Математичні Студії. Т.65, №2 Matematychni Studii. V.65, No.2
ISSN 1027-4634 (print) ISSN 2411-0620 (online)

M. SHEREMETA, Y. TRUKHAN*

ON ANALYTIC SOLUTIONS OF A DIFFERENTIAL EQUATION OF SHAH TYPE IN THE UNIT DISK. BOUNDEDNESS OF l-INDEX

M. Sheremeta, Y. Trukhan. *On analytic solutions of a differential equation of Shah type in the unit disk. Boundedness of l-index*, Mat. Stud. **65** (2026), 157–168.

For an analytic in the unit disk $\mathbb{D}$ solution of the form $f(z) = F(1/(1-z))$, where F is an entire transcendental function, of the differential equation $(1-z)^n w'' + a(1-z)^m w' + bw = 0$ (with $n > m \geq 0$, $a \in \mathbb{R}$, $b \in \mathbb{R}$) the boundedness of l-index of f in $\mathbb{D}$ and l-index of F in $\mathbb{C}$ are investigated.

1. Introduction. An analytic univalent in $\mathbb{D} = \{z\colon |z| < 1\}$ function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known ([1, p.203]) that the condition $\mathrm{Re}\,\{1 + zf''(z)/f'(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the convexity of f. By W. Kaplan ([2]) the function f is said to be close-to-convex in $\mathbb{D}$ (see also [1, p.583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\mathrm{Re}\,(f'(z)/\Phi'(z)) > 0$ $(z \in \mathbb{D})$. Function (1) is close-to-convex in $\mathbb{D}$ if and only if the function

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \quad g_n = f_n/f_1, \tag{2}$$

is close-to-convex in $\mathbb{D}$. We remark also, that the function (2) is said to be starlike in $\mathbb{D}$, if $g(\mathbb{D})$ is starlike domain regarding the origin.

S.M. Shah ([3]) indicated conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 of the differential equation

$$z^2 w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0, \tag{3}$$

under which there exists an entire transcendental solution (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$. In particular he obtained the following result.

Theorem 1. *If $-1 \leq \beta_0 < 0$, $\beta_1 > 0$ and $\beta_1 + \gamma_2 = \gamma_0 = \gamma_1 = 0$, then equation (3) has an entire solution (2) such that all derivatives $g^{(n)}$ $(n \geq 0)$ are close-to-convex in $\mathbb{D}$ and $\ln M_g(r) = (1 + o(1))|\beta_0| r$ as $r \to +\infty$, where $M_g(r) = \max\{|g(z)|\colon |z| = r\}$.*

The convexity and starlikeness of solutions of equation (3) were studied in [4–7].

Let $0 < R \leq +\infty$ and l be a positive continuous function on $[0, R)$, which satisfies

$$l(r) > \beta/(R-r)\ (r \in (0, R)),\ \beta = \mathrm{const} > 1.$$

2020 *Mathematics Subject Classification:* 34A30, 30C45, 30D20.
Keywords: differential equation; analytical function; l-index.
doi:10.30970/ms.65.1.157-168

*Corresponding author: Y. Trukhan

An analytic in $\mathbb{D}_R$ function f is said ([8]) to be of bounded l-index if there exists $N \in \mathbb{Z}_+$ such that

$$\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \le \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)} : 0 \le k \le N\right\}. \tag{4}$$

for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{D}_R := \{z\colon |z| < R\}$. The least such integers is called the l-index of f and is denoted by $N(l, f; \mathbb{D}_R)$. If there exists $N \in \mathbb{Z}_+$ such that (4) holds for all $n \in \mathbb{Z}_+$ and $z \in G \subset \mathbb{D}_R$ then the function f is said to be of bounded l-index on (or in) G, and the l-index is denoted by $N(l, f; G)$.

The l-index boundedness of entire solutions of equation (3) was studied in [9–12].

In [13], three analogues of the Shah equation are considered, which have analytical solutions in the disk $\mathbb{D}$. Among them there is the equation

$$(1-z)^3 w'' + a(1-z)w' + bw = 0. \tag{5}$$

The following theorem has been proven ([13]).

Theorem 2. *If $a < 0$, $b < 0$ and $a + b \ge -3$ then equation (5) has an analytic in $\mathbb{D}$ solution $f(z) = F(1/(1-z))$ such that $\ln M_f(r) = (1 + o(1))|a|/(1-r)$ as $r \uparrow 1$ and the function $F(t) = t + \sum_{n=2}^{\infty} F_n t^n$ is entire and close-to-convex in $\mathbb{D}$. Moreover, the function f has a bounded l-index $N(l, f; \mathbb{D}) \le 1$ with $l(|z|) = 4/(1-|z|)^2$.*

A more general equation

$$(1-z)^n w'' + a(1-z)^m w' + bw = 0, \quad (n > m \ge 0,\ a \in \mathbb{R},\ b \in \mathbb{R}), \tag{6}$$

than (5) was considered in [14]. As in [13], the solutions of this equation were constructed in the form

$$f(z) = F\Big(\frac{1}{1-z}\Big) = \sum_{k=0}^{\infty} \frac{F_k}{(1-z)^k} \tag{7}$$

where F is an entire transcendental function. Remark that there was formulated a conjecture ([15]) that for an entire function f the function $H(z) = f(1/(1-z)^n)$, $n \in \mathbb{N}$, is of bounded l-index in $\mathbb{D}$ with $l(|z|) = \beta/(1-|z|)^{n+1}, \beta > 1$, if and only if f is of bounded index. Later it was completely proved in [16]. The class of such functions (7) we denote by $\mathfrak{S}$. It is proved ([14]) that equation (6) can have solutions belonging to the class $\mathfrak{S}$ only in cases if either $n = 2$ and $m = 0$ or $n = 3$ and $m = 0$ or $n = 3$ and $m = 2$, and in these cases the following theorem is correct.

