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PUTNAM-FUGLEDE COMMUTATIVITY AND THE RANGE-KERNEL ORTHOGONALITY OF AN ELEMENTARY OPERATOR II

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Let $T,S\in\mathcal{L}(H)$ be commuting operators on a Hilbert space H, where T is a w-hyponormal operator satisfying $\ker T\subseteq\ker T^*$, and S is normal. Consider the elementary operator $\psi_{T,S}$ defined by $\psi_{T,S}(X)=TXT^*-SXS^*,\ X\in\mathcal{L}(H)$. It is shown that (1) For all $X\in\mathcal{L}(H),\ TXT^*=SXS^*$ implies $T^*XT=S^*XS$, i.e., $\ker\psi_{T,S}\subseteq\ker\psi_{T^*,S^*}$; and (2) For all $X,Y\in\mathcal{L}(H)$ with $Y\in\ker(\psi_{T,S})$ we have
$$\|TXT^*-SXS^*+Y\|\geq\|Y\|,$$
which means the range of $\psi_{T,S}$ is orthogonal to its kernel, if and only if $\ker T\cap\ker S=\{0\}$. These results are further extended to the elementary operator $\Psi\in\mathcal{L}(\mathcal{L}(H))$ defined by $\Psi(X)=AXB-CXD$, where $A,B^*\in\mathcal{L}(H)$ are w-hyponormal operators satisfying $\ker A\subseteq\ker A^*$ and $\ker B^*\subseteq\ker B$, and $C,D\in\mathcal{L}(H)$ are normal operators that commute with A and B respectively. This extension broadens classical results by Turnšek, as well as the Putnam-Fuglede property, and the theorem of Weiss, by involving four operators, not all of which are normal.

1. Introduction. Let H be a separable complex Hilbert space. Let $\mathcal{L}(H)$ be the algebra of all bounded linear operators on H. The classical Putnam–Fuglede theorem states the following if $A,B\in\mathcal{L}(H)$ are normal operators and there exists an operator $X\in\mathcal{L}(H)$ such that $AX=XB$, then $A^*X=XB^*$. Extending the Putnam–Fuglede theorem beyond normal operators has received much attention. Many generalizations have appeared in recent years. A related version says: if A,B are normal operators and $AXB=X$, then $A^*XB^*=X$. Several references, such as [2, 12, 17, 18], present extensions of this result to broader classes of operators.

Weiss [19] gave an interesting extension of the Putnam–Fuglede theorem involving four normal operators. He showed that if (A,C) and (B,D) are pairs of commuting normal operators on H, then $AXB=CXD$ implies $A^*XB^*=C^*XD^*$ for all $X\in\mathcal{L}(H)$. This result was extended by Furuta [6] to the case where A,B^*,C, and D^* are hyponormal, under the assumption that X belongs to the Hilbert–Schmidt class. More precisely, if $CA^*=A^*C$ and $BD^*=D^*B$, then $AXB=CXD$ implies $A^*XB^*=C^*XD^*$ for every Hilbert–Schmidt operator X. A similar result appears in [2, Theorem 2.5] for a broader class of operators, under suitable commutation relations and extra conditions. Although Weiss's theorem is mathematically appealing, it has received little further development in later works.

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Given subspaces M and N of a Banach space V with norm $\|.\|$, then M is said to be orthogonal to N in the sense of Birkhof–James, denoted $M \perp N$, if $\|m+n\| \geqslant \|n\|$ for all $m \in M$ and $n \in N$. The range-kernel orthogonality of elementary operators has been studied by a number of authors over recent decades. Duggal [3] considered the elementary operator $\Delta_{A,B} \in \mathcal{L}(\mathcal{L}(H))$ defined by $\Delta_{A,B}(X) = AXB - X$, to prove that if $A, B \in \mathcal{L}(H)$ are normal operators, then $\operatorname{ran}(\Delta_{A,B}) \perp \ker \Delta_{A,B}$. These results have been extended to a diversity of elementary operators $\Psi \in \mathcal{L}(\mathcal{L}(H))$, where $\Psi(X) = AXB - CXD$, for a variety of choices of tuples of commuting operators (A, C) and (B, D) (see [5, 7, 9, 10, 16] for further references). In particular, Turnšek [16] proved that if A, C, respectively B, D, are nonzero normal commuting operators, then

$$\operatorname{ran}(\Psi) \perp \ker \Psi \quad \text{if and only if} \quad \ker A \cap \ker C = \ker B^* \cap \ker D^* = \{0\}.$$

