Математичні Студії. Т.65, №2 Matematychni Studii. V.65, No.2
ISSN 1027-4634 (print) ISSN 2411-0620 (online)

O. T. KHOLYAVKA, O. M. BUHRII, T. M. BOKALO, M. M. BOKALO

UNIQUE SOLVABILITY OF INITIAL-BOUNDARY-VALUE PROBLEM FOR SECOND ORDER EQUATIONS OF KIRCHHOFF TYPE WITH VARIABLE EXPONENTS OF NONLINEARITY

O. T. Kholyavka[1], O. M. Buhrii[2], T. M. Bokalo[3], M. M. Bokalo[4]. *Unique solvability of initial-boundary-value problem for second order equations of Kirchhoff type with variable exponents of nonlinearity*, Mat. Stud. **65** (2026), 169–181.

The paper devoted to investigation strong solutions of the initial-boundary-value problem for some equations of the Kirchhoff type with avariable exponent of nonlinearity. As we know the equations of the Kirchhoff type of the second order with variable exponents of the nonlinearity are not studied yet. Problems for nonlinear partial differential equations with variable exponents of the nonlinearity are investigated in the generalized Lebesgue and Sobolev spaces.

In present paper we investigate strong solutions of the initial-boundary value problem for these equations. This article continues the research which starts in the paper *Adv. Math. Sci. App.,* **23** *(2013), 509–528*, where we investigated the equation with strong damping. We found the sufficient conditions of the existence and uniqueness of the strong solution to given problem. The proof is based on the Faedo-Galerkin method, the derivation of a priori energy estimates, and the use the embedding results for Lebesgue spaces with variable exponent of nonlinearity. The obtained results can be applied to further studies of global solvability and long-time behavior of solutions for our problem.

1. Introduction. A typical example of the equations being studied here is

$$u_{tt} - \Big(\alpha + \beta \int_0^\ell |u_x(y,t)|^2\, dy\Big)^\gamma u_{xx} + g|u|^{p(x)-2}u = 0, \quad (x,t) \in (0,\ell)\times(0,T), \qquad (1)$$

where $\alpha, \beta, T, \ell, \gamma, g$ are positive numbers, p is some function such that $p(x) > 1$ for all $x \in (0,\ell)$. The function p is called variable exponent of the nonlinearity for equation (1).

The study of problems for Kirchhoff-type equations with variable exponents of nonlinearity is significantly complicated by the combination of a nonlocal operator and nonlinearity with a variable degree. For a correct investigation of such problems, it is natural to employ Lebesgue and Sobolev spaces with variable exponents, which provide an appropriate functional framework for formulating and describing the obtained results. Therefore, problems for nonlinear partial differential equations with variable exponents of the nonlinearity are investigated in some special classes of the functions namely in the generalized Lebesgue and Sobolev spaces (see [6], [20]).

2020 *Mathematics Subject Classification:* 35L75, 35L35, 35D30, 35D35.
Keywords: nonlinear equation; Kirchhoff equation; generalized Lebesgue and Sobolev spaces; variable exponents of the nonlinearity.
doi:10.30970/ms.65.2.169-181

*Corresponding author: O. Kholyavka

In the case $\gamma = 0$, type (1) equations are considered in [14], [18] etc. For $\gamma > 0$ and $g = 0$, equation (1) was considered by G. Kirchhoff in [11] as the mathematical model which describes the process of the small transverse vibration of the string better then the linear equation $u_{tt} - u_{xx} = 0$ (it takes into account the change of the string length). Problems for the equations of the Kirchhoff type with constant exponents of nonlinearity were investigated in [3], [9], [17], [22], [23] etc.

In present paper we consider the equation of type (1) with variable exponent of the nonlinearity. We investigate strong solutions of the initial-boundary value problem for these equations. This article continues the research which starts in [8], where we investigate the equation with strong damping, which corresponds to equation (1). As we know the equations of the Kirchhoff type of the second order with variable exponents of the nonlinearity are not studied yet. We found sufficient conditions of the existence and uniqueness of the strong solution to given problem. The paper is organized as follows. In Section 2, we formulate the considered problem and main results. The auxiliary statements are given in Section 3. Finally, in Section 4 we prove the main statements.

2. Statement of problem and formulation of main results. Let $n \in \mathbb{N}$ and $T > 0$ be some numbers, $\Omega \subset \mathbb{R}^n$ be a bounded domain with the smooth enough boundary $\partial\Omega$. Put $Q_\tau := \Omega \times (0, \tau)$, $\Sigma_\tau := \partial\Omega \times (0, \tau)$ and $\Omega_\tau := \{(x, \tau) \mid x \in \Omega\}$ for $\tau > 0$.

We investigate the initial-boundary value problem for nonlinear equations of the Kirchhoff type with variable exponents of the nonlinearity:

$$u_{tt} - M\Big(\int_\Omega |\nabla u(y,t)|^2\, dy\Big)\, \Delta u + g(x,t)|u|^{p(x)-2}u = f(x,t), \quad (x,t) \in Q_T, \tag{2}$$

$$u|_{\Sigma_T} = 0, \tag{3}$$

$$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \quad x \in \Omega, \tag{4}$$

where M, p, g, f, u_0, u_1 are given functions, u is an unknown function.

Here we consider strong solutions to problem (2)–(4), and for their definitions and investigation we need the following functional spaces and notations.

For every $r \in L^\infty(\Omega)$, we denote $r_- := \operatorname*{ess\,inf}_{x\in\Omega} r(x)$, $r_+ := \operatorname*{ess\,sup}_{x\in\Omega} r(x)$. By definition, put $L^\infty_{>\omega}(\Omega) := \{r \in L^\infty(\Omega) \,|\, r_- > \omega\}$, $L^\infty_{\geqslant\omega}(\Omega) := \{r \in L^\infty(\Omega) \,|\, r_- \geqslant \omega\}$. If $r \in L^\infty_{>1}(\Omega)$, then we take $r'(x) := \frac{r(x)}{r(x)-1}$ for a.e. $x \in \Omega$.

Let $L^q(\Omega)$, $W_0^{m,q}(\Omega)$, $W^{m,q}(\Omega)$, where $m \in \mathbb{N}$, $q \in [1, \infty]$, be the standard Lebesgue and Sobolev spaces respectively (see [1, p. 22, 45]). Put $H_0^1(\Omega) := W_0^{1,2}(\Omega)$, $H^2(\Omega) := W^{2,2}(\Omega)$. Suppose that $r \in L^\infty_{\geqslant 1}(\Omega)$. The linear space of all classes of equivalent measurable functions $v\colon \Omega \to \mathbb{R}$ such that $\rho_r(v, \Omega) < \infty$, where $\rho_r(v, \Omega) := \int_\Omega |v(x)|^{r(x)}\, dx$, is called *a generalized Lebesgue space* and is denoted by $L^{r(\cdot)}(\Omega)$. The space $L^{r(\cdot)}(\Omega)$ is a Banach space with respect to the norm $\|v\|_{L^{r(\cdot)}(\Omega)} := \inf\big\{\lambda > 0 \,|\, \rho_r(v/\lambda, \Omega) \leqslant 1\big\}$. Similarly we define the generalized Lebesgue space $L^{r(\cdot)}(Q_\tau)$ for any $\tau > 0$. The properties of these spaces were widely studied in [6].

