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MIXED EXPONENTIAL STATISTICAL STRUCTURES AND THEIR APPROXIMATION OPERATORS

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The paper examines the construction and analysis of a new class of mixed exponential statistical structures that combine the properties of stochastic models and linear positive operators. The aim of the study is to introduce and analyze a generalized family of mixed exponential statistical structures and their corresponding linear positive operators, which include known operators as particular cases. We define two auxiliary statistical structures $\mathbf{B}$ and $\mathbf{H}$ through differential relations between their elements, and construct the main Phillips-type structure. Recurrent relations for the central moments are obtained, their properties are established, and the convergence and approximation accuracy of the constructed operators are investigated. The proposed approach allows mixed exponential structures to be viewed as a generalization of known statistical systems, providing a unified analytical and stochastic description. The results demonstrate that mixed exponential statistical structures can be used to develop new classes of positive operators with controllable preservation and approximation properties. The proposed methodology forms a basis for further research in constructing multidimensional statistical structures, analyzing operators in weighted spaces, and studying their asymptotic characteristics.

1. Introduction. In modern probability theory and mathematical statistics, studies of linear positive operators associated with various classes of distributions of random variables play a significant role. Such operators have deep theoretical significance and are widely used in problems of function approximation, modeling of random processes, and construction of numerical methods. Of particular interest are operators generated by stochastic structures that generalize classical distributions: Binomial, Poisson, Negative Binomial, Gamma, and Beta distributions.

The problem of constructing and studying the properties of positive operators has a long history in analysis. This direction was initiated by the work of R. S. Phillips, which proposed an inversion formula for the Laplace transform $f(x) = \int_0^\infty e^{-xt} d\mu(t)$, and showed the connection between semigroups of linear operators $T_t = \exp(tA)$ and statistical structures ([1]).

Further development of these ideas led to the emergence of operators of the J.L. Durrmeyer type ([2])

$$D_n(f;x) = \sum_{k=0}^{n} f_k \binom{n}{k} x^k (1-x)^{n-k}, \quad f_k = (n+1) \int_0^1 f(t) \binom{n}{k} t^k (1-t)^{n-k} dt.$$

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They combine a discrete and an integral component and provide accurate approximation of continuous functions.

Within the branch of Durrmeyer-type operators and their extensions, modifications, in particular the beta-type operators have attracted considerable attention. Thus, in the work of Deo (2008), direct results were obtained for the beta-operator variant of Durrmeyer's operators ([3])

$$B_n^{(\alpha,\beta)}(f;x) = (n+\alpha+\beta+1)\sum_{k=0}^{n} f_k B(k+\alpha+1, n-k+\beta+1),$$

where $B(\cdot,\cdot)$ is the beta function.

Subsequently, numerous variations of such operators emerged — Bernstein-Durrmeyer, Szász-Durrmeyer, Beta-Durrmeyer, which became basic tools in the study of the properties of positive linear operators ([4–6]). Works devoted to the quantitative estimates of convergence for operators that preserve certain functions play a special role. In particular, in Bigou (2019) ([7]), inequalities of the type $\|L_n(f) - f\| \le C\omega\big(f; \frac{1}{\sqrt{n}}\big)$, where $\omega(f;\delta)$ is the modulus of continuity, were obtained, which significantly expands the class of known estimates. These results significantly expand the class of known convergence estimates and create a basis for the construction of new generalized operators, which include the mixed exponential structures considered in this work.

In parallel, another direction is developing — the modeling of stochastic processes using mixed exponential distributions, which combine several exponential components and are used in reliability statistics, risk analysis, and queuing theory ([8]). Such models are distinguished by flexibility and analytical convenience, but the question of their integration with the operator approach remains open.

Thus, despite significant progress in the study of positive operators and mixed exponential distributions, the task of constructing a unified theoretical scheme that combines these directions remains topical. It is precisely such a scheme — in the form of mixed exponential statistical structures — that is proposed in this work.

The research method is the introduction and study of a generalized family of mixed exponential statistical structures and their corresponding linear positive operators, which encompass known operators as special cases.

2. Statistical Structures B. Let us denote by $b_{n,k}(x)$, $k \in \{0,1,2,\ldots\}$ an integer-valued statistical structure that depends on parameters $x \in X, n \in N$, and is such that

$$b(x)\frac{d}{dx}b_{n,k}(x) = (k - nx)b_{n,k}(x), \tag{1}$$

where $b(x)$ is a non-negative function, which we shall call the covariance characteristic of the structure. We shall call this structure the structure **B**.

