Математичні Студії. Т.66, №1 Matematychni Studii. V.66, No.1
ISSN 1027-4634 (print) ISSN 2411-0620 (online)

V. S. ILKIV, Z. M. NYTREBYCH

ON THE CRITERIA FOR EXTRACTING A FACTOR FROM A MATRIX POLYNOMIA

V. S. Ilkiv, Z. M. Nytrebych. *On the criteria for extracting a factor from a matrix polynomia*, Mat. Stud. **66** (2026), 3–10.

The problem of extracting a monic factor with a prescribed determinant from a matrix polynomial, i.e., the factorization problem, is studied. Based on Kazimirskii's well-known criterion, new factorization criteria using only the coefficients of the matrix polynomial and those of the prescribed determinant are established. An algorithm for determining such a factor is also proposed, and examples illustrating one of the criteria are presented.

New forms of a criterion for extracting a monic factor with a given determinant from a matrix polynomial have been obtained. The algorithm for calculating the coefficients of the desired factor, which consists in solving a (reduced) system of matrix equations, does not involve finding the roots of the characteristic polynomial or the adjugate matrix. It uses only the coefficients of the given matrix polynomial and the coefficients of the determinant of the factor.

1. Introduction. The problem of extracting a factor from a matrix polynomial $A(z)$, i.e., obtaining a factorization

$$A(z) = B(z)D(z),$$

where $B(z)$ and $D(z)$ are matrix polynomials of positive degree, arises in many areas of mathematics and its applications. In particular, matrix polynomial factorization plays an important role in the control theory of linear systems, the theory of systems of differential equations, and many other branches of mathematics, as well as in the study of wave diffraction, dynamic elasticity, fracture mechanics, and geophysics.

The study of the solvability of elliptic problems led Ya. B. Lopatynsky to establish the following divisibility criterion for matrix polynomials [1, 2].

Theorem 1 (Lopatynsky). *Let $A(z) = A_0 z^s + A_1 z^{s-1} + \cdots + A_s$, where $A_0, A_1, \ldots, A_s$ are complex $p \times p$ matrices, $s \geq 2$, and $\det A_0 \neq 0$. Assume that among the ps roots of the equation $\det A(z) = 0$ one can select pk roots, where $k < s$, that are distinct from the remaining roots. Then there exists a matrix polynomial $B(z) = B_0 z^k + B_1 z^{k-1} + \cdots + B_k$, with $\det B_0 \neq 0$, whose determinant has precisely the selected pk roots and which is a left divisor of $A(z)$ if and only if the matrix*

$$\operatorname{Res}_+\Big(z^{\alpha+\beta-2} A^{-1}(z)\Big)_{\substack{\alpha=1,\ldots,s\\ \beta=1,\ldots,k}},$$

2020 *Mathematics Subject Classification:* 15A23.
Keywords: matrix polynomial; matrix factorization; monic factor; factorization criteria.
doi:10.30970/ms.66.1.3-10
*Corresponding author: V. S. Ilkiv

where $\mathrm{Res}_+(S)$ *denotes the sum of the residues of the matrix* S *at the selected* pk *roots, has rank* pk*.*

2. Basic notations and auxiliary results. For the polynomial

$$\xi = \xi(z) = (z - z_1)^{t_1} \cdots (z - z_l)^{t_l},$$

of degree $t = t_1 + \cdots + t_l \geq 1$, with distinct complex roots $z_1, \ldots, z_l$, and for a matrix-valued function $G = G(z)$ analytic in a domain containing the points $z_1, \ldots, z_l$, let $M_G(\xi)$ denote the matrix of values of G on the root system of the polynomial ξ, i.e., the block column matrix consisting of t square blocks,

$$M_G(\xi) = (G_1, \ldots, G_l)^T \qquad G_j = \mathrm{col}\left(\tfrac{1}{(\alpha-1)!} G^{(\alpha-1)}(z_j)\right)_{\alpha=1}^{t_j}.$$

In the particular case $\xi = z^t$, we write $M_G(\xi) = \mathrm{col}\left(\frac{G^{(\alpha-1)}(0)}{(\alpha-1)!}\right)_{\alpha=1}^{t}$ simply as $M_G[t]$.