Theorem 3. *In the cases if either $n = 2$ and $m = 0$ or $n = 3$ and $m = 0$ or $n = 3$ and $m = 2$, differential equation (6) has solutions f of the form (7) that belong to the class $\mathfrak{S}$ and have the following properties.*

For $n = 2$, $m = 0$ and $b = -2$, if $-3 \le a < 0$ then

$$F(t) = t + \sum_{k=2}^{\infty} F_k t^k, \quad t = \varrho e^{i\theta}, \tag{8}$$

with

$$F_k = \frac{-a(k-1)}{k(k+1)-2} F_{k-1}, \quad k \ge 2, \tag{9}$$

is a close-to-convex function in $\mathbb{D}$ and $\ln M_F(\varrho) = (1 + o(1))|a|\varrho$ as $\varrho \to +\infty$. If $|a| \le 30/19$ then function F is starlike in $\mathbb{D}$ and if $|a| \le 29/20$ then it is convex in $\mathbb{D}$.

For $n = 3$, $m = 0$ and $a = 0$, if $-3 \le b < 0$ then

$$F(t) = \frac{2}{|b|} + t + \sum_{k=2}^{\infty} F_k t^k, \quad t = \varrho e^{i\theta}, \tag{10}$$

with

$$F_{k+1}=\frac{-b}{(k+1)(k+2)}F_k,\quad k\ge 2. \tag{11}$$

is a close-to-convex function in $\mathbb{D}$ *and* $\ln M_F(\varrho)=2(1+o(1))\sqrt{|b|\varrho}$ *as* $\varrho\to+\infty$. *If* $|b|\le 24/11$ *then the function* $F(t)-2/|b|$ *is starlike in* $\mathbb{D}$ *and if* $|b|\le 48/41$ *then the function* F *is convex in* $\mathbb{D}$.

For $n=3$, $m=2$ *and* $2+a>0$, *if* $-(3+a)\le b<0$ *then*

$$F(t)=\frac{2+a}{|b|}+t+\sum_{k=2}^{\infty}F_kt^k,\quad t=\varrho e^{i\theta}, \tag{12}$$

with

$$F_{k+1}=\frac{-b}{(k+1)(k+2+a)}F_k,\quad k\ge 2, \tag{13}$$

is a close-to-convex function in $\mathbb{D}$ *and* $\ln M_F(\varrho)=2(1+o(1))\sqrt{|b|\varrho}$ *as* $\varrho\to+\infty$. *If* $|b|/(a+3)+|b|/(2a+8)\le 1$ *then the function* $F(t)-(2+a)/|b|$ *is starlike in* $\mathbb{D}$ *and if* $2|b|/(a+3)+3|b|/(4a+16)\le 1$ *then function* F *is convex in* $\mathbb{D}$.

If $n=3$, $m=0$, but $a<0$ then Theorem 3 is supplemented by the following statement.

Theorem 4. *Let* $n=3$, $m=0$, $a<0$ *and* $b<0$. *Then differential equation* (6) *has solution* f *of the form* (7) *that belong to the class* $\mathfrak{S}$, *where the function* F *is given by equality* (10) *with* $F_2=|b|/6$,

$$F_{k+1}=\frac{|b|}{(k+1)(k+2)}F_k+\frac{|a|(k-1)}{(k+1)(k+2)}F_{k-1},\quad k\ge 2, \tag{14}$$

and $\ln M_F(\varrho)=(1+o(1))|a|\varrho^2/2$ *as* $\varrho\to+\infty$. *If* $11|b|/24+9|a|/20\le 1$ *then the function* $F(t)-2/|b|$ *is starlike in* $\mathbb{D}$, *if* $41|b|/48+23|a|/20\le 1$ *then the function* F *is convex in* $\mathbb{D}$.

Here we examine the boundedness of the l-index in all cases considered in Theorems 3 and 4.

2. l-Index boundedness of function (7)**.** To study the boundedness of the l-index of a function f one can use the fact that this function is a solution of one or another differential equation. In this case, we will take into account the fact that if $l_1(|z|)\le l_2(|z|)$ then [8, p.23] the inequality $N(l_1,f;G)\le N$ implies the inequality $N(l_2,f;G)\le N$.

At first, let $n=2$, $m=0$, $b=-2$ and $-3\le a<0$, i.e. f satisfies the equation $(1-z)^2w''+aw'-2w=0$ and, therefore,

$$(1-z)^2f''(z)+af'(z)-2f(z)\equiv 0,\quad -3\le a<0. \tag{15}$$

We put $l(|z|)=A/(1-|z|)^2$, $A=1+\sqrt{2}$. Then for all $z\in\mathbb{D}$

$$\frac{|f''(z)|}{2!l^2(|z|)}\le\frac{|a|}{2l(|z|)(1-|z|)^2}\frac{|f'(z)|}{1!l(|z|)}+\frac{|b|}{2l^2(|z|)(1-|z|)^2}|f(z)|\le\frac{3}{2A}\frac{|f'(z)|}{1!l(|z|)}+\frac{1}{Al(|z|)}|f(z)|\le$$

$$\le\left(\frac{2}{A}+\frac{1}{A^2}\right)\max\left\{\frac{|f'(z)|}{1!l(|z|)},|f(z)|\right\}\le\max\left\{\frac{|f'(z)|}{1!l(|z|)},|f(z)|\right\}. \tag{16}$$

For $j \ge 1$ from (15) we have

$$(1-z)^2 f^{(j+2)}(z) + (a - 2j(1-z)) f^{(j+1)}(z) + (j(j-1)-2) f^{(j)}(z) \equiv 0,$$

whence as above

$$\frac{|f^{(j+2)}(z)|}{(j+2)!l^{j+2}(|z|)} \le$$

$$\le \frac{3+2j}{(j+2)(1-|z|)^2 l(|z|)} \frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)} + \frac{|j(j-1)-2|}{(j+1)(j+2)(1-|z|)^2 l^2(|z|)} \frac{|f^{(j)}(z)|}{j!l^j(|z|)} \le$$

$$\le \frac{2}{A} \frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)} + \frac{1}{A^2} \frac{|f^{(j)}(z)|}{j!l^j(|z|)} \le \max\left\{ \frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)}, \frac{|f^{(j)}(z)|}{j!l^j(|z|)} \right\}. \tag{17}$$

In view of (16) and (17) the following statement is true.