Throughout the paper we assume that the date-in of problem (2)–(4) satisfies the following conditions:

(A$_1$): $M \in C^1([0 + \infty))$, $M(s) \geqslant m_0 = \text{const} > 0$ for every $s \geqslant 0$;

(A$_2$): $p \in L^\infty(\Omega)$, $\operatorname*{ess\,inf}_{x\in\Omega} p(x) \geqslant 2$; $g \in L^\infty(Q_T)$, $\operatorname*{ess\,inf}_{(x,t)\in Q_T} g(x,t) \geqslant 0$;

(A$_3$): $f \in L^2(Q_T)$; $u_0 \in H_0^1(\Omega)$; $u_1 \in L^2(\Omega)$.

A strong (local) solution to problem (2)–(4) is called a function $u\colon Q_{\widehat{T}} \to \mathbb{R}$, where $\widehat{T} \in (0,T]$ is some number, if

$$u \in C([0,\widehat{T}]; H_0^1(\Omega)) \cap L^\infty(0,\widehat{T}; H^2(\Omega)) \cap L^{p(\cdot)}(Q_{\widehat{T}}), \tag{5}$$

$$u_t \in C([0,\widehat{T}]; L^2(\Omega)) \cap L^\infty(0,\widehat{T}; H_0^1(\Omega)), \quad u_{tt} \in L^2(Q_{\widehat{T}}), \tag{6}$$

u satisfies equation (2) and conditions (4) almost everywhere.

According to our notation, $p_- = \operatorname*{ess\,inf}_{x\in\Omega} p(x)$, $p_+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x)$. For every $n \in \mathbb{N}$, $n \geqslant 3$, we put

$$\widehat{p}(n) := \begin{cases} 2 + \frac{4}{n-2}, & \text{if } 3 \leqslant n \leqslant 5; \\ 2 + \frac{2}{n-4}, & \text{if } n \geqslant 6. \end{cases} \tag{7}$$

Theorem 1 (uniqueness of the strong solution)**.** *Suppose that*

(B): *$p_- \geqslant 2$, and, if $n \geqslant 3$, then $p_+ \leqslant \widehat{p}(n)$.*

Then if $u\colon Q_{T_1} \to \mathbb{R}$ and $\widetilde{u}\colon Q_{T_2} \to \mathbb{R}$ are strong (local) solutions to problem (2)–(4), then $u = \widetilde{u}$ a.e. on $Q_{\min\{T_1,T_2\}}$.

Theorem 2 (existence of the strong solution)**.** *Suppose that $\partial\Omega \in C^2$ and the following hold:*

(C$_1$): *$p \in C^1(\overline{\Omega})$, $p_- \geqslant 2$, and, if $n \geqslant 3$, then $p_+ < 2 + \min\{\frac{1}{2}, \frac{2}{n-2}\}$;*

(C$_2$): *$g \in W^{1,\infty}(Q_T)$ and either $g(x,t) \geqslant g_- = \text{const} > 0$ for a. e. $(x,t) \in Q_T$, or $g(x,t) \geqslant 0$ and $g_t(x,t) \leqslant 0$ for a. e. $(x,t) \in Q_T$;*

(C$_3$): *$u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$, $f \in L^2(0,T; H_0^1(\Omega))$.*

Then for some $\widehat{T} \in (0,T]$ problem (2)–(4) has a solution defined on $Q_{\widehat{T}}$.

3. Auxiliary statements. By definition, put $(v,w)_{L^2(\Omega)} := \int_\Omega v(x)w(x)\,dx$, $|v|_{L^2(\Omega)} := \sqrt{(v,v)_{L^2(\Omega)}}$. We consider the space $H_0^1(\Omega)$ with respect to the norm $\|w\| := |\nabla w|_{L^2(\Omega)} \equiv \left(\int_\Omega |\nabla w|^2\,dx\right)^{1/2}$. For the sake of simplicity, let $\int_{\Omega_t} v\,dx$ be $\int_\Omega v(x,t)\,dx$ for arbitrary $t \geqslant 0$ and $v \in C(\overline{Q_T})$. Let for every Banach spaces X and Y the notation $X \circlearrowleft Y$ means that X is continuous embedded in Y. Let X' is a dual to X.

The following statements are needed for the sequel.

Proposition 1 ([1], p. 97)**.** *The following hold:*

(i) if $1 \leqslant n \leqslant 2$, then $H_0^1(\Omega) \circlearrowleft L^q(\Omega)$ for each $q \in [1;\infty)$, and, if $n \geqslant 3$, then $H_0^1(\Omega) \circlearrowleft L^q(\Omega)$ for each $q \in [1; \frac{2n}{n-2}]$;

(ii) if $1 \leqslant n \leqslant 4$, then $H^2(\Omega) \circlearrowleft L^q(\Omega)$ for each $q \in [1;\infty)$, and, if $n \geqslant 5$, then $H^2(\Omega) \circlearrowleft L^q(\Omega)$ for each $q \in [1; \frac{2n}{n-4}]$.

Remark 1. A similar proposition as the Proposition 1 is correct with the replacement of $H_0^1(\Omega)$ by $H^1(\Omega)$ in the case of $\partial\Omega \in C^2$.

Proposition 2 ([12], p. 599–600)**.** *The following hold:*
(i) if $r \in L^{\infty}_{>1}(\Omega)$, *then* $L^{r(\cdot)}(\Omega)$ *is a reflexive space, and* $(L^{r(\cdot)}(\Omega))' \cong L^{r'(\cdot)}(\Omega)$.
(ii) if $r, q \in L^{\infty}_{\geqslant 1}(\Omega)$ *and* $r(x) \geqslant q(x)$ *for a.e.* $x \in \Omega$, *then* $L^{r(\cdot)}(\Omega) \circlearrowleft L^{q(\cdot)}(\Omega)$.

Remark 2. Suppose that the function p satisfies condition $(\mathbf{C}_1)$ of Theorem 2. First let us consider the case $n \geqslant 3$. Then condition $(\mathbf{C}_1)$ yields that the inequality $2(p_+ - 2) < \min\{1, \frac{4}{n-2}\}$ holds. Take $\varkappa \in \big(2(p_+ - 2), \min\{1, \frac{4}{n-2}\}\big)$. Since $\varkappa < \frac{4}{n-2}$, we have $2 + \varkappa < \frac{2n}{n-2}$ and item (i) of Proposition 1 implies that

$$H^1_0(\Omega) \circlearrowleft L^{2+\varkappa}(\Omega). \tag{8}$$

Similarly, it is proved (see Remark 1) that in the case of $\partial\Omega \in C^2$ the correct embedding is

$$H^1(\Omega) \circlearrowleft L^{2+\varkappa}(\Omega). \tag{9}$$

Since $\varkappa > 2(p_+ - 2)$, we get that $1 > 2(p_+ - 2)\,\frac{1}{\varkappa}$, $2 + \varkappa > 2(p_+ - 2)\,\frac{2+\varkappa}{\varkappa}$, and so, using (8) and item (ii) of Proposition 2, we obtain

$$H^1_0(\Omega) \circlearrowleft L^{q(\cdot)}(\Omega), \quad \text{where} \;\; q(x) := \max\left\{1,\, 2(p(x) - 2)\frac{2+\varkappa}{\varkappa}\right\}, \; x \in \Omega. \tag{10}$$

Further, if either $n = 1$ or $n = 2$, then take arbitrary $\varkappa \in (0, 1)$. Then item (i) of Proposition 1 implies that (8) and (10) also hold.