Remark. Not every non-negative function $b(x)$ can be the covariance characteristic of structure **B**; for example, the functions $b(x) = 1,\ b(x) = x^2$ cannot be.

The structure **B** can be constructed as follows. Let us take the power series

$$\omega(y) = \sum_{k=0}^{\infty} a_k y^k$$

with non-negative coefficients, $a_k \ge 0,\ 0 \le y < R,\ R$ — the radius of convergence of the series. Let us consider a random variable ξ, which can take non-negative integer values with probabilities

$$P\{\xi = k\} := b_k \frac{y^k}{(\omega(y))^n},\ k \in \{0,1,2,\ldots\}, \tag{2}$$

where b_k are the coefficients of the power series expansion of the function $(\omega(y))^n,\ n \in N$.

The sequence (1) defines a statistical structure with parameters y and n, where

$$\mathrm{M}\,\xi = ny\frac{\omega'(y)}{\omega(y)}, \text{ and } \mathrm{D}\,\xi = ny\frac{d\,\mathrm{M}\,\xi}{dy}.$$

Let us denote by $y(x)$ the function inverse to the function (it exists because $\mathrm{D}\,\xi \geq 0$)

$$x = y\frac{\omega'(y)}{\omega(y)}, \tag{3}$$

and by $b(x) := \frac{y(x)}{y'(x)}$. Then the mathematical expectation $\mathrm{M}\,\xi = nx$, and the variance $\mathrm{D}\,\xi = nb(x)$. As a result, we obtain a family of distributions that will depend on the parameter x

$$b_{n,k}(x) := b_k\frac{(y(x))^k}{(\omega(y(x)))^n}.$$

Let us prove that the functions $b_{n,k}(x)$ satisfy the relation (1). We have

$$\log b_{n,k}(x) = \log b_k + k\log y(x) - n\log\omega(y(x)).$$

Therefore

$$\frac{b'_{n,k}(x)}{b_{n,k}(x)} = k\frac{y'(x)}{y(x)} - n\frac{y'(x)\omega'(y(x))}{\omega(y(x))} = \frac{k}{b(x)} - n\frac{y'(x)}{y(x)}\frac{y(x)\omega'(y(x))}{\omega(y(x))},$$

and taking into account (3) we obtain (1) from here.

Lemma 1. *Let $I(x)$ be the Fisher information of structure $\boldsymbol{B}$. Then $I(x) = n/b(x)$.*

Proof. Indeed,

$$I(x) = \sum_{k=0}^{\infty}\left(\frac{d\log b_{n,k}(x)}{dx}\right)^2 b_{n,k}(x) = (b(x))^{-2}\sum_{k=0}^{\infty}(k-nx)^2 b_{n,k}(x) = \frac{n}{b(x)}.$$

□

Lemma 2. *Let $\beta_m(x)$ be the central moments of the distribution. Then the following recurrence relation holds*

$$\beta_{m+1}(x) = b(x)\Big(\frac{d\beta_m(x)}{dx} + nm\beta_{m-1}(x)\Big),\ \beta_0(x) = 1,\ \beta_1(x) = 0. \tag{4}$$

Proof. Indeed, $\beta_m(x) = \sum_{k=0}^{\infty} b_{n,k}(x)(k-nx)^m$. Then

$$\frac{d\beta_m(x)}{dx} = \sum_{k=0}^{\infty}\frac{db_{n,k}(x)}{dx}(k-nx)^m - \sum_{k=0}^{\infty} mnb_{n,k}(x)(k-nx)^{m-1} =$$
$$= \sum_{k=0}^{\infty}(b(x))^{-1}b_{n,k}(x)(k-nx)^{m+1} - \sum_{k=0}^{\infty} mnb_{n,k}(x)(k-nx)^{m-1},$$

and (4) follows from here. □

Corollary 1. $\beta_m(x) = \begin{cases} c_m n^r(b(x))^r + O(n^{r-1}), & \text{if } m = 2r; \\ c_m n^r(b(x))^r b'(x) + O(n^{r-1}), & \text{if } m = 2r+1, \end{cases}$

where

$$c_m = \begin{cases} (2r-1)!!, & \text{if } m = 2r; \\ \frac{1}{2}(2r)!!\sum\limits_{i=0}^{r-1}\frac{(2i+1)!!}{(2i)!!}, & \text{if } m = 2r+1. \end{cases}$$

Example 1. The function $\omega(y) = 1 + y$ generates the structure

$$b_{n,k}(x) = C_n^k x^k(1-x)^{n-k},\ 0 \leq x \leq 1,\ b(x) = x(1-x),$$

i.e., the Binomial distribution with parameters n and x.