For a matrix-valued function H and $g \in \mathbb{N}$, we define the accompanying matrix $\lfloor H, g \rceil$ to be the block row matrix consisting of g blocks,

$$\lfloor H, g \rceil = \lfloor H, g \rceil(z) = \big(H(z)\ zH(z)\ \ldots\ z^{g-1}H(z)\big). \tag{1}$$

If H is a matrix polynomial of degree p, then $\lfloor H, g \rceil$ is a matrix polynomial of degree $p + g - 1$.

For an arbitrary polynomial ξ, define the matrix $W_{g,n}(\xi)$ by

$$W_{g,n}^{\top}(\xi) = M_{\lfloor I_n, g \rceil}(\xi), \tag{2}$$

where $^{\top}$ denotes matrix transposition and $\lfloor I_n, g \rceil$ is the accompanying matrix associated with the identity matrix I_n. The matrix $W_{g,n}(\xi)$ has size $gn \times tn$ and rank $n \min(g, t)$. It is a block matrix of block size $g \times t$, where t is the degree of the polynomial ξ. If $p < g$, then $W_{p,n}(\xi)$ consists of the first p block rows of $W_{g,n}(\xi)$.

We next consider several properties of the matrices $M_G(\xi)$, $\lfloor H, g \rceil$, and $W_{g,n}(\xi)$.

Let $G = \tilde{G}\,\hat{G}$, where $\tilde{G}$ and $\hat{G}$ are matrix-valued functions analytic in a domain containing the points $z_1, \ldots, z_l$, and suppose that $\tilde{G}$ has a zero of multiplicity t_j at the point z_j for $j = 1, \ldots, l$, i.e., $\tilde{G}$ is divisible by ξ. Then $M_G(\xi) = 0$.

Indeed,

$$\frac{1}{(\alpha-1)!} G^{(\alpha-1)}(z) = \sum_{\beta=1}^{\alpha} \frac{\tilde{G}^{(\beta-1)}(z)}{(\beta-1)!} \frac{\hat{G}^{(\alpha-\beta)}(z)}{(\alpha-\beta)!}, \qquad \alpha = 1, \ldots, t_j,$$

and these matrices vanish at the point z_j, since $\tilde{G}^{(\beta-1)}(z_j) = 0$ for $\beta = 1, \ldots, t_j$ and $j = 1, \ldots, l$.

In particular,

$$M_{\xi\hat{G}}(\xi) = 0. \tag{3}$$

For the product of a matrix-valued function $H = H(z)$ and a matrix polynomial $G = G(z)$ of degree $p - 1$, the identity

$$H(z)G(z) = H(z)G(0) + zH(z)G'(0) + \cdots + z^{p-1}H(z)\frac{G^{(p-1)}(0)}{(p-1)!} = \lfloor H, p \rceil M_G[p] \tag{4}$$

holds. Consequently, $G(z) = \lfloor I, p \rceil M_G[p]$, where I is the identity matrix.

From (4) and the obvious identity $M_{HK}(\xi) = M_H(\xi)K$, which holds for every constant matrix K, we obtain

$$M_{HG}(\xi) = M_{\lfloor H,p \rceil}(\xi) M_G[p], \qquad M_G(\xi) = W_{p,h}^{\top}(\xi) M_G[p], \qquad \lfloor FH, p \rceil = F \lfloor H, p \rceil, \qquad (5)$$

where h is the number of rows of the matrix G, and $F = F(z)$ is a matrix-valued function. In the particular case,

$$M_{HG}[t] = M_{\lfloor H,p \rceil}[t] M_G[p].$$

The mappings

$$G \mapsto M_G(\xi), \qquad G \mapsto \lfloor G, p \rceil,$$

are linear.

If $H = (H_1 \ H_2)$, then

$$M_H(\xi) = \big(M_{H_1}(\xi) \ M_{H_2}(\xi)\big),$$

and if $G = \operatorname{col}(G_1, G_2)$, then

$$\lfloor G, p \rceil = \operatorname{col}\big(\lfloor G_1, p \rceil, \lfloor G_2, p \rceil\big).$$

Continuing Lopatynsky's work, P. S. Kazimirskii established in [3] a factorization criterion based on the two special matrices introduced above: the matrix of values on the root system of a polynomial and the accompanying matrix.