Proposition 3 ([12], [15])**.** *Let* $r \in L^{\infty}_{\geqslant 1}(\Omega)$,

$$S_r(z) := \max\{z^{r_-}, z^{r_+}\}, \quad S_{1/r}(z) := \max\{z^{1/r_-}, z^{1/r_+}\}, \quad z \geqslant 0.$$

Then for arbitrary measurable function $v\colon \Omega \to \mathbb{R}$ *the following hold:*

(i) if $\rho_r(v, \Omega) < +\infty$, *then* $\|v\|_{L^{r(\cdot)}(\Omega)} \leqslant S_{1/r}(\rho_r(v, \Omega))$;

(ii) if $\|v\|_{L^{r(\cdot)}(\Omega)} < +\infty$, *then* $\rho_r(v, \Omega) \leqslant S_r(\|v\|_{L^{r(\cdot)}(\Omega)})$.

Proposition 4 ([8])**.** *Let* $q \in L^{\infty}(\Omega)$, $r \in L^{\infty}_{\geqslant 1}(\Omega)$, *and* $0 \leqslant q(x) \leqslant r(x)$. *Then there exist positive constants* C_1, C_2 *such that for every* $v \in L^{r(\cdot)}(\Omega)$ *we have*

$$\int_{\Omega} |v(x)|^{q(x)}\, dx \leqslant C_1 + C_2 \|v\|^{\max\{1, q_+\}}_{L^{r(\cdot)}(\Omega)}. \tag{11}$$

Proposition 5 ([16], part IV, §2, Lemma 3)**.** *If* $\partial\Omega \in C^2$, *then*

(i) if $v \in H^1_0(\Omega)$ *and* $\Delta v \in L^2(\Omega)$, *then* $v \in H^2(\Omega)$ *and*

$$\|v\|_{H^2(\Omega)} \leqslant C_3 \left(|v|_{L^2(\Omega)} + |\Delta v|_{L^2(\Omega)}\right), \tag{12}$$

where C_3 *is a some positive constant;*

(ii) there exist a sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ *of the positive numbers and a sequence* $\{w^j\}_{j\in\mathbb{N}} \subset C^2(\overline{\Omega})$ *of the linearly independent functions whose finite linear combinations are dense in the space* $H^2(\Omega) \cap H^1_0(\Omega)$, *and we have*

$$-\Delta w^j = \lambda_j\, w^j \quad \text{in} \quad \Omega, \quad j \in \mathbb{N}. \tag{13}$$

Remark 3. Suppose that $\mathcal{O} \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) is a open set, $z \in C^1(\overline{\mathcal{O}})$, $r \in C^1(\overline{\mathcal{O}}) \cap L^\infty_{>1}(\mathcal{O})$. Then, for each $j \in \{1, \dots, N\}$, we have

$$(|z|^{r-2}z)_{y_j} = (r-1)|z|^{r-2}z_{y_j} + |z|^{r-2}z \ln|z|\, r_{y_j} \quad \text{on} \quad \{y \in \mathcal{O} \mid |z(y)| > 0\}. \tag{14}$$

Indeed, let's put $h(u,v) := |u|^{v-2}u$, $u \neq 0, v > 1$. Then we have $\frac{\partial h(u,v)}{\partial u} = (v-1)|u|^{v-2}$, $\frac{\partial h(u,v)}{\partial v} = |u|^{v-2}u \ln|u|$. Therefore, by the rule of differentiation of the composition of functions of many variables, for function $|z|^{r-2}z := h(z,r)$ we obtain (14).

For the sake of convenience we shall write $u(t)$ instead of $u(\cdot,t)$ etc.

4. Proofs of main results.

Proof of Theorem 1. We use the method of proof by contradiction. Assume that $u\colon Q_{T_1} \to \mathbb{R}$ and $\widetilde{u}\colon Q_{T_2} \to \mathbb{R}$ be strong solutions to problem (2)–(4), $u \neq \widetilde{u}$ on positive measure subset of $Q_{\min\{T_1,T_2\}}$. Without loss of generality, it can be assumed that $T_1 = T_2 = T$. Denote by R the positive number such that

$$\max\{\, \|u\|_{L^\infty(0,T;H^2(\Omega))},\ \|\widetilde{u}\|_{L^\infty(0,T;H^2(\Omega))}, \|u_t\|_{L^\infty(0,T;H^1_0(\Omega))},\ \ \|\widetilde{u}_t\|_{L^\infty(0,T;H^1_0(\Omega))} \,\} \leqslant R. \tag{15}$$

Throughout the proof of Theorem 1, C_k, $k = 1, 2, 3, \dots$, denote the constants which may depend on R, but they do not depend on u, $\widetilde{u}$ directly. We also shall write p instead of $p(\cdot)$.

Note (see Propositions 1, 2 [8]) that

$$[H^2(\Omega) \cap H^1_0(\Omega)] \circlearrowleft H^1_0(\Omega) \circlearrowleft L^{p_+}(\Omega) \circlearrowleft L^p(\Omega) \circlearrowleft L^{p_-}(\Omega) \circlearrowleft L^2(\Omega). \tag{16}$$

Embeddings (16) imply that for every $n \in \mathbb{N}$ we have $L^\infty(0,T;H^1_0(\Omega)) \circlearrowleft L^{p_+}(Q_T) \circlearrowleft L^p(Q_T)$, and hence $u_t, \widetilde{u}_t \in L^p(Q_T)$.

Set $w := u - \widetilde{u}$, $\mu(t) := M(\|u(t)\|^2)$, $\widetilde{\mu}(t) := M(\|\widetilde{u}(t)\|^2)$, $t \in [0,T]$. Take $\tau \in (0,T]$. Then, it is easy to obtain the equality

$$\int_{Q_\tau} \Big[w_{tt}w_t - \mu\,\Delta w\, w_t \Big]\, dxdt = \int_{Q_\tau} \Big[(\mu - \widetilde{\mu})\,\Delta\widetilde{u}\, w_t + g(|\widetilde{u}|^{p-2}\widetilde{u} - |u|^{p-2}u)w_t \Big]\, dxdt. \tag{17}$$

Clearly, that

$$\int_{Q_\tau} w_{tt}w_t\, dxdt = \frac{1}{2}\int_{\Omega_\tau} |w_t|^2\, dx.$$

Similarly as in the proof of Theorem 1 ([8, §3]) we get

$$-\int_{Q_\tau} \mu\,\Delta w\, w_t\, dxdt \geqslant \frac{m_0}{2}\|w(\tau)\|^2 - \frac{C_4}{2}\int_0^\tau \|w(t)\|^2\, dt,$$

$$\int_{Q_\tau} (\mu - \widetilde{\mu})\,\Delta\widetilde{u}\, w_t\, dxdt \leqslant C_5 \int_0^\tau \Big(\|w(t)\|^2 + |w_t(t)|^2_{L^2(\Omega)} \Big)\, dt,$$

$$\int_{Q_\tau} g(|\widetilde{u}|^{p-2}\widetilde{u} - |u|^{p-2}u)w_t\, dxdt \leqslant C_6 \int_0^\tau \Big(\|w(t)\|^2 + |w_t(t)|^2_{L^2(\Omega)} \Big)\, dt.$$