Example 2. The function $\omega(y) = e^y$ generates the structure

$$b_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!},\ 0 \le x < \infty,\ b(x) = x,$$

i.e., the Poisson distribution with parameter nx.

Example 3. The function $\omega(y) = 1/(1-y)$ generates the structure

$$b_{n,k}(x) = C^k_{n+k-1}x^k(1+x)^{-n-k},\ 0 \le x < \infty,\ b(x) = x(1+x),$$

i.e., the Negative Binomial distribution with parameters n and x.

Example 4. The function $\omega(y) = (1-\sqrt{1-4y})/(2y)$ generates the structure

$$b_{n,k}(x) = \frac{n}{2k+n}C^k_{2k+n}x^k(1+x)^{n+k}(1+2x)^{-n-2k},\quad 0 \le x < \infty,$$
$$b(x) = x(1+x)(1+2x),$$

i.e., the Catalap distribution with parameters n and x (see [9, p. 31]).

2. Statistical Structures H. Let us consider another auxiliary statistical structure, which is defined by the family of densities $h_{n,k}(t), k \in \{0,1,2,\ldots\}$, with parameters $k \in N, n \in N$, and such that

$$h(t)\frac{d}{dt}h_{n,k}(t) = (k-nt)h_{n,k}(t), \tag{5}$$

where $h(t)$ is a non-negative function, which we shall call the covariance characteristic of the structure. We shall call this structure the structure **H**.

The structure **H** can be constructed as follows. Let us take a measure $\mu(t)dt$ (absolutely continuous with respect to the Lebesgue measure) for which the Laplace transform $u(s) = \int_R e^{-s\tau}\mu(\tau)d\tau$ exists. Let $t = -u'(s)/u(s)$ and let us denote by $s(t)$ the inverse function to $t = t(s)$, and by $h(t) := -1/s'(t)$. Then

$$h_{n,k}(t) = c_{n,k}e^{-s(t)k}(u(s(t)))^{-n},\ \text{ where } c_{n,k} = \Big(\int_R e^{-s(t)k}(u(s(t)))^{-n}dt\Big)^{-1}$$

(if the integral converges).

Let us prove that the functions $h_{n,k}(x)$ satisfy the relation (1). We have:

$\log h_{n,k}(t) = \log c_{n,k} - ks(t) - n\log u(s(t))$. Therefore $\frac{h'_{n,k}(t)}{h_{n,k}(t)} = -ks'(t) - ns'(t)\frac{u'(s(t))}{u(s(t))}$, and (5) follows from here.

Lemma 3. *The structure **H** exists if $h(t) = at^2 + bt + c$, and the vector (a,b,c) can take the following values:* $(-1,1,0),\ (1,1,0),\ (0,1,0),\ (1,0,0),\ (0,0,1),\ (1,0,1)$.

Proof. We simply point out these structures, for which the relation (5) is verified directly.

Structure (-1,1,0):

$$h_{n,k}(t) = (n+1)C^k_n t^k(1-t)^{n-k},\ k \in \{0,1,2,\ldots\}, 0 \le t \le 1,\ h(t) = t(1-t).$$

Structure (1,1,0):

$$h_{n,k}(t) = (n-1)C^k_{n+k}t^k(1+t)^{-n-k},\ k \in \{0,1,2,\ldots\}, 0 < t < \infty,\ h(t) = t(1+t),\ n > 2.$$

Structure (0,1,0):

$$h_{n,k}(t) = \frac{e^{-nt}n^{k+1}t^k}{k!},\ k \in \{0,1,2,\ldots\}, 0 < t < \infty, h(t) = t.$$

Structure (1,0,0):

$$h_{n,k}(t) = \frac{k^{n-1}t^{-n}}{\Gamma(n-1)}e^{-k/n},\ k \in \{0,1,2,\ldots\}, 0 < t < \infty,\ h(t) = t^2,\ n > 2.$$

