $x, y, z \in D$. So, Proposition 13 implies that $LO_{A_a \leftarrow D} \int O_{D^0}$ is a g-dimonoid. Taking into account that O_{D^0} and $LO_{A_a \leftarrow D}$ are rectangular semigroups, we conclude that the g-dimonoid $LO_{A_a \leftarrow D} \int O_{D^0}$ is rectangular as well. Since $0 \vdash 0 = 0 \neq a = 0 \dashv 0$, it follows that the g-dimonoid $LO_{A_a \leftarrow D} \int O_{D^0}$ is not abelian. Furthermore, because $0 \vdash (0 \dashv 0) = 0 \notin A$ and $(0 \vdash 0) \dashv 0 = 0 \dashv 0 = a \in A$, axiom (D_2) fails to hold, and hence $(D, \dashv, \vdash)$ is not a dimonoid.

Let ψ be an arbitrary automorphism of the dimonoid $(D, \dashv, \vdash)$. Then ψ is an automorphism of the semigroup $(D, \dashv)$ and ψ is an automorphism of the semigroup $(D, \vdash)$. Since automorphisms preserve zeros and 0 is a zero of $(D, \vdash)$, we conclude that $\psi(0) = 0$. Taking into account that automorphisms preserves left zeros and A is a left zero subsemigroup of $(D, \dashv)$, we conclude that $\psi(A) = A$, and thus $\psi(D \setminus A) = D \setminus A$. Because $0 \notin A$, we obtain $\psi(a) = \psi(0 \dashv a) = \psi(0) \dashv \psi(a) = 0 \dashv \psi(a) = a$.

On the other hand, let f be any bijection of D such that $f(A) = A$, $f(a) = a$, and $f(0) = 0$. If $x \in A$, then $f(x) \in A$, and hence $f(x \dashv y) = f(x) = f(x) \dashv f(y)$ for all $y \in D$. In the case $x \in D\setminus A$ we have that $f(x) \in D\setminus A$, and thus $f(x \dashv y) = f(a) = a = f(x) \dashv f(y)$ for all $y \in D$. Moreover, $f(x \vdash y) = f(0) = 0 = f(x) \vdash f(y)$ for all $x, y \in D$. It follows that any bijection of A that preserves a and any bijection of $D \setminus A$ that preserves 0 generate an automorphism of $LO_{A_a \leftarrow D} \int O_{D^0}$. Therefore, $\mathrm{Aut}\,(LO_{A_a \leftarrow D} \int O_{D^0}) \cong S_{A\setminus\{a\}} \times S_{D\setminus(A\cup\{0\})}$.

For $|D| > 1$, the commutative semigroup O_{D^0} has no identity, and therefore $\mathrm{Halo}\,(LO_{A_a \leftarrow D} \int O_{D^0}) = \varnothing$.

Since in the case $|A| > 1$ the semigroup $LO_{A_a \leftarrow D}$ contains at least two left zeros, and hence it is not commutative, we conclude that the g-dimonoid $LO_{A_a \leftarrow D} \int O_{D^0}$ is not commutative as well. □

Dually, one can prove the following proposition and theorem.

Proposition 14. *Let $(D, \dashv)$ be a null semigroup with zero 0 and $(D, \vdash)$ be an arbitrary semigroup. An algebraic structure $(D, \dashv, \vdash)$ is a g-dimonoid if and only if $x \vdash y \vdash z = 0 \vdash z$ for all $x, y, z \in D$.*