Then, from equality (17) we have

$$|w_t(\tau)|^2_{L^2(\Omega)} + \|w(\tau)\|^2 \leqslant C_7 \int_0^\tau \Big(|w_t(t)|^2_{L^2(\Omega)} + \|w(t)\|^2 \Big)\, dt, \quad \tau \in [0,T].$$

The Gronwall Lemma yields $|w_t(\tau)|^2_{L^2(\Omega)} + \|w(\tau)\|^2 \leqslant 0$, hence $w_t = \nabla w = 0$ a.e. on Q_T, i.e. $w = \text{const}$ a.e. on Q_T. From condition (3) we get $w = 0$ a.e. on Q_T, i.e., $u = \widetilde{u}$ a.e. on Q_T. This contradiction proves Theorem 1. □

Proof of Theorem 2. The solution will be constructed via Faedo-Galerkin's method. Next, to reduce the entries, we will write p instead of $p(\cdot)$ correspondingly.

Step 1 (construction of the Faedo-Galerkin approximations). Let the set $\{w^j\}_{j\in\mathbb{N}}$ is taken from Proposition 5. Then $\{w^j\}_{j\in\mathbb{N}}$ is complete system in $H^2(\Omega)\cap H_0^1(\Omega)\cap L^p(\Omega)$ (see Proposition 5 and embeddings (16)). For each $m\in\mathbb{N}$ we take $\varphi_1^m,\ldots,\varphi_m^m$ such that the function

$$u^m(x,t)=\sum_{k=1}^m \varphi_k^m(t)w^k(x),\quad (x,t)\in Q_T, \tag{18}$$

satisfies the following equalities

$$\int_{\Omega_t}\Big[u_{tt}^m - M\Big(\|u^m(t)\|^2\Big)\Delta u^m + g|u^m|^{p-2}u^m\Big]w^j\,dx=\int_{\Omega_t} fw^j\,dx,\quad t\in(0,T),\quad j=\overline{1,m}, \tag{19}$$

$$u^m(0)=u_0^m,\qquad u_t^m(0)=u_1^m. \tag{20}$$

Here

$$u_0^m(x):=\sum_{k=1}^m \alpha_k^m w^k(x),\quad u_1^m(x):=\sum_{k=1}^m \widetilde{\alpha}_k^m w^k(x),\quad x\in\Omega,$$

where the numbers α_k^m, $\widetilde{\alpha}_k^m$ $(k=\overline{1,m})$ satisfy the following conditions:

$$u_0^m \underset{m\to\infty}{\longrightarrow} u_0 \quad\text{in}\quad H^2(\Omega)\cap H_0^1(\Omega),\qquad u_1^m \underset{m\to\infty}{\longrightarrow} u_1 \quad\text{in}\quad H_0^1(\Omega).$$

According to the theorems of the existence and extension of the solution to the Cauchy problem for the systems of the ordinary differential equations we obtain the existence the mentioned functions $\varphi_1^m,\ldots,\varphi_m^m$.

Let us show that the sequence $\{u^m\}_{m\in\mathbb{N}}$ converges to the solution of problem (2)–(4). Throughout the proof of Theorem 2, C_k, $k=1,2,3,\ldots$, denotes the constants which is independent of m.

Step 2 (first estimates). By definition, put

$$\mu_m(t):=M(\|u^m(t)\|^2),\quad t\in[0,T],\quad m\in\mathbb{N}. \tag{21}$$

Take $\tau\in(0,T]$. Multiplying the j-th equation of system (19) by the function $(\varphi_j^m(t))'$, summing by j from 1 to m, and integrating in $t\in[0,\tau]$, we get

$$\int_{Q_\tau}\Big[u_{tt}^m u_t^m-\mu_m\Delta u^m u_t^m+g|u^m|^{p-2}u^m u_t^m\Big]dxdt=\int_{Q_\tau} fu_t^m\,dxdt. \tag{22}$$

Clearly, that

$$\int_{Q_\tau} u_{tt}^m u_t^m\,dxdt=\frac12\int_\Omega |u_t^m(\tau)|^2\,dx-\frac12\int_\Omega |u_1^m|^2\,dx,$$

$$\int_{Q_\tau} fu_t^m\,dxdt\leqslant\frac12\int_{Q_\tau}|f|^2\,dxdt+\frac12\int_{Q_\tau}|u_t^m|^2\,dxdt. \tag{23}$$

Similarly as in the proof of Theorem 2 ([8, §3]) we get the estimates

$$-\int_{Q_\tau}\mu_m\Delta u^m u_t^m\,dxdt\geqslant\frac{m_0}{2}\int_{\Omega_\tau}|\nabla u^m|^2\,dx-\frac12\widetilde{M}\Big(\|u_0^m\|^2\Big), \tag{24}$$

where $\widetilde{M}(\xi) := \int_0^\xi M(s)\,ds,\ \ \xi \geqslant 0$.

Assume that the case $g(x,t) \geqslant g_- > 0$ of condition $(\mathbf{C}_2)$ is satisfied. Then

$$\int_{Q_\tau} g|u^m|^{p-2}u^m u_t^m\,dxdt \geqslant \frac{g_-}{p_+}\int_{\Omega_\tau}|u^m|^p\,dx - \frac{g_+}{p_-}\int_\Omega |u_0^m|^p\,dx - \frac{g_+^1}{p_-}\int_{Q_\tau}|u^m|^p\,dxdt, \qquad (25)$$

where $g_+ := \operatorname*{ess\,sup}_{(x,t)\in Q_T} g(x,t)$, $\ g_+^1 := \operatorname*{ess\,sup}_{(x,t)\in Q_T}|g_t(x,t)|$. Using (23)–(25), by (22) we obtain

$$\frac{1}{2}\int_{\Omega_\tau}|u_t^m|^2\,dx + \frac{g_-}{p_+}\int_{\Omega_\tau}|u^m|^p\,dx + \frac{m_0}{2}\int_{\Omega_\tau}|\nabla u^m|^2\,dx \leqslant \frac{1}{2}\int_\Omega |u_1^m|^2\,dx + \frac{1}{2}\widetilde{M}\Big(\|u_0^m\|^2\Big)+$$
$$+\frac{g_+}{p_-}\int_\Omega |u_0^m|^p\,dx + \frac{1}{2}\int_{Q_\tau}|f|^2\,dxdt + \frac{1}{2}\int_{Q_\tau}|u_t^m|^2\,dxdt + \frac{g_+^1}{p_-}\int_{Q_\tau}|u^m|^p\,dxdt. \qquad (26)$$

The Gronwall Lemma and (26) yield that

$$\int_{\Omega_\tau}[|u_t^m|^2 + |u^m|^p]\,dx \leqslant C_8, \quad \tau\in[0,T]. \qquad (27)$$

Combining (26), (27) and Proposition 3, we get

$$\max\Big\{\|u^m\|_{L^\infty(0,T;H_0^1(\Omega)\cap L^p(\Omega))}, \quad \|u_t^m\|_{L^\infty(0,T;L^2(\Omega))}\Big\} \leqslant C_9. \qquad (28)$$

Now consider the case $g(x,t) \geqslant 0$ and $g_t(x,t) \leqslant 0$ for a.e $(x,t)\in Q_T$ (see condition $(\mathbf{C}_2)$). Similarly as in the proof of Theorem 2 [8, §3] we obtain the estimates

$$\max\Big\{\|u^m\|_{L^\infty(0,T;H_0^1(\Omega))}, \quad \|u_t^m\|_{L^\infty(0,T;L^2(\Omega))}\Big\} \leqslant C_{10}, \qquad (29)$$

$$0 < m_0 \leqslant \mu_m(t) \leqslant C_{11}, \quad |\mu_m'(t)| \leqslant C_{12}\|u_t^m(t)\|, \quad t\in[0,T], \qquad (30)$$

where m_0 is taken from condition $(\mathbf{A}_1)$.