Structure (0,0,1):

$$h_{n,k}(t) = \sqrt{\frac{n}{2\pi}}\exp\left(-\frac{(k-nt)^2}{2n}\right),\ k \in \{0,1,2,\ldots\}, -\infty < t < \infty,\ h(t) = 1.$$

Structure (1,0,1):

$$h_{n,k}(t) = c_{n,k}\exp(k\operatorname{arctg}(t))(1+t^2)^{-n/2},\ k\in\{0,1,2,\ldots\},\ n>2,$$
$$-\infty<t<\infty,\ h(t)=1+t^2,$$

where
$$(c_{n,k})^{-1} = \frac{2\operatorname{sh}(k\pi/2)(2m-2)!}{(k^2+4)(k^2+16)\cdots(k^2+(2m-2)^2)},\quad \text{if } n=2m,$$
$$(c_{n,k})^{-1} = \frac{2\operatorname{ch}(k\pi/2)(2m)!}{(k^2+1)(k^2+9)\cdots(k^2+(2m-1)^2)},\quad \text{if } n=2m+1. \qquad \square$$

Remark. If the argument t of the density $h_{n,k}(t), k\in\{0,1,2,\ldots\}$, is specified on an interval different from the entire number line, it is assumed that outside this interval the density is zero.

Further we will use structures with a quadratic covariance characteristic.

Lemma 4. *If α_1 denotes the initial moment of the distribution $\boldsymbol{H}$, then $\alpha_1 = \frac{k+b}{n-2a}$.*

Proof. Let us rewrite the relation (5) as $nth_{n,k}(t) = kh_{n,k}(t) - (at^2+bt+c)h'_{k,t}(t)$ and integrate. We get

$$n\alpha_1 = k - \int_{-\infty}^{\infty}(at^2+bt+c)h'_{k,t}(t)dt =$$
$$= k - \Big(\big[(at^2+bt+c)h_{k,t}(t)\big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}(2at+b)h_{n,k}(t)dt\Big) = k+2a\alpha_1+b.$$

The statement of the lemma follows from here. $\square$

Lemma 5. *If $\nu_m(x)$ denotes the central moments of the distribution $\boldsymbol{H}$, then the following recurrence relation holds: $\nu_0=1,\nu_1=0$, and for each $m\geq 2$*

$$\nu_m(k) = \frac{1}{n-a(m+1)}\left(((m(2a\alpha_1+b)-(n\alpha_1-k))\nu_{m-1}+(m-1)(a\alpha_1^2+b\alpha_1+c)\nu_{m-2}\right), \tag{6}$$

Proof. Let us rewrite the relation (5) as

$$(n\alpha_1-k)h_{n,k}(t)+n(t-\alpha_1)h_{n,k}(t) = -(at^2+bt+c)h'_{k,t}(t) =$$
$$= -(a\alpha_1^2+b\alpha_1+c+(2a\alpha_1+b)(t-\alpha_1)+a(t-\alpha_1)^2)h'_{k,t}(t).$$

Multiplying both sides of this equality by $(t-\alpha_1)^{m-1}$ and integrating, we obtain

$$(n\alpha_1-k)\nu_{m-1}+n\nu_m = (at^2+bt+c)h_{k,t}(t)\big|_{-\infty}^{\infty}+(m-1)(a\alpha_1^2+b\alpha_1+c)\nu_{m-2}+a(m+1)\nu_m.$$

(6) follows from here. In particular, if $m=2$, then

$$\nu_2 = \frac{(k+b)(ak+bn-ab)}{(n-2a)^2(n-3a)}+\frac{c}{n-3a}.$$

$\square$

3. Structures of Phillips Type.

Definition 1. A *statistical structure of Phillips type* will be called the family of densities $p_n(x,t) = \sum_{k=0}^{\infty} b_{n,k}(x)h_{n,k}(t)$, which depends on the parameter $x, x\in X$, where X is the image of the interval $[0,R)$ under the mapping $x=x(y)$ (see [4]).

Let the function f be defined and integrable on the entire number line. Let us denote by $P_n(f,x)$ the mathematical expectation of the random variable $f(\eta)$, where the distribution of the random variable η is given by the density $p_n(x,t)$, i.e.,

$$P_n(f,x) = \sum_{k=0}^{\infty} b_{n,k}(x)\int_{-\infty}^{\infty} f(t)h_{n,k}(t)dt. \tag{7}$$

It is useful to view $P_n(f,x)$ as an operator; it is linear and positive. We shall call the function $b_{n,k}(x)$ the discrete kernel of the operator, and the function $h_{n,k}(t)$ the continuous kernel.