Let $A(z)$ be a square matrix polynomial over the field $\mathbb{C}$ of the form

$$A = A(z) = A_0 z^m + A_1 z^{m-1} + \cdots + A_m, \qquad (6)$$

where $A_0, A_1, \ldots, A_m$ are complex matrices of order n. Assume that $\det A = \Delta = \Delta(z)$ has the form

$$\Delta = \varphi\psi = \varphi(z)\psi(z), \qquad (7)$$

where φ is a monic polynomial of degree rn, and a monic polynomial matrix $B = B(z)$ is a left divisor of A:

$$A = BD, \qquad (8)$$

where

$$B = B(z) = B_0 z^r + B_1 z^{r-1} + \cdots + B_r, \qquad B_0 = I_n, \qquad \det B = \varphi, \qquad (9)$$

$B_1, \ldots, B_r$ are matrices of order n, $D = D(z)$ is a matrix polynomial with $\det D = \psi$, and I_n is the identity matrix of order n.

Let $S = \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n)$ denote the Smith normal form of A, where

$$S = P^{-1} A Q^{-1} = \Phi\Psi, \qquad (10)$$

with $\Phi = \operatorname{diag}(\varphi_1, \ldots, \varphi_n)$ and $\Psi = \operatorname{diag}(\psi_1, \ldots, \psi_n)$ diagonal matrices with monic entries. Moreover, $\varepsilon_j \mid \varepsilon_{j+1}$ for $j = 1, \ldots, n-1$.

Let $\Sigma = \{\Phi_1, \ldots, \Phi_t\}$ be the set of matrices Φ satisfying (10) and $\det \Phi = \varphi$.

For $\Phi = \operatorname{diag}(\varphi_1, \ldots, \varphi_n)$, consider the lower triangular matrix $W(\Phi)$ with unit diagonal and entries

$$W(\Phi)_{ij} = \frac{\varphi_i k_{ij}}{(\varphi_i, \varepsilon_j)},$$

where $(\varphi_i, \varepsilon_j)$ denotes the greatest common divisor of the polynomials φ_i and ε_j,

$$k_{ij} = \begin{cases} 0, & (\varphi_i, \varepsilon_j) = \varphi_j; \\ k_{ij0} + k_{ij1} z + \cdots + k_{ijh} z^h, & (\varphi_i, \varepsilon_j) \neq \varphi_j, \end{cases}$$

$$h = h_{ij} = \deg \frac{(\varphi_i, \varepsilon_j)}{\varphi_j} - 1, \qquad i = 2, \dots, n, \quad j = 1, \dots, i-1,$$

where k_{ijl} are parameters introduced in [4, p. 181] over the field $\mathbb{C}$.

The field obtained by adjoining the coefficients of all polynomials k_{ij} is denoted by $\mathbb{C}(k)$. The following two theorems are considered over this field [5, p. 190, 222].

Theorem 2. *Let condition* (7) *be satisfied. The matrix polynomial A admits a factorization of the form* (8) *if and only if*

$$\max_{\Phi \in \Sigma} \operatorname{rank} M_{\lfloor W(\Phi)P, r+1 \rceil}(\varphi) = rn. \tag{11}$$

Theorem 3. *Let condition* (7) *be satisfied. If*

$$\operatorname{rank} M_{\lfloor W(\Phi_j)P, r+1 \rceil}(\varphi) = rn, \qquad j = 1, \dots, s,$$

then

$$A = B_1 D_1 = \cdots = B_s D_s, \qquad \det B_1 = \cdots = \det B_s = \varphi, \tag{12}$$

where $B_j(z) = I_n z^r + N_{j1} z^{r-1} + \cdots + N_{jr}$, and the coefficients N_{jl} are determined as solutions of the matrix equations

$$M_{\lfloor W(\Phi_j)P, r+1 \rceil}(\varphi) \left(N_{jr}, \dots, N_{j1}, I_n\right)^T = 0, \qquad j = 1, \dots, s. \tag{13}$$

Remark 1. By assigning admissible values from $\mathbb{C}$ to the parameters k_{ijl} of the matrices $W(\Phi_1), \dots, W(\Phi_s)$, one obtains monic polynomial matrices over $\mathbb{C}[z]$ that are left divisors of A and whose Smith forms coincide with one of the Smith forms $\Phi_1, \dots, \Phi_s$.