Theorem 4. *Let A be a nonempty proper subset of a set D such that $a \in A$ and $0 \in D \setminus A$. Then $O_{D^0} \int RO_{A_a \leftarrow D} = (D, \dashv, \vdash)$, where $(D, \dashv)$ is a null semigroup with zero 0 and*

$$x \vdash y = \begin{cases} y & \text{if } y \in A; \\ a & \text{if } y \in D \setminus A, \end{cases}$$

is a nonabelian rectangular g-dimonoid which is not a dimonoid, and $\mathrm{Aut}\,(O_{D^0} \int RO_{A_a \leftarrow D}) \cong S_{A\setminus\{a\}} \times S_{D\setminus(A\cup\{0\})}$, $\mathrm{Halo}\,(O_{D^0} \int RO_{A_a \leftarrow D}) = \varnothing$*. Moreover, if* $|A| > 1$*, then* $O_{D^0} \int RO_{A_a \leftarrow D}$ *is a noncommutative g-dimonoid.*

6. Two-element g-dimonoids and their automorphism groups. In this section we describe, up to isomorphism, all two-element g-dimonoids and their automorphism groups.

It is well-known that there are exactly five pairwise nonisomorphic semigroups having two elements: the multiplicative cyclic group $C_2 = \{-1, 1\}$, the linear semilattice $L_2 = \{0, 1\}$ with min-operation, the null semigroup $O_2 = \{0, 1\}$ with zero 0, the left zero semigroup LO_2 with operation $xy = x$, and the right zero semigroup RO_2 with operation $xy = y$.

Theorem 5. *Up to isomorphism, there exist* 9 *two-element g-dimonoids, among which* 4 *dimonoids are abelian. Moreover, up to isomorphism, there are* 4 *commutative g-dimonoids of order* 2*: three abelian trivial g-dimonoids and one nonabelian nontrivial g-dimonoid. Furthermore, there are* 7 *rectangular g-dimonoids of order* 2*, including two commutative iso-dual g-dimonoids, two pairs of dual noncommutative nonabelian g-dimonoids and a single noncommutative abelian g-dimonoid. Additionally, there exist exactly* 5 *pairwise nonisomorphic two-element trivial g-dimonoids.*

Proof. In the sequel, we divide our investigation into cases. In the case of a semigroup $(\{a, b\}, *)$, we shall find all pairwise nonisomorphic dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv)$ is isomorphic to $(\{a, b\}, *)$.

Cases C_2 and L_2. According to Proposition 1, if a semigroup $(D, \dashv)$ possesses a left identity or a semigroup $(D, \vdash)$ possesses a right identity, then the operations of a g-dimonoid $(D, \dashv, \vdash)$ coincide. Therefore, up to isomorphism, there exist a unique g-dimonoid $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong C_2$ or $(D, \vdash) \cong C_2$, and this g-dimonoid is the trivial dimonoid C_2. Similarly, L_2 is a unique g-dimonoid in the class of g-dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong L_2$ or $(D, \vdash) \cong L_2$. The trivial g-dimonoids C_2 and L_2 are commutative and abelian.

Note, that in all the remaining cases, the semigroups LO_2, RO_2, and O_2 are rectangular.

Case LO_2. If $(D, \vdash) \cong RO_2$, then by Proposition 11 we obtain the noncommutative abelian rectangular g-dimonoid $LO_2 \int RO_2$. According to Proposition 8, $LO_2 \int RO_2$ is the unique g-dimonoid in the class of abelian g-dimonoids $(D, \dashv, \vdash)$ satisfying $(D, \dashv) \cong LO_2$ and $(D, \vdash) \cong RO_2$. It follows from Corollary 8 that $\mathrm{Aut}\,(LO_2 \int RO_2) = \mathrm{Aut}\,(LO_2) = S_2 \cong C_2$.

In the case $(D, \vdash) \cong O_2$, Proposition 11 yields the noncommutative nonabelian rectangular g-dimonoid $LO_2 \int O_2$. By Proposition 6, $LO_2 \int O_2$ is the unique g-dimonoid in the class of dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong LO_2$ and $(D, \vdash) \cong O_2$. Consequently, by Corollary 7, $\mathrm{Aut}\,(LO_2 \int O_2) = \mathrm{Aut}\,(O_2) \cong C_1$.