This result was later extended by Morjane et al. [11] to the class of w-hyponormal operators by considering the elementary operator

$$\phi_{T,S}(X) = TXS^* - SXT^*,$$

where T is a w-hyponormal operator satisfying $\ker(T) \subseteq \ker(T^*)$, and S is a normal operator such that $TS = ST$. These assumptions allow the generalization of Weiss's theorem and the extension of the range-kernel orthogonality result to the elementary operator Ψ. Recall that an operator T is said to be w-hyponormal if

$$|\widetilde{T}| \geq |T| \geq |(\widetilde{T})^*|,$$

where $|T| = (T^*T)^{1/2}$, and $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$ denotes the Aluthge transform of T, with $T = U|T|$ being the polar decomposition of T (see [2, 14, 15]).

A bounded linear operator T on a complex Hilbert space H is said to be *p-hyponormal* $(0 < p \leq 1)$ if $(TT^*)^p - (T^*T)^p$ is a positive operator. If $p = 1$, then T is called a *hyponormal* operator. If $p = 1/2$, then T is called a *semi-hyponormal* operator. Every subnormal operator (in particular, a normal operator) is hyponormal, and every hyponormal operator is a paranormal convexoid operator. Not every paranormal operator is, however, hyponormal. We say that T is *w_*-hyponormal*, and write $T \in w_*\text{-}H$, if T is w-hyponormal and satisfies $\ker(T) \subseteq \ker(T^*)$. Rashid shows in [13, Example 3.2] that the inclusion $\ker(T) \subseteq \ker(T^*)$ may fail for w-hyponormal operators, and that the Fuglede–Putnam theorem does not generally hold in this class, unlike for w_*-hyponormal operators (see [2, 13]).

In this paper, we consider the elementary operator $\psi_{T,S}(X) = TXT^* - SXS^*$, where $T \in w_*\text{-}H$ and S is a normal operator such that $TS = ST$. Using various techniques, we show that $\ker \psi_{T,S} \subset \ker \psi_{T^*,S^*}$. This result implies that if $\Psi \in \mathcal{L}(\mathcal{L}(H))$ is the elementary operator defined by $\Psi(X) = AXB - CXD$, where $A, B^* \in w_*\text{-}H$, and C, D are normal operators such that $AC = CA$ and $BD = DB$, then $\ker \Psi \subset \ker \Psi_*$, where $\Psi_* \colon \mathcal{L}(H) \to \mathcal{L}(H)$ is given by $\Psi_*(X) = A^*XB^* - C^*XD^*$. This provides a generalization of both the Putnam-Fuglede property (by taking $C = D = I$) and Weiss's theorem. Another aim of this paper is to investigate the orthogonality between the range and the kernel of the elementary operator Ψ, thereby extending Duggal's inequality [3]. We conclude this section with some notations.

For X a linear operator acting on Banach space E, we denote by X^*, $\ker(X)$, $\ker X^{\perp}$, $R(X)$ and $X|_M$ respectively the adjoint, the kernel, the orthogonal complement of the kernel, the range of X and the restriction of X to an invariant subspace M. For g and ω two vectors in H, we define $g \otimes \omega \in \mathcal{L}(H)$ as follows

$$g \otimes \omega(x) = \langle x, \omega \rangle g \text{ for all } x \in H.$$

Recall that an operator $T \in \mathcal{L}(H)$ is said to be hyponormal if $T^*T \geq TT^*$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators [8]. An operator T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ for $p \in]0,1]$ and an operator T is said to be log-hyponormal if T is invertible and $\log|T| \geq \log|T^*|$. p-hyponormal and log-hyponormal operators are defined as extension of hyponormal operator. The classes of log- and w-hyponormal operators were introduced and their properties were studied in [1]. In particular, it was shown in [1] that the class of w-hyponormal operators contains both p-and log-hyponormal operators. Moreover, it is easy to see that w-hyponormal operators T are paranormal (i.e., $\|Tx\|^2 \leqslant \|T^2x\|$ for all unit vectors $x \in H$). The w-hyponormal operators have some interesting properties, amongst them that the restriction of a w-hyponormal operator to an invariant subspace is again a w-hyponormal operator, the inverse of an invertible w-hyponormal operator is again w-hyponormal.