Using formulas (6)–(9), we formulate the following factorization criterion, which is a weaker version of Theorem 3.6 in [5, p. 135]. For completeness, we include a brief proof.

Theorem 4. *Let $A = A(z)$ be a regular matrix polynomial* ($\det A \neq 0$) *of the form* (6), *and let φ and ψ be relatively prime polynomials of degrees rn and $(m-r)n$, respectively.*

A factorization of the form (8) *exists if and only if the system of linear algebraic matrix equations*

$$M_{\lfloor A_*, r+1 \rceil}(\varphi)\, X = 0, \tag{14}$$

where $M_G(\varphi)$ denotes the matrix of values of G on the root system of φ, A_* is the adjugate matrix, and $\lfloor H, l \rceil$ is the accompanying matrix, admits a solution of the form

$$X = \operatorname{col}(X_r, \dots, X_1, X_0), \qquad X_0 = I_n.$$

Moreover, the mapping

$$X \longmapsto B(z) = B_X(z) = X_0 z^r + X_1 z^{r-1} + \cdots + X_r$$

establishes a one-to-one correspondence between solutions of (14) *and factors $B(z)$ in* (8), (9).

Proof. Necessity. Assume that a factorization $A = BD$ exists, where $B_0 = I_n$ and $\det B = \varphi$. Taking adjugates, we obtain $A_* = D_* B_*$. Multiplying this identity on the right by B and using

$$B_* B = (\det B) I_n = \varphi I_n,$$

we obtain $A_* B = \varphi D_*$. The left-hand side can be written as

$$A_* B = \lfloor A_*, r+1 \rceil \operatorname{col}(B_r, \dots, B_1, B_0).$$

Let λ be a root of $\varphi(z)$ of multiplicity $k \in \mathbb{N}$. Then

$$\frac{d^j}{dz^j}[\varphi(z) D_*(z)]\Big|_{z=\lambda} = 0, \qquad j = 0, 1, \dots, k-1.$$

The same relations hold for $A_* B$, hence $M_{A_* B}(\varphi) = 0$. Therefore,

$$M_{\lfloor A_*, r+1 \rceil}(\varphi) \operatorname{col}(B_r, \dots, B_1, B_0) = 0,$$

i.e., system (14) admits a solution $X = \operatorname{col}(B_r, \dots, B_1, B_0)$.
Sufficiency. Assume that system (14) admits a solution

$$X = \operatorname{col}(X_r, \dots, X_1, X_0), \qquad X_0 = I_n.$$

Define $B(z) = X_0 z^r + X_1 z^{r-1} + \dots + X_r$. Then according to (5) equation (14) is equivalent to $M_{A_* B}(\varphi) = 0$. By the definition of $M_G(\varphi)$,

$$\frac{d^j}{dz^j}[A_*(z) B(z)]\Big|_{z=\lambda} = 0, \qquad j = 0, 1, \dots, k-1,$$

for every root λ of $\varphi(z)$. Hence $A_*(z)B(z)$ is divisible by $\varphi(z)$, i.e.,

$$A_*(z) B(z) = \varphi(z) F(z)$$

for some matrix polynomial $F(z)$.

Taking determinants gives

$$\det(A_* B) = (\det A)^{n-1} \det B = \varphi^{n-1} \psi^{n-1} \det B, \qquad \det(\varphi F) = \varphi^n \det F,$$

hence

$$\psi^{n-1} \det B = \varphi \det F.$$

Since φ and ψ are relatively prime, φ divides $\det B$. Because $\det B$ is monic of degree nr, we conclude that $\det B = \varphi$, $\det F = \psi^{n-1}$. Thus $A_* B = (\det B) F$, $A_* = F B_*$.