Thus, if $f(t)=t$, we obtain the mathematical expectation $\mathbf{M}\eta$, and in the case of the densities given above, $\mathbf{M}\eta=\alpha(x)=(nx+b)/(n-2a), n>2a$, indeed,

$$\alpha(x)=\sum_{k=0}^{\infty} b_{n,k}(x)\int_{-\infty}^{\infty} t h_{n,k}(t)dt=\sum_{k=0}^{\infty} b_{n,k}(x)\alpha_1=\sum_{k=0}^{\infty} b_{n,k}(x)\frac{k+b}{n-2a}=$$
$$=\sum_{k=0}^{\infty} b_{n,k}(x)\frac{(k-nx)+(nx+b)}{n-2a}=\sum_{k=0}^{\infty} b_{n,k}(x)\frac{k-nx}{n-2a}+$$
$$+\sum_{k=0}^{\infty} b_{n,k}(x)\frac{nx+b}{n-2a}=0+\frac{nx+b}{n-2a}\cdot 1=\frac{nx+b}{n-2a}.$$

If $f(t)=(t-\mathbf{M}\eta)^m$, then $P_n(f,x)=\mu_m(x)$ is the central moment of the m-th order of the random variable η.

The paper investigates statistical structures of Phillips type and the operator $P_n(f,x)$.

Theorem 1. *Let the covariance characteristic of structure H be a polynomial of degree r. Then for the central moments of the random variable η the following recurrence relation holds*

$$b(x)(\mu_m'(x)+m\alpha'(x)\mu_{m-1}(x))=n\mu_{m+1}(x)+n(\alpha(x)-x)\mu_m(x)-\sum_{i=0}^{r}\frac{m+i}{i!}h^{(i)}(\alpha(x))\mu_{m+i-1}(x).$$

Proof. First, let us establish the following auxiliary statement.

Let the function $f(t)$ be a polynomial of degree r, and the function $p(t)$ be continuously differentiable on the interval $[a,b]$ and $f(a)p(a)=f(b)p(b)=0$. Then

$$\int_a^b f(t)p'(t)(t-x)^m dt=-\sum_{i=0}^{r}(m+i)\frac{f^{(i)}(x)}{i!}\int_a^b p(t)(t-x)^{m+i-1}dt.$$

Indeed, by Taylor's formula $f(t)=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i$. Therefore

$$\int_a^b f(t)p'(t)(t-x)^m dt=\int_a^b\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}p'(t)(t-x)^{m+i}dt=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_a^b p'(t)(t-x)^{m+i}dt=$$
$$=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\left(p(t)(t-x)^{m+i}\Big|_a^b-(m+i)\int_a^b p(t)(t-x)^{m+i-1}dt\right)=$$
$$=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\left(p(b)(b-x)^{m+i}-p(a)(a-x)^{m+i}-(m+i)\int_a^b p(t)(t-x)^{m+i-1}dt\right)=$$
$$=(b-x)^m p(b)f(b)-(b-x)^m p(b)f(b)-\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\left((m+i)\int_a^b p(t)(t-x)^{m+i-1}dt\right)=$$
$$=-\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\left((m+i)\int_a^b p(t)(t-x)^{m+i-1}dt\right).$$

Since the central moments of the m-th order are found by the formula

$$\mu_m(x)=\sum_{k=0}^{\infty} b_{n,k}(x)\int_{-\infty}^{\infty}(t-\alpha(x))^m h_{n,k}(t)dt,$$

we obtain from here

$$b(x)\mu'_m(x)=\sum_{k=0}^{\infty}b_{n,k}(x)(k-nx)\int_{-\infty}^{\infty}(t-\alpha(x))^m h_{n,k}(t)dt-mb(x)\alpha'(x)\mu_{m-1}(x),$$

and from here

$$b(x)(\mu'_m(x)+m\alpha'(x)\mu_{m-1}(x))=\sum_{k=0}^{\infty}b_{n,k}(x)\int_{-\infty}^{\infty}(k-nt)(t-\alpha(x))^m h_{n,k}(t)dt+$$

$$+\sum_{k=0}^{\infty}b_{n,k}(x)\int_{-\infty}^{\infty}(n(t-\alpha(x))+n(\alpha(x)-x))(t-\alpha(x))^m h_{n,k}(t)dt=$$

$$=\sum_{k=0}^{\infty}b_{n,k}(x)\int_{-\infty}^{\infty}h(t)(t-\alpha(x))^m h'_{n,k}(t)dt+n\mu_{m+1}(x)+n(\alpha(x)-x)\mu_m(x).$$