Proposition 1. *If the function f satisfies* (6) *with $n=2$, $m=0$, $b=-2$ and $-3 \le a < 0$ then it is of bounded l-index $N(l,f;\mathbb{D}) \le 1$ with $l(|z|) = (1+\sqrt{2})/(1-|z|)^2$.*

Now let $n=3$, $m=0$, $a=0$ and $-3 \le b < 0$. Then $(1-z)^3 f''(z) + bf(z) \equiv 0$ and for $l(|z|) = A/(1-|z|)^{3/2}$ with $A \ge \sqrt{3/2}$ we have

$$\frac{|f''(z)|}{2!l^2(|z|)} \le \frac{3}{2A^2}|f(z)| \le |f(z)|. \tag{18}$$

As above $(1-z)^3 f'''(z) - 3(1-z)^2 f''(z) + bf'(z) \equiv 0$, whence for $A \ge (1+\sqrt{3})/2$ we get

$$\frac{|f'''(z)|}{3!l^3(|z|)} \le \frac{1}{A}\frac{|f''(z)|}{2!l^2(|z|)} + \frac{1}{2A^2}\frac{|f'(z)|}{1!l(|z|)} \le \max\left\{ \frac{|f''(z)|}{2!l^2(|z|)}, \frac{|f'(z)|}{1!l(|z|)} \right\}. \tag{19}$$

For $j \ge 2$ now we have $(1-z)^3 f^{(j+2)}(z) - 3j(1-z)^2 f^{(j+1)}(z) + (3j(j-1)(1-z)+b) f^{(j)}(z) - -j(j-1)(j-2) f^{(j-1)}(z) \equiv 0$, whence as above for $A = 4$ we get

$$\frac{|f^{(j+2)}(z)|}{(j+2)!l^{j+2}(|z|)} \le \frac{3j}{(j+2)A}\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)} + \frac{3j(j-1)+3}{(j+2)(j+1)A^2}\frac{|f^{(j)}(z)|}{j!l^j(|z|)} +$$

$$+\frac{j(j-1)(j-2)}{(j+2)(j+1)jA^3}\frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)} \le \max\left\{ \frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)}, \frac{|f^{(j)}(z)|}{j!l^j(|z|)}, \frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)} \right\}. \tag{20}$$

In view of (18), (19) and (20) the following statement is true.

Proposition 2. *If the function f satisfies* (6) *with $n=3$, $m=0$, $a=0$ and $-3 \le b < 0$ then it is of bounded l-index $N(l,f;\mathbb{D}) \le 1$ with $l(|z|) = 4/(1-|z|)^{3/2}$.*

Finally, let $n=3$, $m=2$, $-2 < a < 0$ and $-(3+a) \le b < 0$. Then

$$(1-z)^3 f''(z) + a(1-z)^2 f'(z) + bf(z) \equiv 0, \quad (|a| \le 2, |b| \le 3). \tag{21}$$

We put $l(|z|) = A/(1-|z|)^{3/2}$. If $A \ge (1+\sqrt{7})/2$ then for all $z \in \mathbb{D}$ as above we obtain

$$\frac{|f''(z)|}{2!l^2(|z|)} \le \frac{|a|}{2l(|z|)(1-|z|)}\frac{|f'(z)|}{1!l(|z|)} + \frac{|b|}{2l^2(|z|)(1-|z|)^3}|f(z)| \le$$

$$\leq \left(\frac{1}{A}+\frac{3}{2A^2}\right)\max\left\{\frac{|f'(z)|}{1!l(|z|)}, |f(z)|\right\} \leq \max\left\{\frac{|f'(z)|}{1!l(|z|)}, |f(z)|\right\}. \tag{22}$$

From (21) it follows that

$$(1-z)^3 f'''(z)+(a-3)(1-z)^2 f''(z)+(b-2a(1-z))f'(z)\equiv 0,$$

whence for $A \geq (5+\sqrt{67})/6$ we get

$$\frac{|f'''(z)|}{3!l^3(|z|)} \leq \frac{|a-3|}{3l(|z|)(1-|z|)}\frac{|f''(z)|}{2!l^2(|z|)}+\frac{|b|+2|a|}{3!l^2(|z|)(1-|z|)^3}\frac{|f'(z)|}{1!l(|z|)} \leq$$

$$\leq \frac{5}{3A}\frac{|f''(z)|}{2!l^2(|z|)}+\frac{7}{6A^2}\frac{|f'(z)|}{1!l(|z|)} \leq \max\left\{\frac{|f''(z)|}{2!l^2(|z|)}, \frac{|f'(z)|}{1!l(|z|)}\right\}. \tag{23}$$

For $j \geq 2$ from (21) also we have

$$(1-z)^3 f^{(j+2)}(z)-(3j-a)(1-z)^2 f^{(j+1)}(z)+$$
$$+((3j(j-1)-2ja)(1-z)+b)f^{(j)}(z)-(j(j-1)(j-2)-aj(j-1))f^{(j-1)}(z)\equiv 0,$$

whence as above we get

$$\frac{|f^{(j+2)}(z)|}{(j+2)!l^{j+2}(|z|)} \leq \frac{3j+|a|}{(j+2)(1-|z|)l(|z|)}\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)}+$$
$$+\frac{3j(j-1)+2|a|j+|b|}{(j+2)(j+1)l^2(|z|)(1-|z|)^3}\frac{|f^{(j)}(z)|}{j!l^j(|z|)}+\frac{j(j-1)(j-2+|a|)}{(j+2)(j+1)jl^3(|z|)(1-|z|)^3}\frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)} \leq$$
$$\leq \frac{3j+|a|}{(j+2)A}\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)}+\frac{3j(j-1)+2|a|j+|b|}{(j+2)(j+1)A^2}\frac{|f^{(j)}(z)|}{j!l^j(|z|)}+$$
$$+\frac{j(j-1)(j-2+|a|)}{(j+2)(j+1)jA^3}\frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)} \leq$$
$$\leq \left(\frac{3}{A}+\frac{3}{A^2}+\frac{1}{A^3}\right)\max\left\{\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)}, \frac{|f^{(j)}(z)|}{j!l^j(|z|)}, \frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)}\right\} \leq$$
$$\leq \max\left\{\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)}, \frac{|f^{(j)}(z)|}{j!l^j(|z|)}, \frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)}\right\} \tag{24}$$

provided $A=4$. In view of (22), (23) and (24) the following statement is true.