2. Main results.

Lemma 1 ([2])**.** *If $A, B \in w_*$-H are such that $[A,B] = [A^*,B] = 0$ and B is invertible, then $AB^{-1} \in w_*$-H.*

Theorem 1. *Let $T \in w_*$-H and $S \in \mathcal{L}(H)$ is normal such that $TS = ST$. If T or S is injective, then the following assertions holds:*

(i) $\ker \psi_{T,S} \subset \ker \psi_{T^*,S^*}$.

(ii) *For all $X, Y \in \mathcal{L}(H)$ such that $Y \in \ker(\psi_{T,S})$ we have* $\|\psi_{T,S}(X) + Y\| \geq \|Y\|$.

Proof. Firstly, assume that S is injective and for a natural number n, let $\Gamma_n = \{\lambda \in \mathbb{C} \colon |\lambda| \leqslant 1/n\}$, and let $E_S(\Gamma_n)$ denote the corresponding spectral projection. Set $I - E_S(\Gamma_n) = P_n$; then $P_n \to I$ in the strong topology. Since $TS = ST$, the Fuglede's Theorem implies $TS^* = S^*T$, and so $R(P_n)$ reduces both T and S. Hence

$$T = T_{1,n} \oplus T_{2,n} \text{ and } S = S_{1,n} \oplus S_{2,n} \text{ on } H_n = H = \ker(P_n) \oplus R(P_n)$$

where $T_{i,n}$ are w_*-hyponormal $(i = 1,2)$, $S_{1,n}$ is normal and $S_{2,n}$ is invertible normal. Now, and for $Y \in \ker(\psi_{T,S})$, let $Y_n = P_nYP_n$, hence $Y_n \longrightarrow Y$ weakly (even, strongly). Also, if we set $R_n = T_{2,n}S_{2,n}^{-1}$, then we have

$$\begin{aligned} P_n\psi_{T,S}(Y)P_n = P_n\left(TYT^* - SYS^*\right)P_n = T_{2,n}\left(P_nYP_n\right)T_{2,n}^* - S_{2,n}\left(P_nYP_n\right)S_{2,n}^* = \\ = T_{2,n}Y_nT_{2,n}^* - S_{2,n}Y_nS_{2,n}^* = S_{2,n}\left(R_nY_nR_n^* - Y_n\right)S_{2,n}^*, \end{aligned}$$

which means that $Y_n \in \ker(\Delta_{R_n,R_n^*})$, and since R_n is w_*-hyponormal by Lemma 1, then [2, Lemma 2.4] implies that $Y_n \in \ker(\Delta_{R_n^*,R_n})$. Hence

$$P_n\psi_{T^*,S^*}(Y)P_n = P_n\left(T^*YT - S^*YS\right)P_n \quad = S_{2,n}^*\left(R_n^*Y_nR_n - Y_n\right)S_{2,n} = 0,$$

and as result $Y \in \ker(\psi_{T^*,S^*})$. Secondly, Lemma 4 from [4] ensures that

$$\left\|\Delta_{R_n,R_n^*}\left(Z_n\right) + Y_n\right\| \geq \|Y_n\| \text{ for all } Z_n \in \mathcal{L}\left(R(P_n)\right).$$

Thus for $Z_n = S_{2,n}X_nS_{2,n}^*$, we would have

$$\left\|T_{2,n}X_nT_{2,n}^* - S_{2,n}X_nS_{2,n}^* + Y_n\right\| \geq \|Y_n\|$$

for all $X_n = P_n X P_n \in \mathcal{L}(R(P_n))$. It follows that

$$\|\psi_{T,S}(X)+Y\| \geq \|P_n(\psi_{T,S}(X)+Y)P_n\| = \left\|T_{2,n}X_nT_{2,n}^* - S_{2,n}X_nS_{2,n}^* + Y_n\right\| \geq \|Y_n\|$$

Therefore, since $\|Y_n\| \longrightarrow \|Y\|$,

$$\|\psi_{T,S}(X)+Y\| \geq \|Y\| \text{ for all } X \in \mathcal{L}(H) \text{ and } Y \in \ker(\psi_{T,S}).$$