It remains to use the auxiliary statement. We obtain

$$\int_{-\infty}^{\infty}h(t)(t-\alpha(x))^m h'_{n,k}(t)dt=-\sum_{i=0}^{r}\frac{(m+i)!}{i!}h^{(i)}(\alpha(x))\int_a^b p_{n,k}(t)(t-\alpha(x))^{m+i-1}dt.$$

□

And the following statement follows from here.

Corollary. *If $h(t)=at^2+bt+c$, then for the central moments of the Phillips structure we will have the following recurrence relation*

$$(n-a(m+2))\mu_{m+1}(x)=b(x)\mu'_m(x)+\mu_m(x)((m+1)(2a\alpha(x)+b)-n(\alpha(x)-x))+$$
$$+\mu_{m-1}(x)(m(a(\alpha(x))^2+b\alpha(x)+c)+mb(x)\alpha'(x)).$$

In particular, if $m=1$, we obtain from here

$$\mu_2(x)=\frac{1}{n-3a}\left(a(\alpha(x))^2+b\alpha(x)+c+b(x)\alpha'(x)\right)=$$
$$=\frac{1}{n-3a}\Big(a\Big(\frac{nx+b}{n-2a}\Big)^2+b\frac{nx+b}{n-2a}+b(x)\frac{n}{n-2a}\Big),\quad n>3a.$$

Example 5. Let the density be

$$h_{n,k}(t)=e^{-nt}\frac{n^{k+1}t^{k+1}}{k!}.$$

Then the central moments $\mu_m(x)$ of the random variable η satisfy the recurrence relation

$$\mu_{m+1}(x)=\frac{1}{n}\Big(b(x)\frac{d\mu_m(x)}{dx}+m\mu_m(x)+m\mu_{m-1}(x)\big(b(x)+x+1/n\big)\Big),\ \mu_0=1,\mu_1=0.\quad (8)$$

From here $\mu_2(x)=(x+b(x))/n+1/n^2$.

Example 6. Let the density be

$$h_{n,k}(t)=(n-1)C^k_{n+k+1}\frac{t^k}{(1+t)^{n+k}}.$$

Then the central moments $\mu_m(x)$ of the random variable η satisfy the recurrence relation

$$\mu_{m+1}(x)=\frac{1}{n-m-2}\Bigg(b(x)\frac{d\mu_m(x)}{dx}+\frac{nm(2x+1)}{n-2}\mu_m(x)+$$
$$+\Big(\Big(\frac{nx+1}{n-2}\Big)^2+\frac{nx+1}{n-2}+\frac{n}{n-2}b(x)\Big)m\mu_{m-1}(x)\Bigg),\qquad n>m+2,\ \mu_0=1,\ \mu_1=0.$$

From here

$$\mu_2(x)=\frac{1}{n-3}\Big(\Big(\frac{nx+1}{n-2}\Big)^2+\frac{nx+1}{n-2}+\frac{n}{n-2}b(x)\Big).$$

4. Approximation Properties.

Theorem 2. *Let $f \in H^\omega$. Then for $n > 3a$*

$$|P_n(f,x) - f(x)| \le$$
$$\le \omega(1/\sqrt{n})\Big(1 + \frac{n}{n-3a}\Big(a\Big(\frac{nx+b}{n-2a}\Big)^2 + b\frac{nx+b}{n-2a} + b(x)\frac{n}{n-2a}\Big)\Big) + \omega\left(\left|\frac{2ax+b}{n-2a}\right|\right).$$