Proposition 3. *If the function f satisfies* (6) *with $n=3$, $m=2$, $-2<a<0$ and $-(3+a)\leq b<0$ then it is of bounded l-index $N(l,f;\mathbb{D})\leq 1$ with $l(|z|)=4/(1-|z|)^{3/2}$.*

Uniting Propositions 1, 2 and 3, we obtain the following theorem.

Theorem 5. *Suppose that f is a solution of differential equation* (6).

If $n=2$, $m=0$, $b=-2$ and $-3\leq a<0$ then f is of bounded l-index $N(l,f;\mathbb{D})\leq 1$ with $l(|z|)=(1+\sqrt{2})/(1-|z|)^2$.

If either $n=3$, $m=0$, $a=0$ and $-3\leq b<0$ or $n=3$, $m=2$, $-2\leq a<0$ and $-(3+a)\leq b<0$ then f is of bounded l-index $N(l,f;\mathbb{D})\leq 1$ with $l(|z|)=4/(1-|z|)^{3/2}$.

As in [14] the case $n = 3$, $m = 0$, $a < 0$ and $b < 0$ (see Theorem 4) we will consider separately. In this case

$$(1-z)^3 f''(z) + af'(z) + bf(z) \equiv 0, \tag{25}$$

whence for $l(|z|) = A_1/(1-|z|)^3$, where A_1 is the positive root of the equation $\frac{|a|}{2A_1} + \frac{|b|}{2A_1^2} = 1$, we have

$$\frac{|f''(z)|}{2!l^2(|z|)} \le \frac{|a|}{2l(|z|)(1-|z|)^3}\frac{|f'(z)|}{1!l(|z|)} + \frac{|b|}{2l^2(|z|)(1-|z|)^3}|f(z)| \le \frac{|a|}{2A_1}\frac{|f'(z)|}{1!l(|z|)} + \frac{|b|}{2A_1^2}|f(z)| \le$$
$$\le \left(\frac{|a|}{2A_1} + \frac{|b|}{2A_1^2}\right)\max\left\{\frac{|f'(z)|}{1!l(|z|)}, |f(z)|\right\} \le \max\left\{\frac{|f'(z)|}{1!l(|z|)}, |f(z)|\right\}. \tag{26}$$

From (25) it follows that

$$(1-z)^3 f'''(z) + (a - 3(1-z)^2)f''(z) + bf'(z) \equiv 0, \tag{27}$$

whence for $l(|z|) = \frac{A_2}{(1-|z|)^3}$, where A_2 is the positive root of the equation $\frac{|a|+3}{3A_2} + \frac{|b|}{6A_2^2} = 1$, we get similarly

$$\frac{|f'''(z)|}{3!l^3(|z|)} \le \frac{|a|+3}{3l(|z|)(1-|z|)^3}\frac{|f''(z)|}{2!l^2(|z|)} + \frac{|b|}{6l^2(|z|)(1-|z|)^3}\frac{|f'(z)|}{1!l(|z|)} \le$$
$$\le \frac{|a|+3}{3A_2}\frac{|f''(z)|}{2!l^2(|z|)} + \frac{|b|}{6A_2^2}\frac{|f'(z)|}{1!l(|z|)} \le \max\left\{\frac{|f''(z)|}{2!l^2(|z|)}, \frac{|f'(z)|}{1!l(|z|)}\right\}. \tag{28}$$

From (27) it follows that

$$(1-z)^3 f''''(z) + (a - 6(1-z)^2)f'''(z) + (b + 6(1-z))f''(z) \equiv 0,$$

whence for $l(|z|) = \frac{A_3}{(1-|z|)^3}$, where A_3 is the positive root of the equation $\frac{|a|+6}{4A_3} + \frac{|b|+6}{12A_3^2} = 1$, we get

$$\frac{|f''''(z)|}{4!l^4(|z|)} \le \frac{|a|+6}{4l(|z|)(1-|z|)^3}\frac{|f'''(z)|}{3!l^3(|z|)} + \frac{|b|+6}{12l^2(|z|)(1-|z|)^3}\frac{|f''(z)|}{2!l^2(|z|)} \le$$
$$\le \frac{|a|+6}{4A_3}\frac{|f'''(z)|}{3!l^3(|z|)} + \frac{|b|+6}{12A_3^2}\frac{|f''(z)|}{2!l^2(|z|)} \le \max\left\{\frac{|f'''(z)|}{3!l^3(|z|)}, \frac{|f''(z)|}{2!l^2(|z|)}\right\}. \tag{29}$$

Finally, for $j \ge 3$ from (25) we obtain

$$(1-z)^3 f^{(j+2)}(z) + (a - 3j(1-z)^2)f^{(j+1)}(z) +$$
$$+ (b + 3j(j-1)(1-z))f^{(j)}(z) - j(j-1)(j-2)f^{(j-1)}(z) \equiv 0$$

whence for $l(|z|) = A_4/(1-|z|)^3$, where A_4 is the positive root of the equation $\frac{|a|+15}{5A_4} + \frac{|b|+60}{20A_4^2} + \frac{1}{A_4^3} = 1$, we get

$$\frac{f^{(j+2)}(z)|}{(j+2)!l^{j+2}(|z|)} \le \frac{|a|+3j}{(j+2)l(|z|)(1-|z|)^3}\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)} +$$
$$+ \frac{|b|+3j(j-1)}{(j+2)(j+1)l^2(|z|)(1-|z|)^3}\frac{|f^{(j)}(z)|}{j!l^j(|z|)} + \frac{j(j-1)(j-2)}{j(j+1)(j+2)l^3(|z|)(1-|z|)^3}\frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)} \le$$

$$\begin{aligned} &\le \frac{|a|+3j}{(j+2)A_4}\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)}+\frac{|b|+3j(j-1)}{(j+2)(j+1)A_4^2}\frac{|f^{(j)}(z)|}{j!l^j(|z|)}+\\ &+\frac{(j-1)(j-2)}{(j+1)(j+2)A_4^3}\frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)}\le\frac{|a|+15}{5A_4}\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)}+\frac{|b|+60}{20A_4^2}\frac{|f^{(j)}(z)|}{j!l^j(|z|)}+\\ &+\frac{1}{A_4^3}\frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)}\le\max\left\{\frac{|f^{(j+1)}(z)|}{(j+1)!l^{j+1}(|z|)},\frac{|f^{(j)}(z)|}{j!l^j(|z|)},\frac{|f^{(j-1)}(z)|}{(j-1)!l^{j-1}(|z|)}\right\}. \end{aligned} \tag{30}$$

Due to inequalities (26), (28), (29) and (30) the following theorem is correct.