In the remaining case, we obtain the noncommutative nonabelian rectangular trivial g-dimonoid LO_2. By Proposition 6, LO_2 is the unique g-dimonoid in the class of g-dimonoids $(D, \dashv, \vdash)$ satisfying $(D, \dashv) \cong (D, \vdash) \cong LO_2$.

Case RO_2. Each element of the semigroup RO_2 is a left identity. According to Proposition 1, if a semigroup $(D, \dashv)$ has a left identity, then the two operations of a g-dimonoid $(D, \dashv, \vdash)$ coincide. Hence, by Proposition 7, the trivial noncommutative nonabelian rectangular g-dimonoid RO_2 is the unique g-dimonoid in the class of g-dimonoids $(D, \dashv, \vdash)$ with $(D, \dashv) \cong RO_2$. This dimonoid is dual to LO_2.

Case O_2. If $(D, \vdash)$ is a right zero semigroup, then Proposition 12 yields the noncommutative nonabelian rectangular g-dimonoid $O_2 \int RO_2$, which is unique in the class of g-dimonoids $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong O_2$ and $(D, \vdash) \cong RO_2$, in accordance with Proposition 7. This g-dimonoid is dual to the g-dimonoid $LO_2 \int O_2$, and hence by Corollary 6, $\mathrm{Aut}\,(O_2 \int RO_2) = \mathrm{Aut}\,(LO_2 \int O_2) \cong C_1$.

According to Proposition 1, if $(D, \vdash)$ contains a right identity, then the operations of a g-dimonoid $(D, \dashv, \vdash)$ coincide. Since each element of the semigroup LO_2 is a right identity, there does not exist a g-dimonoid $(D, \dashv, \vdash)$ such that $(D, \dashv) \cong O_2$ and $(D, \vdash) \cong LO_2$.

In the final case, we obtain by Theorem 2 the abelian commutative regular trivial g-dimonoid O_2 and the nonabelian commutative regular iso-dual nontrivial g-dimonoid $O_{D^a} \int O_{D^b}$ with $\mathrm{Aut}\,(O_{D^a} \int O_{D^b}) \cong C_1$. □

Table 1 lists all pairwise nonisomorphic two-element g-dimonoids and their corresponding automorphism groups.

D	C_2	L_2	O_2	LO_2	RO_2	$LO_2 \smallint RO_2$	$LO_2 \smallint O_2$	$O_2 \smallint RO_2$	$O_{D^a} \smallint O_{D^b}$
$\mathrm{Aut}(D)$	C_1	C_1	C_1	C_2	C_2	C_2	C_1	C_1	C_1

Table 1: Nonisomorphic 2-element g-dimonoids and their automorphism groups.

7. Number of g-dimonoids of small order. The determination of the numbers $\mathrm{s}(n)$ and $\mathrm{cs}(n)$, representing respectively the counts of all pairwise nonisomorphic semigroups of order n and all pairwise nonisomorphic commutative semigroups of order n, is a difficult combinatorial problem. The functions $\mathrm{s}(n)$ and $\mathrm{cs}(n)$ grow very rapidly as n tends to infinity. The sequences $(\mathrm{s}(n))$ and $(\mathrm{cs}(n))$ are listed in the On-Line Encyclopedia of Integer Sequences as entries A027851 and A001426, respectively. All currently known exact values of these sequences are presented in Tables 2 and 3.

n	1	2	3	4	5	6	7	8	9
$\mathrm{s}(n)$	1	5	24	188	1915	28634	1627672	3684030417	105978177936292

Table 2: Number of nonisomorphic semigroups up to order 9.

n	1	2	3	4	5	6	7	8	9	10
$\mathrm{cs}(n)$	1	3	12	58	325	2143	17291	221805	11545843	3518930337

Table 3: Number of nonisomorphic commutative semigroups up to order 10.