Proof. We consistently obtain

$$|P_n(f,x) - f(x)| \le |P_n(f,x) - f(\alpha(x))| + |f(\alpha(x)) - f(x)| \le$$
$$\le \sum_{k=0}^{\infty} b_{n,k}(x) \int_{-\infty}^{\infty} |f(t) - f(\alpha(x))| h_{n,k}(t)dt + \omega(|\alpha(x) - x|) \le$$
$$\le \sum_{k=0}^{\infty} b_{n,k}(x) \int_{-\infty}^{\infty} \omega(|t - \alpha(x)|) h_{n,k}(t)dt + \omega(|\alpha(x) - x|) \le$$
$$\le \omega(\delta) \sum_{k=0}^{\infty} b_{n,k}(x) \int_{-\infty}^{\infty} \left(1 + \left[\delta^{-1}|t - \alpha(x)|\right]\right) h_{n,k}(t)dt + \omega(|\alpha(x) - x|) \le$$
$$\le \omega(\delta)\Big(1 + \int_{t:\delta^{-1}|t-\alpha(x)|\ge 1} \left[\delta^{-1}|t - \alpha(x)|\right] \Big(\sum_{k=0}^{\infty} b_{n,k}(x) h_{n,k}(t)\Big)dt\Big) + \omega(|\alpha(x) - x|) \le$$
$$\le \omega(\delta)\Big(1 + \sum_{k=0}^{\infty} b_{n,k}(x)\delta^{-2} \int_{-\infty}^{\infty} (t - \alpha(x))^2 h_{n,k}(t)dt\Big) + \omega(|\alpha(x) - x|) =$$
$$= \omega(\delta)(1 + \delta^{-2}\mu_2(x)) + \omega(|\alpha(x) - x|).$$

If we now take $\delta = 1/\sqrt{n}$, we obtain the statement of the theorem. □

Remark 1. If $b_{n,k}(x) = e^{-nx}(nx)^k/k!$, and $h_{n,k}(t) = e^{-nt}\frac{n^{k+1}t^{k+1}}{k!}$, then the operator $P_n(f,x)$ turns into the slightly modified Phillips operator ([1]), which was used by him to obtain one of the formulas for the inversion of the Laplace transform.

This fact explains the name of the statistical structure under study and the corresponding operator. For such operators, the inequality follows from Theorem 2

$$|P_n(f,x) - f(x)| \le \omega(1/\sqrt{n})(1 + x + x^2),\ x \ge 0.$$

Remark 2. There are a number of studies on the approximation properties of operators of type $P_n(f,x)$, where specific functions generated by distributions such as the Binomial, Negative Binomial, Gamma, and Beta distributions are taken as the functions $b_{n,k}(x)$ and $h_{n,k}(t)$. For example, if the discrete kernel is taken as

$$b_{n,k}(x) = C_n^k x^k (1-x)^{n-k},\ 0 \le x \le 1,$$

and the continuous kernel is taken as

$$h_{n,k}(t) = (n+1)C_n^k t^k (1-t)^k,\ 0 \le t \le 1,$$

then we obtain the Bernstein-Durrmeyer polynomials (see [2]).

For these polynomials, the inequality follows from Theorem 2

$$|P_n(f,x) - f(x)| \le \frac{1}{4}\omega(1/\sqrt{n}) + \omega(1/n),\ 0 \le x \le 1.$$

If

$$b_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!},\ 0 \le x < \infty, \quad h_{n,k}(t) = (n-1)C_{n+k}^k t^k (1+t)^{-n-k},\ 0 < t < \infty,$$

then we obtain the Szász-Baskakov operators [4]. For such operators, the inequality follows from Theorem 2

$$|P_n(f,x)-f(x)| \le \omega(1/\sqrt{n})\left(1+\frac{n^2x^2+2n^2x-2nx+n-2}{(n-2)^2(n-3)}n\right)+\omega\left(\frac{2x+1}{n-2}\right),$$

for $x \ge 0,\ n > 3$. If

$$b_{n,k}(x)=C_{n+k}^k x^k(1+x)^{-n-k},\ 0<x<\infty,\ h_{n,k}(t)=(n-1)C_{n+k}^k t^k(1+t)^{-n-k},\ 0<t<\infty,$$

then we obtain the Durrmeyer-Beta operators ([3]); for such operators, the inequality follows from Theorem 2

$$|P_n(f,x)-f(x)| \le \omega(1/\sqrt{n})\Big(1+\frac{n}{n-3}\Big(\left(\frac{nx+1}{n-2}\right)^2+\frac{nx^2+2nx+1}{n-2}\Big)\Big)+\omega\left(\left|\frac{2x+1}{n-2}\right|\right),$$

for $x>0, n>3$.

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