Theorem 6. *Suppose that f is a solution of differential equation* (6) *with $n=3$, $m=0$, $a<0$ and $b<0$. Then f is of bounded l-index $N(l,f;\mathbb{D})\le 1$ with $l(|z|)=A/(1-|z|)^3$, where $A=\max\{A_1,A_2,A_3,A_4\}$.*

3. l-Index boundedness of the function F. We will need the following lemma [13].

Lemma 1. *If $0<R<\infty$, $0<\eta<1$, function* (2) *is analytic in $\overline{\mathbb{D}}_R=\{z\colon |z|\le R\}$ and*

$$\sum_{k=2}^{\infty}k|g_k|R^{k-1}\le a(R)<1 \tag{31}$$

then $N(l,f;\,\mathbb{D}_{\eta R})\le 1$ with $l(|z|)\equiv\frac{1+a(R)}{(1-\eta)R(1-a(R))}$.

First assume that $n=2$, $m=0$, $b=-2$ and $-3\le a<0$. Then for function (8) with coefficients satisfying (9) we have

$$\begin{aligned} \sum_{k=2}^{\infty}k|F_k|R^{k-1}&\le\sum_{k=2}^{\infty}\frac{3k(k-1)}{k(k+1)-2}|F_{k-1}|R^{k-1}=\frac{6}{4}|F_1|R+\sum_{k=3}^{\infty}\frac{3k(k-1)}{k(k+1)-2}|F_{k-1}|R^{k-1}=\\ &=\frac{3}{2}R+\sum_{k=2}^{\infty}\frac{3(k+1)R}{(k+2)(k+1)-2}k|F_k|R^{k-1}\le\frac{3}{2}R+\sum_{k=2}^{\infty}\frac{9R}{10}k|F_k|R^{k-1}, \end{aligned}$$

whence for $R<5/12$ we get

$$\left(1-\frac{9R}{10}\right)\sum_{k=2}^{\infty}k|F_k|R^{k-1}\le\frac{3}{2}R,$$

i.e. (31) holds with $a(R)=15R/(10-9R)<1$.

Therefore, by Lemma 1 we obtain the following statement.

Proposition 4. *For function* (8) *$N(l,F;\,\mathbb{D}_{\eta R})\le 1$ with $l(|z|)\equiv\frac{5+3R}{(1-\eta)R(5-12R)}$, where $0<\eta<1$ and $0<R<5/12$.*

If $n=3$, $m=0$, $a=0$ and $-3\le b<0$ then for function (10) with coefficients satisfying (11) we have similarly

$$\sum_{k=1}^{\infty}(k+1)|F_{k+1}|R^k\le\sum_{k=1}^{\infty}\frac{3}{k(k+2)}k|F_k|R^k\le R+\frac{3R}{8}\sum_{k=1}^{\infty}(k+1)|F_{k+1}|R^k,$$

whence for $R<8/11$ we get

$$\sum_{k=1}^{\infty}(k+1)|F_{k+1}|R^k\le a(R)=\frac{8R}{8-3R}<1,$$

and, thus, by Lemma 1 we obtain the following statement.

Proposition 5. *For function* (10) $N(l, F(t)-\frac{2}{|b|}; \mathbb{D}_{\eta R}) \le 1$ *with* $l(|z|) \equiv \frac{8+5R}{(1-\eta)R(8-11R)}$, *where* $0<\eta<1$ *and* $0<R<8/11$.

Finally, if $n=3$, $m=2$, $-2<a<0$ and $-(3+a)\le b<0$ then for function (12) with coefficients satisfying (13) we have

$$\sum_{k=1}^{\infty}(k+1)|F_{k+1}|R^k \le \sum_{k=1}^{\infty}\frac{3}{k^2}k|F_k|R^k = 3R+\sum_{k=1}^{\infty}\frac{3R}{(k+1)^2}(k+1)|F_{k+1}|R^k \le$$
$$\le 3R+\frac{3R}{4}\sum_{k=1}^{\infty}(k+1)|F_{k+1}|R^k,$$

whence for $R<4/15$ we get

$$\sum_{k=1}^{\infty}(k+1)|F_{k+1}|R^k \le a(R)=\frac{12R}{4-3R}<1.$$

Therefore, by Lemma 1 the following statement is true.

Proposition 6. *For function* (12) $N(l, F(t)-(2+a)/|b|; \mathbb{D}_{\eta R}) \le 1$ *with* $l(|z|)\equiv(4+9R)/((1-\eta)R(4-15R))$, *where* $0<\eta<1$ *and* $0<R<4/15$.

Now we investigate the l-index boundedness of the function F in $\mathbb{C}\setminus\mathbb{D}_{\eta R}$. To do this, we use the methodology used in the previous section.