Multiplying on the left by F_* yields $F_* A_* = \psi^{n-1} B_*$, and multiplying on the right by A gives $\varphi F_* = \psi^{n-2} B_* A$. Hence $F_* = \psi^{n-2} D$, where $D(z)$ is a matrix polynomial.

Multiplying by B we obtain $B F_* = \psi^{n-2} A$. Substituting $F_* = \psi^{n-2} D$, we get $A = BD$, which is the required factorization. $\square$

Remark 2. Since $(\varphi, \psi) = 1$, the factorization $A = BD$ is either unique or does not exist. This follows from Theorem 5.1 (p. 151) and Theorem 4.1 (p. 123) of [4].

The above divisibility criteria for matrix polynomials involve the roots of polynomials, which in general can be computed only approximately, as well as inverse or adjugate matrices. These features limit their applicability. Therefore, we develop alternative criteria that avoid these drawbacks.

More generally, the factorization of matrix functions has been extensively studied and continues to attract considerable attention. A comprehensive overview can be found in the monographs [6, 5, 7, 8, 4].

A preliminary announcement of the results presented in this paper appeared in [9].

3. New criteria. To obtain new criteria for extracting a factor from a matrix polynomial, we will transform formula (14). A similar approach was employed in [10, 11] to derive the ellipticity condition (Lopatynsky condition) for boundary value problems of elliptic systems. In this context, the theory of partial differential equations is closely related to algebraic problems in matrix theory [12]. If the Lopatynsky condition is not satisfied, the problem of small denominators (divisors) may arise, the solution of which requires methods of metric Diophantine approximations from number theory [11, 13, 14].

Let us fix the positive integers n, m, r in formulas (6)–(9), where $n > 1$, $m > 1$, and $m > r$. Henceforth, we assume $I = I_n$ and introduce the following indices:

$$q = (n-1)(m-r)+1, \qquad q_1 = q + rn = (n-1)m + r + 1. \tag{15}$$

Obviously, q and q_1 are positive integers such that $q > 1$ and $q_1 > 3$.

Theorem 5 (The first criterion)**.** *If $G_1 = \lfloor A_*, r+1\rceil$, $G_2 = \lfloor \varphi I, q\rceil$, $G = (G_1\ G_2)$, the polynomials φ and ψ in* (6) *are coprime, then the criterion of factorization* (8) *is the solvability of the system of linear matrix equations*

$$M_G[q_1]\begin{pmatrix} X \\ Y \end{pmatrix} = 0, \tag{16}$$

where the matrix $Y = \operatorname{col}(Y_{q-1}, \dots, Y_1, Y_0)$ has square blocks Y_j of order n.

Proof. The system (14) can be rewritten as $M_{G_1}(\varphi)X = 0$, and the matrix $M_{G_2}(\varphi)$ can be rewritten as $M_{\lfloor \varphi I, q\rceil}(\varphi)$ or $M_{\varphi\lfloor I, q\rceil}(\varphi)$. By the formula (3) we have $M_{G_2}(\varphi) = 0$, therefore

$$M_{G_1}(\varphi)X = \big(M_{G_1}(\varphi)\ M_{G_2}(\varphi)\big)\begin{pmatrix} X \\ Y \end{pmatrix} = M_G(\varphi)\begin{pmatrix} X \\ Y \end{pmatrix}.$$

Taking into account that the matrix polynomials G_1, G_2 and G have the same degree $q_1 - 1$, using the factorization formula (5), we obtain $M_G(\varphi) = W^\top_{q_1,n}(\varphi) M_G[q_1]$.

We represent the matrix $(0\ I_{qn})M_G[q_1]$ in the form $(G_3\ G_4)$, where

$$G_3 = (0\ I_{qn})M_{\lfloor A_*, r+1\rceil}[q_1], \qquad G_4 = (0\ I_{qn})M_{\lfloor \varphi I, q\rceil}[q_1],$$

and

$$\varphi(z) = \varphi_0 z^{rn} + \varphi_1 z^{rn-1} + \dots + \varphi_{rn}, \quad \varphi_0 = 1, \quad \varphi_{-1} = \ldots = \varphi_{1-q} = 0,$$

then $G_4 = \big(\varphi_{j-i} I\big)_{i,j=1,\dots,q}$ is a square block-Toeplitz triangular matrix of order qn and $\det G_4 = 1$.