Step 3 (second estimates). Multiplying the j-th equation of system (19) by $\lambda_j(\varphi_j^m(t))'$, using (13), (18), and summing by j from 1 to m, we deduce

$$\int_{\Omega_t}\Big[u_{tt}^m - \mu_m(t)\Delta u^m + g|u^m|^{p-2}u^m\Big]\Big(-\Delta u_t^m\Big)dx = \int_{\Omega_t} f\Big(-\Delta u_t^m\Big)\,dx, \quad t\in[0,T]. \qquad (31)$$

It is obvious that

$$\int_{\Omega_t} u_{tt}^m\Big(-\Delta u_t^m\Big)\,dx = \frac{1}{2}\frac{d}{dt}\|u_t^m(t)\|^2, \qquad (32)$$

$$-\int_{\Omega_t}\mu_m(t)\Delta u^m\Big(-\Delta u_t^m\Big)\,dx = \frac{1}{2}\mu_m(t)\frac{d}{dt}|\Delta u^m(t)|^2_{L^2(\Omega)}, \qquad (33)$$

$$\int_{\Omega_t} f\Big(-\Delta u_t^m\Big)\,dx = \int_{\Omega_t}\nabla f\cdot\nabla u_t^m\,dx. \qquad (34)$$

By definition, put $D_t^m := \{x\in\Omega \mid u^m(x,t)=0\}$, $m\in\mathbb{N}$, $t\in[0,T]$. Then, using (14), we get

$$\int_{\Omega_t} g|u^m|^{p-2}u^m\Big(-\Delta u_t^m\Big)\,dx = \int_{\Omega_t}\nabla\Big(g|u^m|^{p-2}u^m\Big)\cdot\nabla u_t^m\,dx =$$

$$= \int_{\Omega_t} |u^m|^{p-2}u^m \, \nabla g \cdot \nabla u_t^m \, dx + \int_{\Omega_t\setminus D_t^m} g(p-1)|u^m|^{p-2} \, \nabla u^m \cdot \nabla u_t^m \, dx +$$

$$+ \int_{\Omega_t\setminus D_t^m} g\, |u^m|^{p-2}u^m \ln|u^m| \, \nabla p \cdot \nabla u_t^m \, dx =: J_1(t) + J_2(t) + J_3(t). \tag{35}$$

Clearly, that

$$|J_1(t)| \leqslant C_{13} \int_{\Omega_t} |u^m|^{p-1} \, |\nabla u_t^m| \, dx, \tag{36}$$

$$|J_2(t)| \leqslant g_+(p_+-1) \int_{\Omega_t} |u^m|^{p-2} \, |\nabla u^m| \, |\nabla u_t^m| \, dx. \tag{37}$$

By definition, put $B_t^m = \{x \in \Omega \setminus D_t^m \ \mid \ |u^m(x,t)| \leqslant 1\}$, $\widetilde{B_t^m} = (\Omega \setminus D_t^m) \setminus B_t^m$, $m \in \mathbb{N}$, $t \in [0,T]$. Since $z^\alpha \ln z \underset{z\to+0}{\longrightarrow} 0$ if $\alpha > 0$, then we have the estimate

$$\big|\, |u^m|^{p-2}u^m \ln|u^m| \,\big| \leqslant C_{14} \quad \text{a.e. on } \ B_t^m. \tag{38}$$

It is easy to see that for all $\alpha > 0$ and $y \in \mathbb{R}$, $|y| \geqslant 1$, we have

$$\ln|y| \leqslant \frac{1}{\alpha} \, |y|^\alpha. \tag{39}$$

Indeed, inequality $\ln z \leqslant z \ \forall\, z \geqslant 1$ yields (39) if $z = |y|^\alpha$.

Let's take any $\sigma \in (0, \frac{2n-2}{n-2} - p_+)$. In order to estimate the $J_3(t)$ let us use (38) for the integral over B_t^m, and (39) with $\alpha = \sigma$ for the integral over $\widetilde{B_t^m}$. Since $p \in C^1(\overline{\Omega})$, we get

$$|J_3(t)| \leqslant C_{14} g_+ \int_{B_t^m} |\nabla p| \, |\nabla u_t^m| \, dx + \frac{g_+}{\sigma} \int_{\widetilde{B_t^m}} |u^m|^{p-1} \, |u^m|^\sigma \, |\nabla p| \, |\nabla u_t^m| \, dx \leqslant$$

$$\leqslant \int_{\Omega_t} \Big[C_{15} + \frac{C_{16}}{\sigma} |u^m|^{p+\sigma-1} \Big] \, |\nabla u_t^m| \, dx. \tag{40}$$

Thus, from (35), (36), (37), (40), we have

$$\int_{\Omega_t} g|u^m|^{p-2}u^m \left(-\Delta u_t^m \right) dx \leqslant$$

$$\leqslant C_{17} \int_{\Omega_t} \Big[1 + \frac{1}{\sigma} |u^m|^{p+\sigma-1} + |u^m|^{p-1} + |u^m|^{p-2} |\nabla u^m| \Big] \, |\nabla u_t^m| \, dx. \tag{41}$$

Therefore, from (31), (34), (41) it follows that

$$\frac{1}{2} \frac{d}{dt} \, \|u_t^m(t)\|^2 + \frac{1}{2} \, \mu_m(t) \, \frac{d}{dt} \, |\Delta u^m(t)|^2_{L^2(\Omega)} \leqslant \int_{\Omega_t} \mathcal{F}_1 |\nabla u_t^m| \, dx, \tag{42}$$

where

$$\mathcal{F}_1(x,t) := C_{18} \Big(1 + |\nabla f(x,t)| + |u^m(x,t)|^{p+\sigma-1} + |u^m(x,t)|^{p-1} + |u^m(x,t)|^{p-2} \, |\nabla u^m(x,t)| \Big),$$

$(x,t) \in Q_T$. Using the Young inequality, we get

$$\mathcal{F}_1(x,t) \leqslant \mathcal{F}_2(x,t), \quad (x,t) \in Q_T, \tag{43}$$

where

$$\mathcal{F}_2(x,t) := C_{19}\Big(1 + |\nabla f(x,t) + |u^m(x,t)|^{p+\sigma-1} + |u^m(x,t)|^{p-2}\, |\nabla u^m(x,t)|\Big). \tag{44}$$