As above we start with the case if $n=2$, $m=0$, $b=-2$ and $-3\le a<0$. Then if function (7) is a solution of differential equation (6) then for all $t\in\mathbb{C}$

$$t^2F''(t)+(at^2+2t)F'(t)+bF(t)\equiv 0, \tag{32}$$

whence for $|t|\ge\eta R$ and $A\ge 6/\eta R>6$

$$\frac{|F''(t)|}{2!A^2}\le\frac{1}{2A}\left(3+\frac{2}{\eta R}\right)\frac{|F'(t)|}{1!A}+\frac{1}{(A\eta R)^2}|F(t)|\le\left(\frac{3}{2A}+\frac{1}{A\eta R}\right)\frac{|F'(t)|}{1!A}+$$
$$+\frac{1}{(A\eta R)^2}|F(t)|\le\frac{5}{12}\frac{|F'(t)|}{1!A}+\frac{1}{36}|F(t)|\le\max\left\{\frac{|F'(t)|}{1!A},|F(t)|\right\}, \tag{33}$$

where $0<\eta<1$ and $0<R<5/12$. From (32) it follows that $t^2F'''(t)+(at^2+4t)F''(t)+$ $+2atF'(t)\equiv 0$, whence

$$\frac{|F'''(t)|}{3!A^3}\le\frac{1}{3A}\left(3+\frac{4}{\eta R}\right)\frac{|F''(t)|}{2!A^2}+\frac{1}{A^2\eta R}\frac{|F'(t)|}{1!A}\le$$
$$\le\frac{7}{18}\frac{|F''(t)|}{2!A^2}+\frac{1}{36}\frac{|F'(t)|}{1!A}\le\max\left\{\frac{|F''(t)|}{2!A^2},\frac{|F'(t)|}{1!A}\right\}. \tag{34}$$

For $j\ge 2$ from (32) it follows also that

$$t^2F^{(j+2)}(t)+(at^2+2(j+1)t)F^{(j+1)}(t)+(j(j-1+2at+2)-2)F^{(j)}(t)+j(j-1)aF^{(j-1)}(t)\equiv 0,$$

whence

$$\frac{|F^{(j+2)}(t)|}{(j+2)!A^{j+2}}\le\frac{|at+2(j+1)|}{|t|(j+2)A}\frac{|F^{(j+1)}(t)|}{(j+1)!A^{j+1}}+\frac{j|j-1+2at+2|+2}{|t|^2(j+2)(j+1)A^2}\frac{|F^{(j)}(t)|}{j!A^j}+$$

$$
\begin{aligned}
&+\frac{j(j-1)|a|}{|t|^2(j+2)(j+1)jA^3}\frac{|F^{(j-1)}(t)|}{(j-1)!A^{j-1}} \le \frac{1}{j+2}\left(\frac{|a|}{A}+\frac{2(j+1)}{A\eta R}\right)\frac{|F^{(j+1)}(t)|}{(j+1)!A^{j+1}}+\\
&+\frac{1}{(j+1)(j+2)}\left(\frac{2|a|j}{A^2\eta R}+\frac{j(j+1)+2}{(A\eta R)^2}\right)\frac{|F^{(j)}(t)|}{j!A^j}+\\
&+\frac{j(j-1)|a|}{j(j+1)(j+2)A(A\eta R)^2}\frac{|F^{(j-1)}(t)|}{(j-1)!A^{j-1}} \le \frac{11}{24}\frac{|F^{(j+1)}(t)|}{(j+1)!A^{j+1}}+\frac{1}{18}\frac{|F^{(j)}(t)|}{j!A^j}+\\
&+\frac{1}{864}\frac{|F^{(j-1)}(t)|}{(j-1)!A^{j-1}} < \max\left\{\frac{|F^{(j+1)}(t)|}{(j+1)!A^{j+1}}, \frac{|F^{(j)}(t)|}{j!A^j}, \frac{|F^{(j-1)}(t)|}{(j-1)!A^{j-1}}\right\}. \qquad (35)
\end{aligned}
$$

From (33), (34) and (35) we get the following result.

Proposition 7. *If* $n=2$, $m=0$, $b=-2$ *and* $-3\le a<0$ *then function* (8) *is of bounded* l*-index* $N(l,F;\,\mathbb{C}\setminus\mathbb{D}_{\eta R})\le 1$ *with* $l(|z|)\equiv 6/\eta R$, *where* $0<\eta<1$ *and* $0<R<5/12$.

Now let $n=3$, $m=2$, $-2\le a<0$ and $-(3+a)\le b<0$. If function (7) is a solution of differential equation (6) then for all $z\in\mathbb{C}$

$$tF''(t)+(2+a)F'(t)+bF(t)\equiv 0, \qquad (36)$$

whence for $0<\eta<1$ and $0<R<4/15$, $|t|\ge\eta R$ and $A\ge 2/\eta R$

$$\frac{|F''(t)|}{2!A^2}\le\frac{2+a}{2A\eta R}\frac{|F'(t)|}{1!A}+\frac{3+a}{2A^2\eta R}|F(t)|\le\frac{1}{2}\frac{|F'(t)|}{1!A}+\frac{3}{8}|F(t)|\le\max\left\{\frac{|F'(t)|}{1!A},|F(t)|\right\}. \qquad (37)$$

From (36) for $j\ge1$ it follows that $tF^{(j+2)}(t)+(2+a+j)F^{(j+1)}(t)+bF^{(j)}(t)\equiv0$, whence as above we get

$$
\begin{aligned}
&\frac{|F^{(j+2)}(t)|}{(j+2)!A^{j+2}}\le\frac{2+a+j}{(j+2)A\eta R}\frac{|F^{(j+1)}(t)|}{(j+1)!A^{j+1}}+\frac{3+a}{(j+2)(j+1)A^2\eta R}\frac{|F^{(j)}(t)|}{j!A^j}\le\\
&\le\frac{1}{A\eta R}\frac{|F^{(j+1)}(t)|}{(j+1)!A^{j+1}}+\frac{1}{2A^2\eta R}\frac{|F^{(j)}(t)|}{j!A^j}\le\frac{1}{2}\frac{|F^{(j+1)}(t)|}{(j+1)!A^{j+1}}+\frac{1}{8}\frac{|F^{(j)}(t)|}{j!A^j}\le\\
&\le\max\left\{\frac{|F^{(j+1)}(t)|}{(j+1)!A^{j+1}},\frac{|F^{(j)}(t)|}{j!A^j}\right\}. \qquad (38)
\end{aligned}
$$

From (37) and (38) the following statement follows.