The product of the matrices $\begin{pmatrix} W^{\top}_{q_1,n}(\varphi) \\ 0 \quad I_{qn} \end{pmatrix} M_G[q_1]$ equals to $\begin{pmatrix} M_G(\varphi) & 0 \\ G_3 & G_4 \end{pmatrix}$, therefore, the systems $\begin{pmatrix} M_G(\varphi)X \\ G_4^{-1}G_3X+Y \end{pmatrix} = 0$ and

$$\begin{pmatrix} W^{\top}_{q_1,n}(\varphi) \\ 0 \quad I_{qn} \end{pmatrix} M_G[q_1] \begin{pmatrix} X \\ Y \end{pmatrix} = 0, \tag{17}$$

are equivalent.

The system $\begin{pmatrix} M_G(\varphi)X \\ G_4^{-1}G_3X+Y \end{pmatrix} = 0$ is solvable if and only if the system (14) is solvable. Since

$$\det \begin{pmatrix} W^{\top}_{q_1,n}(\varphi) \\ 0 \quad I_{qn} \end{pmatrix} = \det W_{rn,n}(\varphi) = \big(\det W_{rn,1}(\varphi)\big)^n \neq 0,$$

then the systems (16) and (17) are equivalent. □

The matrix $M_G[q_1]$ in formula (16) is independent of the roots of the polynomial φ, unlike the matrix $M_{\lfloor A_*,r+1\rceil}(\varphi)$ in system (14). In the subsequent criterion, we shall also avoid using the polynomial matrix A_*, which is the adjugate of A.

Theorem 6 (Second Criterion)**.** *If the block-row matrix $Q_1(z)$ with $q+r+1$ blocks is of the form $Q_1 = \big(\lfloor \Delta, r+1\rceil \; \lfloor \varphi A, q\rceil\big)$, then system* (16) *and system*

$$M_{Q_1}[q_1+m] \begin{pmatrix} X \\ Y \end{pmatrix} = 0 \tag{18}$$

are equivalent; that is, the criterion of factorization is the solvability of system (18).

Proof. The matrix polynomial Q_1 can be represented as the product:

$$Q_1 = \big(\lfloor AA_*, r+1\rceil \; \lfloor \varphi A, q\rceil\big) = A\big(\lfloor A_*, r+1\rceil \; \lfloor \varphi I, q\rceil\big) = AG,$$

where $q_1 - 1$ is the degree of G. According to formula (5), we obtain:

$$M_{Q_1}[q_1+m] = M_{\lfloor A,q_1\rceil}[q_1+m]M_G[q_1].$$

The block matrix $\lfloor A, q_1\rceil$ of size $n \times q_1 n$ has degree q_1+m-1. Thus, the rank of the matrix $M_{\lfloor A,q_1\rceil}[q_1+m]$ of size $(q_1+m)n \times q_1 n$ coincides with the rank of the square non-degenerate matrix $(0 \;\, I_{q_1 n})M_{\lfloor A,q_1\rceil}[q_1+m]$ (where 0 is of appropriate size), which has the determinant $(\det A_0)^{q_1} \neq 0$.

This implies the equivalence of the systems of matrix linear equations (16) and (18). □

Theorem 7 (Third Criterion)**.** *If $Q = (\lfloor \psi I, r+1\rceil \; \lfloor A, q\rceil)$, the polynomials φ and ψ in* (6) *are coprime, then the criterion of factorization* (8) *consists in the solvability of the system of linear matrix equations:*

$$M_Q[q+m] \begin{pmatrix} X \\ Y \end{pmatrix} = 0. \tag{19}$$

If a solution of the form $\mathrm{col}(X,Y)$ *exists, then the factor is given by*