Now, suppose that T is injective and let $Y \in \ker(\psi_{T,S})$. So, we can write T, S and Y on $H_0 = H = (\ker(S))^\perp \oplus \ker(S)$ as

$$T = N \oplus R, S = S_1 \oplus 0, \text{ and } Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$$

with $NY_1N^* = S_1Y_1S_1^*$, i.e. $Y_1 \in \ker(\psi_{N,S_1})$ and $NY_2R^* = RY_3N^* = RY_4R^* = 0$, hence $Y_2 = Y_3 = Y_4 = 0$. As result, since $N \in w_*\text{-}H$ and S_1 is injective, the first case implies that $Y_1 \in \ker\psi_{N^*,S_1^*}$, which is equivalent to $Y \in \ker\psi_{T^*,S^*}$. More things, let $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in \mathcal{L}(H_0)$, we can also deduce from the first case that

$$\|\psi_{T,S}(X)+Y\| = \left\|\begin{pmatrix} \psi_{N,S_1}(X_1)+Y_1 & * \\ * & * \end{pmatrix}\right\| \geq \|\psi_{N,S_1}(X_1)+Y_1\| \geq \|Y_1\| = \|Y\|,$$

scince the norm of an operator matrix is always at least as large as the norm of its diagonal blocks. □

Remark 1. It is straightforward to verify that Theorem 1 also holds under the assumptions that T is normal and S is w_*-hyponormal, due to the relation $\psi_{T,S} = -\psi_{S,T}$.

We consider the elementary operator $\Psi \in \mathcal{L}(\mathcal{L}(H))$ defined by $\Psi(X) = AXB - CXD$, with $\Psi_*\colon X \in \mathcal{L}(H) \mapsto A^*XB^* - C^*XD^*$. Therefore, we deduce that

Corollary 1. *Let $A, B, C, D \in \mathcal{L}(H)$ such that $AC = CA$ and $BD = DB$. Suppose one of the following conditions holds:*

(i) $A, B^* \in w_*\text{-}H$, *C and D are normal injective operators;*

(ii) $A, B^* \in w_*\text{-}H$ *are injective, C and D are normal operators;*

(iii) $C, D^* \in w_*\text{-}H$, *A and B are normal injective operators;*

(iv) $C, D^* \in w_*\text{-}H$ *are injective, A and B are normal operators.*

Then, we have $\ker\Psi \subset \ker\Psi_*$ *and*

$$\|\Psi(X)+T\| \geq \|T\| \text{ for all } X \in \mathcal{L}(H) \text{ and } T \in \ker\Psi.$$

Proof. Let $T \in \ker\Psi$, and put $\hat{T} = \begin{pmatrix} A & 0 \\ 0 & B^* \end{pmatrix}$, $S = \begin{pmatrix} C & 0 \\ 0 & D^* \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ be defined on $H \oplus H$. This gives us $\hat{T}S = S\hat{T}$ and $Y \in \ker(\psi_{\hat{T},S})$. By our assumptions and Theorem 1, we get $Y \in \ker(\psi_{\hat{T}^*,S^*})$ which is equivalent to $T \in \ker\Psi_*$, furthermore, for all $\hat{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(H \oplus H)$, we have

$$\left\|\psi_{\hat{T},S}(\hat{X})+Y\right\| = \left\|\begin{pmatrix} 0 & AXB - CXD + T \\ 0 & 0 \end{pmatrix}\right\| \geq \|Y\|.$$

Therefore, we obtain the desired result. □

Remark 2. By setting $C = D = I$, we can see that the preceding corollary generalizes Lemma 2.4 from [2]. Moreover, it extends the Duggal inequality [3, Corollary 1] to the case where A and B^* are w_*-hyponormal operators.

Theorem 2. *Let $T \in w_*$-H and $S \in \mathcal{L}(H)$ be normal operator such that $TS = ST$. Then,*

$$TXT^* = SXS^* \implies T^*XT = S^*XS$$

for all $X \in \mathcal{L}(H)$.

Proof. Since T commutes with the normal operator S, Fuglede's Theorem ensures that T^* also commutes with S. Consequently, the subspace $(\ker S)^\perp$ reduces T, allowing us to decompose

$$T = N \oplus R \text{ on } H_0 = H = (\ker(S))^\perp \oplus \ker(S).$$

We can write S and X on H_0 as

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

Since $TS = ST$, it follows that $NS_1 = S_1N$. Therefore, if $TXT^* = SXS^*$, then

$$NX_1N^* = S_1X_1S_1^*,$$

where $N \in w_*$-H and S_1 is a normal injective operator. Applying Theorem 1, we obtain

$$N^*X_1N = S_1^*X_1S_1.$$

Similarly, we have $NX_2R^* = RX_3N^* = RX_4R^* = 0$, that is,

$$0 \cdot (NX_2) = (NX_2)R^*, \quad R(X_3N^*) = (X_3N^*) \cdot 0 \quad \text{and} \quad 0 \cdot (RX_4) = (RX_4)R^*,$$

with $R \in w_*$-H. It follows from [2], Lemma 2.4, that

$$0^* \cdot (NX_2) = (NX_2)R, \quad R^*(X_3N^*) = (X_3N^*) \cdot 0^* \quad \text{and} \quad 0^* \cdot (RX_4) = (RX_4)R.$$