Setting

$$\gamma(t) := \|u^m_t(t)\|^2, \quad \beta(t) := |\Delta u^m(t)|^2_{L^2(\Omega)}, \quad t \in (0,T), \tag{45}$$

from (42)–(44) we obtain

$$\gamma'(t) + \mu_m(t)\beta'(t) \leqslant 2\int_{\Omega_t} \mathcal{F}_2|\nabla u^m_t|\, dx, \quad t \in (0,T). \tag{46}$$

Dividing (46) by $\mu_m(t) > 0$ (see (30)), we get

$$\beta'(t) + \frac{1}{\mu_m(t)}\gamma'(t) \leqslant \frac{2}{\mu_m(t)}\int_{\Omega_t} \mathcal{F}_2|\nabla u^m_t|\, dx. \tag{47}$$

Since

$$\frac{2}{\mu_m(t)}\int_{\Omega_t} \mathcal{F}_2|\nabla u^m_t|\, dx \leqslant \frac{2}{\mu_m(t)}\, |\mathcal{F}_2(t)|_{L^2(\Omega)}\, |\nabla u^m_t(t)|_{L^2(\Omega)} =$$

$$= \frac{2}{\mu_m(t)}\, |\mathcal{F}_2(t)|_{L^2(\Omega)}\sqrt{\gamma(t)} \leqslant |\mathcal{F}_2(t)|^2_{L^2(\Omega)} + \frac{\gamma(t)}{\mu^2_m(t)},$$

from (47) we get

$$\beta'(t) + \frac{1}{\mu_m(t)}\,\gamma'(t) \leqslant |\mathcal{F}_2(t)|^2_{L^2(\Omega)} + \frac{1}{\mu^2_m(t)}\gamma(t). \tag{48}$$

Setting

$$\eta(t) := \beta(t) + \frac{1}{\mu_m(t)}\,\gamma(t), \quad t \in (0,T), \tag{49}$$

we have $\eta'(t) = \beta'(t) + \frac{1}{\mu_m(t)}\,\gamma'(t) - \frac{\mu'_m(t)}{\mu^2_m(t)}\,\gamma(t)$. In view of (48) we obtain for $t \in (0,T)$

$$\eta'(t) \leqslant |\mathcal{F}_2(t)|^2_{L^2(\Omega)} + \frac{1}{\mu^2_m(t)}\,\gamma(t) - \frac{\mu'_m(t)}{\mu^2_m(t)}\,\gamma(t) \leqslant |\mathcal{F}_2(t)|^2_{L^2(\Omega)} + \Big(\frac{1}{\mu_m(t)} + \frac{|\mu'_m(t)|}{\mu_m(t)}\Big)\frac{\gamma(t)}{\mu_m(t)}.$$

Since $\eta(t) \geqslant \frac{\gamma(t)}{\mu_m(t)}$, $t \in (0,T)$, we have

$$\eta'(t) \leqslant |\mathcal{F}_2(t)|^2_{L^2(\Omega)} + \Big(\frac{1}{\mu_m(t)} + \frac{|\mu'_m(t)|}{\mu_m(t)}\Big)\eta(t), \quad t \in (0,T). \tag{50}$$

Taking into account (30) and inequalities

$$\gamma(t) \leqslant \mu_m(t)\eta(t) \leqslant C_{11}\eta(t), \quad |\mu'_m(t)| \leqslant C_{12}\sqrt{\gamma(t)} \leqslant C_{12}\sqrt{C_{11}\eta(t)},$$

from (50) we get

$$\eta'(t) \leqslant |\mathcal{F}_2(t)|^2_{L^2(\Omega)} + \Big(\frac{1}{m_0} + \frac{C_{12}\sqrt{C_{11}\eta(t)}}{m_0}\Big)\eta(t),$$

and so

$$\eta'(t) - \frac{1}{m_0}\,\eta(t) \leqslant |\mathcal{F}_2(t)|^2_{L^2(\Omega)} + C_{20}\,\eta^{3/2}(t), \quad t \in (0,T). \tag{51}$$

From (44) we get

$$|\mathcal{F}_2(t)|^2_{L^2(\Omega)} \leqslant C_{21}\Big(|\mathcal{F}_3(t)|^2_{L^2(\Omega)} + Y(t)\Big), \quad t\in(0,T), \tag{52}$$

where

$$\mathcal{F}_3(x,t) := 1 + |\nabla f(x,t)| + |u^m(x,t)|^{p+\sigma-1}, \ \ (x,t)\in Q_T,$$
$$Y(t) := \int_\Omega |u^m(x,t)|^{2(p-2)}\, |\nabla u^m(x,t)|^2\, dx, \quad t\in(0,T).$$

First let us prove the estimate

$$|\mathcal{F}_3(t)|^2_{L^2(\Omega)} \leqslant C_{22}, \quad t\in(0,T). \tag{53}$$

Note that $2(p+\sigma-1)\leqslant 2(p_+ +\sigma-1) < \frac{2n}{n-2}$. Then $H^1_0(\Omega) \circlearrowleft L^{\max\{1,2(p+\sigma-1)\}}(\Omega)$ (see item (i) of Proposition 1 and Condition $(\mathbf{C}_1)$). Hence, using Proposition 4 and estimates (29), we get

$$\int_{\Omega_t} |u^m|^{2(p+\sigma-1)}\, dx \leqslant C_{23} + C_{24}\|u^m(t)\|^{\max\{1,\,2(p_+ +\sigma-1)\}}_{L^{\max\{1,\,2(p+\sigma-1)\}}(\Omega)} \leqslant$$
$$\leqslant C_{23} + C_{25}\|u^m(t)\|^{\max\{1,2(p_+ +\sigma-1)\}}_{H^1_0(\Omega)} \leqslant C_{26}. \tag{54}$$

Using (54) and the estimate

$$|\mathcal{F}_3(x,t)|^2 \leqslant C_{27}\left(1+|\nabla f(x,t)|^2 + |u^m(x,t)|^{2(p-1+\sigma)}\right), \quad (x,t)\in Q_T,$$

we obtain (53).

Take $\varkappa>0$ from Remark 2. Let us prove the estimate

$$Y(t) \leqslant C_{28}(1+\eta^{3/2}(t)), \quad t\in(0,T). \tag{55}$$

Since $\frac{1}{\frac{2+\varkappa}{\varkappa}} + \frac{1}{\frac{2+\varkappa}{2}} = 1$, the Young inequality implies that

$$Y(t) \leqslant \frac{\varkappa}{2+\varkappa}\, Y_1(t) + \frac{2}{2+\varkappa}\, Y_2(t), \quad t\in(0,T), \tag{56}$$

where

$$Y_1(t) := \int_{\Omega_t} |u^m|^{2(p-2)\frac{2+\varkappa}{\varkappa}}\, dx, \quad Y_2(t) := \int_{\Omega_t} |\nabla u^m|^{2+\varkappa}\, dx, \quad t\in(0,T).$$

Using Proposition 4, (10), and (29), we obtain

$$Y_1(t) \leqslant C_{29} + C_{30}\|u^m\|^{\max\{1,\,2(p_+-2)\frac{2+\varkappa}{\varkappa}\}}_{L^{\max\{1,\,2(p-2)\frac{2+\varkappa}{\varkappa}\}}(\Omega)} \leqslant C_{29} + C_{31}\|u\|^{\max\{1,2(p_+-2)\frac{2+\varkappa}{\varkappa}\}}_{H^1_0(\Omega)} \leqslant C_{32}. \tag{57}$$