Proposition 8. *If* $n=3$, $m=2$, $-2\le a<0$ *and* $-(3+a)\le b<0$ *then function* (12) *is of bounded* l*-index* $N(l,F;\,\mathbb{C}\setminus\mathbb{D}_{\eta R})\le 1$ *with* $l(|z|)\equiv 2/\eta R$, *where* $0<\eta<1$ *and* $0<R<4/15$.

Finally, let $n=3$, $m=0$, $a=0$ and $-3\le b<0$. If function (7) is a solution of differential equation (6) then (36) with $a=0$ holds for all $z\in\mathbb{C}$, whence for $|t|\ge\eta R$ and $A\ge2/\eta R$ we get (37), (38) and the following statement.

Proposition 9. *If* $n=3$, $m=0$, $a=0$ *and* $-3\le b<0$ *then function* (10) *is of bounded* l*-index* $N(l,F;\,\mathbb{C}\setminus\mathbb{D}_{\eta R})\le 1$ *with* $l(|z|)\equiv 2/\eta R$, *where* $0<\eta<1$ *and* $0<R<8/11$.

Uniting Propositions 4 and 7, 5 and 9, 6 and 8, we obtain the following theorem.

Theorem 7. *The following statements are true:*

(i) if $n=2$, $m=0$, $b=-2$ and $-3\le a<0$ then function (8) *is of bounded l-index $N(l,F;\mathbb{C})\le 1$ with $l(|z|)\equiv 36$;*

(ii) if $n=3$, $m=0$, $a=0$ and $-3\le b<0$ then for function (10) *$N(l,F-2/|b|;\mathbb{D}_{4/27})\le 1$ and $N(l,F;\,\mathbb{C}\setminus\mathbb{D}_{4/27})\le 1$ with $l(|z|)\equiv 27/2$;*

(iii) if $n=3$, $m=2$, $-2\le a<0$ and $-(3+a)\le b<0$ then for function (12) *$N(l,F-(2+a)/|b|;\,\mathbb{D}_{2/39})\le 1$ and $N(l,F;\,\mathbb{C}\setminus\mathbb{D}_{2/39})\le 1$ with $l(|z|)\equiv 39$.*

Proof. From Propositions 4 and 7 it follows that function (8) is of bounded l-index $N(l,F;\mathbb{C})\le 1$ with $l(|z|)=\max\left\{\frac{5+3R}{(1-\eta)R(5-12R)},\frac{6}{\eta R}\right\}$, where $0<\eta<1$ and $0<R<5/12$. Choosing $R=1/3$ and $\eta=1/2$ we come to the statement (i).

If we choose $R=8/27$ and $\eta=1/2$ then $\eta R=4/27$ and $(8+5R)/((1-\eta)R(8-11R))= =2/\eta R=27/2$. Therefore, Propositions 5 and 9 imply the statement (ii).

Finally, if we choose $R=4/39$ and $\eta=1/2$ then $\eta R=2/39$ and $2/\eta R=39= =(4+9R)/((1-\eta)R(4-15R))$ Therefore, Propositions 6 and 8 imply the statement (iii). Theorem 7 is proved. □

It remains to study the boundedness of the l-index of function (10) in the case if $n=3$, $m=0$, $a<0$ and $b<0$. Since $F_2=|b|/6$, from (14) we have

$$\sum_{k=2}^{\infty}kF_kR^{k-1}=2F_2R+\sum_{k=2}^{\infty}(k+1)F_{k+1}R^k=\frac{|b|R}{3}+\sum_{k=2}^{\infty}\frac{|b|}{(k+2)}F_kR^k+$$
$$+\sum_{k=2}^{\infty}\frac{|a|(k-1)}{(k+2)}F_{k-1}R^k=\frac{|b|R}{3}+\sum_{k=2}^{\infty}\frac{|b|}{k(k+2)}kF_kR^k+\sum_{k=1}^{\infty}\frac{|a|}{k+3}kF_kR^{k+1}\le$$
$$\le\frac{|b|R}{3}+\sum_{k=2}^{\infty}\frac{|b|R}{8}kF_kR^{k-1}+\frac{|a|R^2}{4}+\sum_{k=2}^{\infty}\frac{|a|R^2}{5}kF_kR^{k-1}.$$

If we put $a(R)=\frac{10(4|b|R+3|a|R^2)}{3(40-5|b|R-8|a|R^2)}$ then from hence we get $\sum_{k=2}^{\infty}kF_kR^{k-1}\le a(R)$. We note that $a(R)<1$ if $54|a|R^2+55|b|R-120<0$, i.e.

$$R<Q(a,b):=(-55|b|+\sqrt{3025|b|^2+25920|a|})/108.$$

Therefore, by Lemma 1 we obtain the following statement.

Proposition 10. *If $n=3$, $m=0$, $a<0$ and $b<0$ then for function* (10) *$N(l,F-2/|b|;\mathbb{D}_{\eta R})\le 1$ with $l(|t|)\equiv\frac{120+6|a|R^2+25|b|R}{(1-\eta)R(120-54|a|R^2-55|b|R)}$, where $0<\eta<1$ and $0<R<Q(a,b)$.*

Note that $R<Q(a,b)\le 1$ if $120|a|\le 54+5|b|$. In what follows we will assume that this condition is satisfied. Then $0<\eta R<1$ for $0<\eta<1$ and $0<R<Q(a,b)$.

For research boundedness of the l-index in the $\mathbb{C}\setminus\mathbb{D}_{\eta R}$, we note that in the current case

$$tF''(t)+(at^2+2)F'(t)+bF(t)\equiv 0, \tag{39}$$

whence for $0<\eta<1$ and $0<R<Q(a,b)$, $|t|\ge\eta R$ and $l(|t|)=A|t|$ with $A\ge A_1:=\frac{|a|}{2}+\frac{1}{(\eta R)^2}+\frac{|b|}{2(\eta R)^3}>1$ we obtain

$$\frac{|F''(t)|}{2!l^2(|t|)}\le\frac{|a||t|^2+2}{2|t|l(|t|)}\frac{|F'(t)|}{1!l(|t|)}+\frac{|b|}{2|t|l^2(|t|)}|F(t)|=\frac{|a||t|^2+2}{2A|t|^2}\frac{|F'(t)|}{1!l(|t|)}+\frac{|b|}{2|t|A^2|t|^2}|F(t)|\le$$

$$\leq \frac{1}{A}\left(\frac{|a|}{2}+\frac{1}{(\eta R)^2}+\frac{|b|}{2(\eta R)^3}\right)\max\left\{\frac{|F'(t)|}{1!l(|t|)}, |F(t)|\right\} \leq \max\left\{\frac{|F'(t)|}{1!l(|t|)}, |F(t)|\right\}. \tag{40}$$