Denote by $\mathrm{gdm}(n)$, $\mathrm{cgdm}(n)$, $\mathrm{agdm}(n)$, and $\mathrm{rgdm}(n)$ the number of all pairwise nonisomorphic g-dimonoids, commutative g-dimonoids, abelian g-dimonoids, and rectangular g-dimonoids of order n, respectively. We were able to calculate these cardinalities for small n. The `Python` code used for these computations is publicly available at `https://gist.github.com/vgavrylkiv/32700f9f64e7e7d81a79f676716b3e64`. This code processes the Cayley tables of all nonisomorphic semigroups of order n obtained with `GAP` using the `Smallsemi` library of semigroups of small order. The corresponding tables were relabeled from $\{1, \ldots, n\}$ to $\{0, \ldots, n-1\}$ for computational convenience and exported to `csv`-files. The code loads the Cayley tables from `csv`-files, generates all their permutations, and checks pairwise combinations for compliance with the system of g-dimonoid axioms. It then eliminates isomorphic cases, retaining only representatives of isomorphism classes, and produces a complete list of pairwise nonisomorphic g-dimonoids of a given order n. The results are saved as operation tables for $\dashv$ and $\vdash$ into a single `csv`-file for further analysis. For $n = 6$, we translated this code from `Python` into `C++` and performed parallel computations.

The results of (computer) calculations are presented in Tables 4–7.

n	1	2	3	4	5
$\mathrm{gdm}(n)$	1	9	78	1693	162465

Table 4: Number of nonisomorphic g-dimonoids up to order 5.

n	1	2	3	4	5	6
$\mathrm{cgdm}(n)$	1	4	22	189	3249	256253

Table 5: Number of nonisomorphic commutative g-dimonoids up to order 6.

Since a g-dimonoid is considered trivial when its two operations coincide, the numbers of all pairwise nonisomorphic nontrivial g-dimonoids and pairwise nonisomorphic commutative nontrivial g-dimonoids of order n are equal to $\mathrm{gdm}(n) - \mathrm{s}(n)$ and $\mathrm{cgdm}(n) - \mathrm{cs}(n)$, respectively.

n	1	2	3	4	5	6
$\mathrm{agdm}(n)$	1	4	17	103	791	10870

Table 6: Number of nonisomorphic abelian g-dimonoids up to order 6.

Since a commutative g-dimonoid is abelian if and only if it is trivial, the number of all pairwise nonisomorphic trivial abelian g-dimonoids of order n is equal to $\mathrm{cs}(n)$, and hence the numbers of all pairwise nonisomorphic nonabelian commutative g-dimonoids and pairwise nonisomorphic noncommutative abelian dimonoids of order n are $\mathrm{cgdm}(n) - \mathrm{cs}(n)$ and $\mathrm{agdm}(n) - \mathrm{cs}(n)$, respectively.

n	1	2	3	4	5	6
$\mathrm{rgdm}(n)$	1	7	27	128	555	2846
$\mathrm{rs}(n)$	1	3	5	10	14	27

Table 7: Number of nonisomorphic rectangular g-dimonoids and semigroups up to order 6.

Table 7 also lists the numbers $\mathrm{rs}(n)$ of all pairwise nonisomorphic rectangular semigroups for $n \leq 6$. Hence, the number of pairwise nonisomorphic rectangular nontrivial g-dimonoids of order n is given by $\mathrm{rgdm}(n) - \mathrm{rs}(n)$.

According to Theorem 2, the numbers of pairwise nonisomorphic noncommutative rectangular g-dimonoids and pairwise nonisomorphic noncommutative rectangular nontrivial g-dimonoids of order $n > 1$ are equal to $\mathrm{rgdm}(n) - 2$ and $\mathrm{rgdm}(n) - \mathrm{rs}(n) - 1$, respectively.

Problem 1. *Determine the numbers* $\mathrm{gdm}(n)$ *for* $n \geq 6$ *and* $\mathrm{cgdm}(n)$, $\mathrm{agdm}(n)$, $\mathrm{rgdm}(n)$ *for* $n \geq 7$.

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