Repeating this reasoning, we conclude that $N^*X_2R = R^*X_3N = R^*X_4R = 0$, and as a result, $T^*XT = S^*XS$. □

Similarly to Corollary 1, we deduce that

Corollary 2. *Let $A, B, C, D \in \mathcal{L}(H)$ satisfy $AC = CA$ and $BD = DB$. If either (i) $A, B^* \in w_*$-H, C and D are normal, or (ii) $C, D^* \in w_*$-H, and A and B are normal, then $AXB = CXD \implies A^*XB^* = C^*XD^*$ for all $X \in \mathcal{L}(H)$.*

Remark 3. Our results generalize the Weiss theorem [19], as well as the Putnam-Fuglede property for w_*-hyponormal operators by taking $C = D = I$.

Proposition 1. *Let $A, B, C, D \in \mathcal{L}(H)$ such that $AC = CA$ and $BD = DB$. If either*

(i) $A, B^*, D^* \in w_*$-H*, C is normal, D is invertible and $BD^* = D^*B$, or*

(ii) $A, B^*, C \in w_*$-H*, D is normal, C is invertible and $AC^* = C^*A$, or*

(iii) $C, B^*, D^* \in w_*$-H, *A is normal, B is invertible and* $BD^* = D^*B$, *or*

(iv) $A, B^*, D^* \in w_*$-H, *B is normal, A is invertible and* $AC^* = C^*A$,

then, $AXB = CXD \implies A^*XB^* = C^*XD^*$ *for all* $X \in \mathcal{L}(H)$.

Proof. It is an immediate consequence of Corollary 2 and Lemma 1. □

Proposition 2. *If $T, S \in w_*$-H are double commuting operators satisfying $|T|S = S|T^*|$, then for all $X \in \mathcal{L}(H)$,*

$$TXT^* = SXS^* \implies T^*XT = S^*XS.$$

Proof. By hypothesis, we have $TS - ST = T^*S - ST^* = 0$, It follows that $(\ker S)^\perp$ reduce T, and $T|_{(\ker S)^\perp}$ is a normal operator. This means

$$T = N \oplus R \text{ on } H_0 = H = (\ker(S))^\perp \oplus \ker(S).$$

Since $\ker S \subseteq \ker S^*$, $\ker S$ reduces S. Hence, we can write S and X on H_0 as follows

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

From $TS = ST$ we have $NS_1 = S_1N$. Thus, if $TXT^* = SXS^*$, then $NX_1N^* = S_1X_1S_1^*$, where $S_1 \in w_*$-H and N is normal. So by Theorem 2 we get

$$N^*X_1N = S_1^*X_1S_1.$$

Moreover, we find that $NX_2R^* = RX_3N^* = RX_4R^* = 0$, with $R \in w_*$-H. Using Lemma 2.4 from [2], the remainder of the proof proceeds similarly to that of Theorem 2. □

Example 1. Let $T \in \mathcal{L}(H)$ be a p-hyponormal operator. Suppose $T = T_1 \oplus T_2$ on the space $H = H_1 \oplus H_2$, where T_1 is the normal part of T and T_2 is the pure part of T; that is, T_2 is p-hyponormal and has no nontrivial invariant subspace $\mathcal{M} \subset H_2$ such that $T_2|_{\mathcal{M}}$ is normal. Define the operator $S = N \oplus 0$ on H, where N is a w_*-hyponormal operator on H_1 that commutes with T_1 (as example $N = T_1$). It is easy to verify that $TS = ST$ and $T^*S = ST^*$ using Fuglede's theorem. Moreover, since $|T^*| = |T_1| \oplus |T_2^*|$, we also have $|T|S = S|T^*|$. Therefore, the operators T and S satisfy the assumptions of Proposition 2. Thus, $TXT^* = SXS^* \implies T^*XT = S^*XS$ for all $X \in \mathcal{L}(H)$.