From (39) it follows that $tF'''(t)+(at^2+3)F''(t)+(2at+b)F'(t)\equiv 0$, whence as above for $A\geq A_2 := \frac{|a|}{3}+\frac{|a|+3}{3(\eta R)^2}+\frac{|b|}{6(\eta R)^3}>1$ we get

$$\frac{|F'''(t)|}{3!l^3(|t|)} \leq \frac{|a||t|^2+3}{3A|t|^2}\frac{|F''(t)|}{2!l^2(|t|)}+\frac{2|a||t|+|b|}{6A^2|t|^3}\frac{|F'(t)|}{1!l(|t|)} \leq$$

$$\leq \frac{1}{A}\left(\frac{|a|}{3}+\frac{1}{(\eta R)^2}+\frac{|a|}{3(\eta R)^2}+\frac{|b|}{6(\eta R)^3}\right)\max\left\{\frac{|F''(t)|}{2!l^2(|t|)}, \frac{|F'(t)|}{1!l(|t|)}\right\} \leq$$

$$\leq \max\left\{\frac{|F''(t)|}{2!l^2(|t|)}, \frac{|F'(t)|}{1!l(|t|)}\right\}. \tag{41}$$

Finally, let $j\geq 2$. Then (39) implies

$$tF^{(j+2)}(t)+(at^2+2+j)F^{(j+1)}(t)+(2jat+b)F^{(j)}(t)+j(j-1)aF^{(j-1)}(t)\equiv 0,$$

whence as above for $A\geq A_3 := \frac{|a|}{4}+\frac{3+|a|}{3(\eta R)^2}+\frac{|b|}{12(\eta R)^3}+\frac{|a|}{10(\eta R)^4}>1$ we get

$$\frac{|F^{(j+2)}(t)|}{(j+2)!l^{j+2}(|t|)} \leq \frac{|a||t|^2+j+2}{(j+2)A|t|^2}\frac{|F^{(j+1)}(t)|}{(j+1)!l^{j+1}(|t|)}+$$

$$+\frac{2j|a||t|+|b|}{(j+2)(j+1)A^2|t|^3}\frac{|F^{(j)}(t)|}{j!l^j(|t|)}+\frac{(j-1)|a|}{(j+2)(j+1)A^3|t|^4}\frac{|F^{(j-1)}(t)|}{(j-1)!l^{j-1}(|t|)} \leq$$

$$\leq \frac{1}{A}\left(\frac{|a|}{4}+\frac{3+|a|}{3(\eta R)^2}+\frac{|b|}{12(\eta R)^3}+\frac{|a|}{10(\eta R)^4}\right)\times$$

$$\times \max\left\{\frac{|F^{(j+1)}(t)|}{(j+1)!l^{j+1}(|t|)}, \frac{|F^{(j)}(t)|}{j!l^j(|t|)}, \frac{|F^{(j-1)}(t)|}{(j-1)!l^{j-1}(|t|)}\right\} \leq$$

$$\leq \max\left\{\frac{|F^{(j+1)}(t)|}{(j+1)!l^{j+1}(|t|)}, \frac{|F^{(j)}(t)|}{j!l^j(|t|)}, \frac{|F^{(j-1)}(t)|}{(j-1)!l^{j-1}(|t|)}\right\}. \tag{42}$$

In view of (40), (41) and (42) the following proposition is proved.

Proposition 11. *If* $n=3$, $m=0$, $a<0$ *and* $b<0$ *then for function* (10) $N(l,F;\mathbb{C}\setminus\mathbb{D}_{\eta R})\leq 1$ *with* $l(|t|)\equiv\max\{A_1,A_2,A_3\}|t|$, *where* $0<\eta<1$ *and* $0<R<Q(a,b)$.

It's easy to check that $\max\{A_1,A_2,A_3\}\leq\frac{|a|}{2}+\frac{|a|+3}{3(\eta R)^2}+\frac{|b|}{2(\eta R)^3}+\frac{|a|}{10(\eta R)^4}$. Therefore, uniting Propositions 10 and 11, we obtain the following theorem.

Theorem 8. *Let* $n=3$, $m=0$, $a<0$, $b<0$ *and* $120|a|\leq 54+5|b|$. *Suppose that* $0<\eta<1$ *and* $0<R<Q(a,b)=\frac{-55|b|+\sqrt{3025|b|^2+25920|a|}}{108}$. *Then for function* (10) $N(l,F-2/|b|;\mathbb{D}_{\eta R})\leq 1$ *with* $l(|t|)\equiv\frac{120+6|a|R^2+25|b|R}{(1-\eta)R(120-54|a|R^2-55|b|R)}$ *and* $N(l,F;\mathbb{C}\setminus\mathbb{D}_{\eta R})\leq 1$ *with* $l(|t|)\equiv(\frac{|a|}{2}+\frac{|a|+3}{3(\eta R)^2}+\frac{|b|}{2(\eta R)^3}+\frac{|a|}{10(\eta R)^4})|t|$.

4. Discussion of Open Problems. Differential equation (10) is the second order equation. Its generalization is the equation $(1-z)^n w^{(k)}+\sum_{j=1}^{k}a_j(1-z)^{m_j}w^{(k-j)}=0$, where $k\geq 3$ and $n>m_1>...>m_k\geq 0$. The problem of finding analytical solutions of this equation, which are convex, close-to-convex, starlike or of bounded l-index, is relevant. Remark that the same problems for similar equations with slightly different coefficients were considered in [17–19].

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Lviv Ivan Franko National University
Lviv, Ukraine

m.m.sheremeta@gmail.com
yurkotrukhan@gmail.com

Received 05.10.2025
Revised 13.03.2026