$$B(z) = Iz^r + X_1 z^{r-1} + \cdots + X_r.$$

Proof. The matrix polynomial Q has degree $q+m-1$. Therefore, from the equality

$$Q_1 = \big(\lfloor \varphi\psi, r+1\rceil \ \lfloor \varphi A, q\rceil\big) = \varphi Q,$$

the following matrix relation holds $M_{Q_1}[q_1+m] = M_{\lfloor \varphi I, q+m\rceil}[q_1+m]M_Q[q+m]$. The rank of the matrix $M_{\lfloor \varphi I, q+m\rceil}[q_1+m]$ of the size $(q_1+m)n \times (q+m)n$ coincides with the rank of the square matrix

$$(0 \ I_{(q+m)n})M_{\lfloor \varphi I, q+m\rceil}[q+m],$$

whose determinant is $\varphi_0^{(q+m)n} = 1$. Consequently, the systems of linear matrix equations (18) and (19) are equivalent. □

It is obvious that by applying the matrix transposition operation, all the above criteria can be adapted to extract the right-hand factor.

References

1. Ya.B. Lopatinskij, *Decomposition of a Polynomial Matrix into a Product,* Nauchn. Zap. L'vov. Polytechn. Inst., Ser. Fiz. Mat. **38** (2) (1953), 3–7.
2. Ya.B. Lopatinskij, *On Some Properties of Polynomial Matrices,* Boundary Value Probl. of Math. Phys., Inst. of Math. AN USSR, Kiev, 1979, 108–116.
3. P.S. Kazimirs'kii, *To the Decomposition of a Polynomial Matrix into Linear Factors,* Dopovidi AN URSR **4** (1964), 446–448.
4. V. Shchedryk, Arithmetic of Matrices over Rings, Akademperiodyka, Kyiv, 2021. doi:10.15407/akademperiodika.430.278
5. P.S. Kazimirs'kii, Factorization of Matrix Polynomials, Pidstryhach Inst. for Appl. Probl. of Mech. and Math. of the NAS of Ukraine, Lviv, 2015.
6. I. Gohberg, P. Lancaster, L. Rodman, Matrix Polynomials, Academic Press, New York, 1982.
7. V.M. Petrychkovych, Generalized Equivalence of Matrices and Its Collections and Factorization of Matrices over Rings, Pidstryhach Inst. for Appl. Probl. of Mech. and Math. of the NAS of Ukraine, Lviv, 2015.
8. V.P. Shchedryk, *Factorization of Matrices over Elementary Divisor Domain,* Algebra Discrete Math. (2) (2009), 79–99.
9. V. Ilkiv, *On the Criterion for Extracting a Factor of a Matrix Polynomial,* 15th Ukraine algebra conference: book of abstracts, Lviv, July 8–12, 2025, p. 58.
10. V.S. Il'kiv, *New Forms of the Lopatynsky Condition,* Ukr. Math. J. **77** (2) (2025), 107–122. doi:10.3842/umzh.v77i2.8534
11. V.S. Ilkiv, P.I. Kalenyuk, *Verification Algorithm for Lopatynsky Condition,* Mathematical Modeling and Computing **11** (4) (2024), 1189–1197. doi:10.23939/mmc2024.04.1189
12. Ya.B. Lopatinskij, *On a Method of Reducing Boundary Problems for a System of Differential Equations of Elliptic Type to Regular Integral Equations,* Ukr. Math. J. **5** (2) (1953), 123–151; Engl. transl., Amer. Math. Transl. (2) **89** (1970), 149–183.
13. V.S. Il'kiv, B.Yo. Ptashnyk, *Problems for Partial Differential Equations with Nonlocal Conditions. Metric Approach to the Problem of Small Denominators,* Ukr. Math. J. **58** (12) (2006), 1847–1875. doi:10.1007/s11253-006-0172-8
14. B.Yo. Ptashnyk, V.S. Il'kiv, I.Ya. Kmit', V.M. Polishchuk, Nonlocal Boundary-Value Problems for Partial Differential Equations, Naukova Dumka, Kyiv, 2002.

Lviv Polytechnic National University, Department of Mathematics
Lviv, Ukraine

ilkivvv@i.ua
znytrebych@gmail.com

Received 19.12.2025
Revised 14.09.2026