From (9), (45), (49), and Proposition 5 we have

$$Y_2(t) \leqslant C_{33}\Big(\sum_{i=1}^n \int_{\Omega_t} |u_{x_i}|^{2+\varkappa}\Big) \leqslant C_{34}\|u^m\|^{2+\varkappa}_{H^2(\Omega_t)} \leqslant C_{35}\Big(\int_{\Omega_t}\left(|u^m|^2 + |\Delta u^m|^2\right)\, dx\Big)^{\frac{2+\varkappa}{2}} \leqslant$$
$$\leqslant C_{36}\left(1+\beta^{\frac{2+\varkappa}{2}}(t)\right) \leqslant C_{37}\left(1+\eta^{\frac{2+\varkappa}{2}}(t)\right). \tag{58}$$

Since $\varkappa \in (0,1)$, we get $\frac{3}{2+\varkappa} > 1$, $\frac{3}{1-\varkappa} > 1$, and $\frac{1}{\frac{3}{2+\varkappa}} + \frac{1}{\frac{3}{1-\varkappa}} = 1$. Then the Young inequality yields that

$$\eta^{\frac{2+\varkappa}{2}}(t) \leqslant C_{38}(1+\eta^{3/2}(t)), \quad t \in (0,T). \tag{59}$$

Thus, (56)–(59) imply (55). Taking into account (52), (53), and (55), from (51) we get

$$\eta'(t) - \frac{1}{m_0}\,\eta(t) \leqslant C_{39} + C_{40}\,\eta^{3/2}(t), \quad t \in [0,T]. \tag{60}$$

Multiplying the inequality (60) by the function $e^{-\frac{t}{m_0}}$, we obtain

$$\Big(e^{-\frac{t}{m_0}}\eta(t)\Big)' \leqslant e^{-\frac{t}{m_0}}\Big(C_{39} + C_{40}\,\eta^{3/2}(t)\Big), \quad t \in [0,T].$$

Integrating last inequality in $t \in (0,\tau) \subset (0,T)$, we get $e^{-\frac{\tau}{m_0}}\eta(\tau) - \eta(0) \leqslant \int_0^\tau e^{-\frac{t}{m_0}}(C_{39} + C_{40}\,\eta^{3/2}(t))\,dt$. Hence

$$\eta(\tau) \leqslant e^{\frac{T}{m_0}}(\eta(0) + C_{39}T) + C_{40}\,e^{\frac{T}{m_0}}\int_0^\tau \eta^{3/2}(t)\,dt. \tag{61}$$

Clearly, $\eta(0) \leqslant \beta(0) + \frac{1}{m_0}\,\gamma(0) = |\Delta u_0^m|^2_{L^2(\Omega)} + \frac{1}{m_0}\,\|u_1^m\|^2 \leqslant C_{41}$. Thus, from (61) we get the inequality

$$\eta(\tau) \leqslant C_{42} + C_{43}\int_0^\tau \eta^{3/2}(t)\,dt, \quad \tau \in [0,T]. \tag{62}$$

Applying to (62) the Bihari Lemma (see [19]), for some $\widehat{T} \in (0,T]$ we get $0 \leqslant \eta(\tau) \leqslant C_{44}$, $\tau \in [0,\widehat{T}]$, where $\widehat{T}, C_{44}$ depends only on C_{42}, C_{43}. Then (see (30)) we have $0 \leqslant \beta(\tau) + \frac{1}{C_{11}}\,\gamma(\tau) \leqslant \beta(\tau) + \frac{1}{\mu_m(\tau)}\,\gamma(\tau) = \eta(\tau) \leqslant C_{44}$, and so

$$0 \leqslant \beta(\tau) \leqslant C_{45}, \quad 0 \leqslant \gamma(\tau) \leqslant C_{46}, \quad \tau \in [0,\widehat{T}]. \tag{63}$$

Note that, since $\partial\Omega \in C^2$, the space $H^2(\Omega)$ has the equivalent norm $\|v\|_{H^2(\Omega)} = \|v\|_{L^2(\Omega)} + \|\Delta v\|_{L^2(\Omega)}$ (see e.g. [5], [13]). Thus, in view of (29), (45), and (63), we have

$$\max\Big\{\|u^m\|_{L^\infty(0,\widehat{T};H^2(\Omega))}, \quad \|u_t^m\|_{L^\infty(0,\widehat{T};H_0^1(\Omega))}\Big\} \leqslant C_{47}. \tag{64}$$

Step 4 (third estimates). Multiplying the j-th equation of system (19) by $(\varphi_j^m(t))''$, summing by $j \in \{1,\dots,m\}$, and integrating in $t \in (0,\tau) \subset (0,\widehat{T})$, we get

$$\int_{Q_\tau}\Big[u_{tt}^m - \mu_m\Delta u^m\Big]u_{tt}^m\,dxdt = \int_{Q_\tau}\Big[f - g|u^m|^{p-2}u^m\Big]u_{tt}^m\,dxdt.$$

Take $\varepsilon > 0$ is arbitrary. Similarly as in [8], we obtain the inequality

$$(1 - 2\varepsilon - h^0\varepsilon - g^0\varepsilon)\int_{Q_\tau}|u_{tt}^m|^2\,dxdt \leqslant$$
$$\leqslant C_{48}\varepsilon^{-1} + C_{11}\varepsilon^{-1}\int_{Q_\tau}|\Delta u^m|^2\,dxdt + \varepsilon^{-1}\int_{Q_\tau}|f|^2\,dxdt, \quad \tau \in [0,\widehat{T}]. \tag{65}$$

Taking $\varepsilon > 0$ sufficiently small, from (65) we have

$$\|u_{tt}^m\|_{L^2(Q_{\widehat{T}})} \leqslant C_{49}. \tag{66}$$

Step 5 (passing to the limit). Taking into account (16), (29), (64), (66), Lemma 1.28 [7, Section 2, §1], Lemma 1.18 [7, Section 2, §5], and the Aubin compact imbedding Theorem (see, for example, either Lemma [2, p. 5042], Theorems 1, 2 [2, p. 5042], or Lemma IV.1 [4, p. 393]), we obtain that there exist a function u and a subsequence $\{u^{m_k}\}_{k\in\mathbb{N}} \subset \{u^m\}_{m\in\mathbb{N}}$ such that u satisfies (5), (6), and

$$u^{m_k} \underset{k\to\infty}{\longrightarrow} u \quad * -\text{weakly in} \quad L^\infty(0,\widehat{T}; H^2(\Omega)\cap H_0^1(\Omega)), \tag{67}$$

$$u^{m_k} \underset{k\to\infty}{\longrightarrow} u \quad \text{weakly in} \quad H^2(Q_{\widehat{T}})\cap L^p(Q_{\widehat{T}}), \tag{68}$$

$$u_t^{m_k} \underset{k\to\infty}{\longrightarrow} u_t \quad * -\text{weakly in} \quad L^\infty(0,\widehat{T}; H_0^1(\Omega)) \tag{69}$$

$$u_t^{m_k} \underset{k\to\infty}{\longrightarrow} u_t \quad \text{weakly in} \quad L^2(0,\widehat{T}; H_0^1(\Omega)), \tag{70}$$