Based on Theorem 1, we now present a generalization of this Theorem by replacing the condition "T or S is injective "with the weaker assumption "$\ker T \cap \ker S = \{0\}$ ".

Theorem 3. *Let $T \in w_*$-H and $S \in \mathcal{L}(H)$ is normal such that $TS = ST$ and suppose that $\ker T \cap \ker S = \{0\}$. Then*

$$\|\psi_{T,S}(X) + Y\| \geq \|Y\|,$$

for every $Y \in \ker(\psi_{T,S})$ and for all $X \in \mathcal{L}(H)$.

Proof. According to Theorem 1, it suffices to consider the case where neither T nor S is injective. We begin by considering the orthogonal decomposition of the Hilbert space $H_0 = H = \ker S^{\perp} \oplus \ker S$. With respect to this decomposition, the operators T and S take the forms

$$T = \begin{pmatrix} N & 0 \\ 0 & R \end{pmatrix}, \qquad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $R = T|_{\ker S}$ is injective due to the assumption $\ker T \cap \ker S = \{0\}$. We now distinguish two cases.

Case 1: $N = 0$. Let $Y \in \ker \psi_{T,S}$, written in block form as $Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix}$. Then the identity $TYT^* = SYS^*$ implies $S_1Y_1S_1^* = 0$ and $RY_4R^* = 0$.

Since both S_1 and R are injective, it follows that $Y_1 = Y_4 = 0$. Now, let $X \in \mathcal{L}(H_0)$ be of the form $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$. Then $\psi_{T,S}(X) + Y = \begin{pmatrix} -S_1X_1S_1^* & Y_2 \\ Y_3 & RX_3R^* \end{pmatrix}$.

Since $|Y| = |Y_3| \oplus |Y_2|$, we obtain $\|\psi_{T,S}(X) + Y\| \geq \max\{\|Y_2\|, \|Y_3\|\} = \|Y\|$.

Case 2: $N \neq 0$. Since $R = T|_{\ker S}$ is injective, but T is not, it follows that $N = T|_{\ker S^{\perp}}$ is not injective. As T is paranormal, the subspace $\ker S^{\perp} \ominus \ker N$ is invariant under T. Consider the orthogonal decomposition $H_1 = H = (\ker S^{\perp} \ominus \ker N) \oplus \ker N \oplus \ker S$. Then T and S decompose as

$$T = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T_2 \end{pmatrix}, \qquad S = \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $T_1 \in w_*\text{-}H$, and all operators T_1, T_2, S_1 and S_2 are injective. Let $Y \in \ker \psi_{T,S}$. It follows from the condition $TYT^* = SYS^*$, that Y can be expressed as

$$Y = \begin{pmatrix} Y_{11} & 0 & 0 \\ 0 & 0 & Y_{23} \\ 0 & Y_{32} & 0 \end{pmatrix},$$

where $Y_{11} \in \ker \psi_{T_1,S_1}$. Now, for any $X = [X_{ij}]_{i,j=1}^3 \in \mathcal{L}(H_1)$, we have

$$\psi_{T,S}(X) + Y = \begin{pmatrix} \psi_{T_1,S_1}(X_{11}) + Y_{11} & * & * \\ * & * & Y_{23} \\ * & Y_{32} & * \end{pmatrix}.$$

Therefore, $\|\psi_{T,S}(X) + Y\| \geq \max\{\|\psi_{T_1,S_1}(X_{11}) + Y_{11}\|, \|Y_{23}\|, \|Y_{32}\|\}$. By Theorem 1, we have $\|\psi_{T_1,S_1}(X_{11}) + Y_{11}\| \geq \|Y_{11}\|$. Thus, $\|\psi_{T,S}(X) + Y\| \geq \max\{\|Y_{11}\|, \|Y_{23}\|, \|Y_{32}\|\} = \|Y\|$, since $|Y| = |Y_{11}| \oplus |Y_{32}| \oplus |Y_{23}|$. □

Corollary 3. *Let $A, B^* \in w_*\text{-}H$ and let $C, D \in \mathcal{L}(H)$ be two normal operators, such that $AC = CA$ and $BD = DB$. Suppose that $\ker A \cap \ker C = \ker B^* \cap \ker D = \{0\}$. Then $\ker \Psi \subset \ker \Psi_*$, and we have $\|\Psi(X) + T\| \geq \|T\|$ for all $T \in \ker \Psi$ and for all $X \in \mathcal{L}(H)$.*

Theorem 4. *Let $T \in w_*\text{-}H$ and S be a normal operator such that $TS = ST$ and $\psi_{T,S} \neq 0$ with T and S are nonzero operators. Then $\|\psi_{T,S}(X) + Y\| \geq \|Y\|$ holds for every $Y \in \ker \psi_{T,S}$ and for all $X \in \mathcal{L}(H)$, if and only if $\ker T \cap \ker S = \{0\}$.*

Proof. By Theorem 3, it is sufficient to demonstrate that the condition $\ker T \cap \ker S = \{0\}$ is necessary for $\mathcal{R}(\psi_{T,S}) \perp \ker \psi_{T,S}$ when T and S are not injective. Assume that $\ker T \cap \ker S \neq \{0\}$. We can decompose $H_0 = H = \ker S^{\perp} \oplus \ker S$, leading to the representations

$$T = \begin{pmatrix} N & 0 \\ 0 & R \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}$$

on H_0. First, suppose that $\ker T \neq \ker S$. Without loss of generality, we can assume that $\ker S \nsubseteq \ker T$. We can further decompose H as follows

$$H_1 = H = \ker S^{\perp} \oplus (\ker S \ominus \ker R) \oplus \ker R$$

where $R = T|_{\ker S}$ is a nonzero operator with a nontrivial kernel. Consequently, we have

$$T = T_1 \oplus T_2 \oplus 0 \quad \text{and} \quad S = S_1 \oplus 0 \oplus 0,$$

where T_2 and S_1 are injective, and $T_1 \in w_*$-H. Let $Y = [Y_{ij}]_{i,j=1}^{3} \in \mathcal{L}(H_1)$. We find that $Y \in \ker \psi_{T,S}$ if and only if $Y_{11} \in \ker \psi_{T_1,S_1}$, $T_1 Y_{12} T_2^{*} = T_2 Y_{21} T_1^{*} = 0$, and $Y_{22} = 0$. Let e be a nonzero vector in H, and choose $X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e \otimes e & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Y_{23} \\ 0 & Y_{32} & Y_{33} \end{pmatrix}$. Then we have

$$\psi_{T,S}(X) + Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_2 e \otimes T_2 e & Y_{23} \\ 0 & Y_{32} & Y_{33} \end{pmatrix}.$$

Since T_2 is injective, the operator $C = T_2 e \otimes T_2 e$ is a nonzero self-adjoint operator of rank one. By applying Lemma 2.4 from [16], we can find operators Y_{23}, Y_{32}, and Y_{33} such that

$$\|\psi_{T,S}(X) + Y\| < \left\| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Y_{23} \\ 0 & Y_{32} & Y_{33} \end{pmatrix} \right\| = \|Y\|.$$

In the case where $\ker T = \ker S$, we have $R = T|_{\ker S} = 0$ and $N = T|_{\ker S^{\perp}}$ is injective. Let $Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \in \mathcal{L}(H_0)$. It follows that $Y \in \ker \psi_{T,S}$ if and only if $Y_{11} \in \ker \psi_{N,S_1}$. We can choose

$$Y = \begin{pmatrix} 0 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} e \otimes e & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we find

$$\psi_{T,S}(X) + Y = \begin{pmatrix} Ne \otimes Ne - S_1 e \otimes S_1 e & Y_2 \\ Y_3 & Y_4 \end{pmatrix}.$$

If $C' = Ne \otimes Ne - S_1 e \otimes S_1 e = 0$ for all $e \in H$, then since both N and S_1 are injective, we would conclude that $S_1 = cN$ for some $|c| = 1$. This would imply that $\psi_{T,S} = 0$, contradicting our assumption. Thus, C' is a nonzero self-adjoint operator of finite rank. We can also apply Lemma 2.4 from [16] to complete the proof. □

Remark 4. 1. The condition $\psi_{T,S} \neq 0$ in Theorem 4 is essential. Consider the case where $T = cS$ with S being a non-injective operator (for example, $S = 0$) and $|c| = 1$. This illustrates that $\mathcal{R}(\psi_{T,S}) \perp \ker \psi_{T,S}$ does not necessarily imply $\ker T \cap \ker S = \{0\}$ when $\psi_{T,S} = 0$.