$$u_{tt}^{m_k} \underset{k\to\infty}{\longrightarrow} u_{tt} \quad \text{weakly in} \quad L^2(Q_{\widehat{T}}), \tag{71}$$

$$u^{m_k} \underset{k\to\infty}{\longrightarrow} u,\ u_t^{m_k} \underset{k\to\infty}{\longrightarrow} u_t,\ u_{x_i}^{m_k} \underset{k\to\infty}{\longrightarrow} u_{x_i} \quad \text{in} \quad C([0,\widehat{T}]; L^2(\Omega)),\ i=\overline{1,n}, \tag{72}$$

$$u^{m_k} \underset{k\to\infty}{\longrightarrow} u, \quad u_t^{m_k} \underset{k\to\infty}{\longrightarrow} u_t, \quad u_{x_i}^{m_k} \underset{k\to\infty}{\longrightarrow} u_{x_i} \quad \text{a.e. on} \quad Q_{\widehat{T}}, \quad i=\overline{1,n}, \tag{73}$$

$$|u^{m_k}|^{p-2}u^{m_k} \underset{k\to\infty}{\longrightarrow} |u|^{p-2}u \quad \text{weakly in} \quad L^{p'(x)}(Q_{\widehat{T}}). \tag{74}$$

From (72) it follows that u satisfies conditions (3), (4).

Now let $\psi \in C([0,\widehat{T}])$. Multiplying the j-th equation of system (19) by the function ψ and integrating in $t \in (0,\widehat{T})$ we get the equality

$$\int_{Q_{\widehat{T}}} \Big[u_{tt}^{m_k} - \mu_{m_k}\Delta u^{m_k} + g|u^{m_k}|^{p-2}u^{m_k} - f \Big] w^j \psi \, dxdt = 0.$$

Letting $k \to \infty$ and taking into account (67)–(74), we see that the function u satisfies equation (2) almost everywhere (notice that we use arbitrariness of the function ψ and that set of linear combinations of the system $\{w^j\}_{j\in\mathbb{N}}$ is dense in the space $H^2(\Omega)\cap H_0^1(\Omega)\cap L^p(\Omega)$). □

References

1. R.A. Adams, Sobolev Spaces, Academic press, 1975. R.A. Adams, J.J.F. Fournier, Sobolev Spaces, 2nd ed., Elsevier/Academic Press, 2003.
2. J.-P. Aubin, *Un Theoreme de Compacite*, Comptes rendus hebdomadaires des seances de l'academie des sciences **256** (24) (1963), 5042–5044.
3. F.E. Bentata, Ie. Zaitsev, *Qualitative Analysis for a New Constrained Hyperbolic p-Kirchhoff Type Problem Involving Free Boundary*, Boundary Value Problems **2025** (76) (2025). doi:10.1186/s13661-025-02070-2
4. F. Bernis, *Existence Results for Doubly Nonlinear Higher Order Parabolic Equations on Unbounded Domain*, Math. Ann. **279** (1988), 373–394. doi:10.1007/BF01456275
5. R. Caccioppoli, *Limitazioni Integrali per Soluzioni Di Unequazioni Lineare Ellitica a Derivate Parziali*, Giornale di Battaglini, Series 4,5, **80** (1950–1951), 186–212.
6. L. Diening, P. Harjulehto, P. Hasto, M. Růžička, LEbesgue and Sobolev Spaces With Variable Exponents, Lecture Notes Math., V.2017, Springer, Heidelberg, 2011, 509 p. doi:10.1007/978-3-642-18363-8
7. H. Gajewski, K. Gröger, K. Zacharias, Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen, Berlin: Akademie-Verlag, 1974.

8. O. Kholyavka, O. Buhrii, M. Bokalo, R. Ayazoglu (Mashiyev), *IniTial-BounDaRy-Value Problem for Third Order Equations of Kirchhoff Type With Variable Exponents of Nonlinearity*, Advances in Math. Sciences and Appl. **23** (2013), 509–528.
9. Da. Kim, Do. Kim, K.-S. Hong, I.H. Jung, *Global Existence and Energy Decay Rates for a Kirchhoff-Type Wave Equation With Nonlinear Dissipation*, Scientific World Journal **2014** 716740, 10 p. doi:10.1155/2014/716740
10. D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Classics in Applied Mathematics, V.31, SIAM, Philadelphia, 2000. doi:10.1137/1.9780898719451
11. G. Kirchhoff, Vorlesungen Über Mechanik, Teubner, Leipzig, 1883. https://archive.org/details/vorlesungenberm00kircgoog
12. O. Kováčik, J. Rákosník, *On Spaces $L^{r(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J. **41** (116) (1991), 592–618.
13. O. Ladyzenskaya, *On closure of the elliptic operator*, Dokl. Akad. Nauk. SSSR **79** (5) (1951), 723–725.
14. S. Lavrenyuk, O. Panat, *The Mixed Problem for a Semilinear Hyperbolic Equation in Generalized Lebesgue Spaces*, Visnyk of the Lviv Univ. Series Mech. and Math. **66** (2006), 243–260.
15. R.A. Mashiyev, O.M. Buhrii, *Existence of Solutions of the Parabolic Variational Inequality With Variable Exponent of Nonlinearity*, J. Math. Anal. Appl. **377** (2011), 450–463. doi:10.1016/j.jmaa.2010.11.059
16. V.P. Mikhailov, Partial Differential Equations, Mir, 1983.
17. N. Pan, P. Pucci, B. Zhang, *Degenerate Kirchhoff-Type Hyperbolic Problems Involving the Fractional Laplacian*, Journal of Evolution Equations, 2017, Springer International Publishing AG. doi:10.1007/s00028-017-0406-2
18. O.T. Panat, O.M. Buhrii, *Some properties of the solutions to the hyperbolic equations with variable exponents of nonlinearity*, Transactions of NAS of Azerbaijan **30**(1)(2010), 155–160. MR 2722059, Zbl 1201.35077
19. M. Pinto, *Integral Inequalities of Bihari-Type and Applications*, Funkcialaj Ekvacioj **33**(1990), 387–403.
20. V. Rădulescu, D. Repovš, Partial Differential Equations With Variable Exponents: Variational Methods and Qualitative Analysis, CRC Press, Boca Raton, London, New York, 2015.
21. R.E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, American Mathematical Society, 1997.
22. Aihui Sun, Hui Xu, *Decay Estimate and Blow-Up for Fractional Kirchhoff Wave Equations Involving a Logarithmic Source*, AIMS Mathematics **10** (6) (2025), 14032–14054. doi:10.3934/math.2025631
23. Y. Ye, *Global Existence of Solutions and Energy Decay for a Kirchhoff-Type Equation With Nonlinear Dissipation*, Journal of Inequalities and Applications **2013** (195) (2013). doi:10.1186/1029-242X-2013-195

[1]Pidstryhach Institute for Applied Problems of Mechanics and Mathematics NASU,
Lviv, Ukraine
[2,3,4]Ivan Franko National University of Lviv,
Lviv, Ukraine

[1]oksana.kholiavka@lnu.edu.ua
[2]oleh.buhrii@lnu.edu.ua
[3]taras.bokalo@lnu.edu.ua
[4]mm.bokalo@gmail.com

Received 03.02.2026
Revised 28.05.2026