2. Duggal showed in [4, Lemma 4] that if $\ker \Delta_{A,B} \subset \ker \Delta_{A^*,B^*}$, then it follows that $\mathcal{R}(\Delta_{A,B}) \perp \ker \Delta_{A,B}$. However, the following example shows that the inclusion $\ker \psi_{T,S} \subset \ker \psi_{T^*,S^*}$ does not imply $\mathcal{R}(\psi_{T,S}) \perp \ker \psi_{T,S}$.

Example 2. Let $H_0 = H \oplus H$ and define the operators

$$T = \begin{pmatrix} I & 0 \\ 0 & R \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 2I & 0 \\ 0 & 0 \end{pmatrix},$$

where R is a w_*-hyponormal non-injective operator on H (for example, $R = 0$). In this case, we have $T \in w_*$-H, S is normal, and $TS = ST$. Therefore, Theorem 2 guarantees that $\ker \psi_{T,S} \subset \ker \psi_{T^*,S^*}$. However, the assumption $\ker T \cap \ker S = \{0\}$ implies that R must be injective, Which is not the case. Since $\psi_{T,S} \neq 0$ ($\psi_{T,S}(I \oplus 0) \neq 0$), it follows from Theorem 4 that $\mathcal{R}(\psi_{T,S})$ is not orthogonal to $\ker \psi_{T,S}$.

Corollary 4. *Let $A, B^* \in w_*$-H and let $C, D \in \mathcal{L}(H)$ be two normal operators such that $AC = CA$ and $BD = DB$, and assume that $\Psi \neq 0$ such that A, B, C and D are nonzero. Then the inequality $\|\Psi(X) + T\| \geq \|T\|$ for all $T \in \ker \Psi$ and for all $X \in \mathcal{L}(H)$, if and only if $\ker A \cap \ker C = \ker B^* \cap \ker D = \{0\}$.*

Remark 5. The condition $\ker A \cap \ker C = \ker B^* \cap \ker D = \{0\}$ in Corollary 4 is not necessary for $\mathcal{R}(\Psi) \perp \ker \Psi$ if at least one of the operators A, B, C, or D is zero. For instance, take $A \in w_*$-H as a nonzero operator with a nontrivial kernel, set $C = 0$, and let $D = I$. In this case, we find that $\Psi = \delta_{A,0} \neq 0$, where $\delta_{A,B} \colon X \in \mathcal{L}(H) \mapsto AX - XB$. The pair $(A, 0)$ possesses the Fuglede-Putnam property according to Lemma 2.4 from [2], which leads to the conclusion that $\mathcal{R}(\Psi) \perp \ker \Psi$ using Lemma 4 from [4].

Proposition 3. *Let $A, B, C, D \in \mathcal{L}(H)$ be nonzero operators such that $AC = CA$, $BD = DB$ and $\Psi \neq 0$.*

1. *If $BD^* = D^*B$ and any of the following conditions hold:*
   - *(i) $A, B^*, D^* \in w_*$-H, C is normal and D is invertible,*
   - *(ii) $C, B^*, D^* \in w_*$-H, A is normal and B is invertible,*

   then the inequality $\|\Psi(X) + T\| \geq \|T\|$ holds for all $T \in \ker \Psi$ and for all $X \in \mathcal{L}(H)$, if and only if $\ker A \cap \ker C = \{0\}$.
2. *If $AC^* = C^*A$ and if one of the following conditions hold:*
   - *(i) $A, B^*, C \in w_*$-H, D is normal and C is invertible,*
   - *(ii) $A, B^*, D^* \in w_*$-H, B is normal and A is invertible,*

   then the inequality $\|\Psi(X) + T\| \geq \|T\|$ holds for all $T \in \ker \Psi$ and for all $X \in \mathcal{L}(H)$, if and only if $\ker B^* \cap \ker D = \{0\}$.

Proof. Under the hypothesis of the case (i), we have $T \in \ker \Psi$ if and only if $T \in \ker \Psi_0$, where $\Psi_0 \colon X \in \mathcal{L}(H) \mapsto AXBD^{-1} - CXI$. By applying Lemma 1 and Corollary 4, we have $\ker A \cap \ker C = \{0\}$, if and only if $\|\Psi_0(Y) + T\| \geq \|T\|$ for all $T \in \ker \Psi_0 = \ker \Psi$ and for all $Y \in \mathcal{L}(H)$, which is equivalent to (by taking $Y = XD$), $\|\Psi(X) + T\| \geq \|T\|$ for all $T \in \ker \Psi$ and for all $X \in \mathcal{L}(H)$. We can prove other cases in the same way. □

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