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V. HLADUN

STABILITY TO PERTURBATIONS OF g–FRACTIONS

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This paper investigates the stability of g–fractions to perturbations. Recurrence formulas for the relative errors of the approximant tails of the g–fraction are established, based on which a formula for the relative error of the approximant is obtained, describing its accumulation in the sequence of approximants. Sufficient conditions for the stability of the g–fraction are established using the majorant estimation technique. These conditions are formulated in terms of a majorant continued fraction that bounds the magnitude of the relative error of the approximant. It is shown that stability is guaranteed if the relative errors of the coefficients and the variable of the fraction are bounded, and the numerical series formed from the parameters of the majorant diverges. The theoretical results are applied to find stability sets in the complex plane using the method of value sets for the approximant tails. In particular, it is proven that under certain conditions on the coefficients, the g–fraction is stable in the closed unit disk. The obtained error estimates can be used for accuracy control in practical problems that use g–fractions.

1. Introduction. Continued fractions are a mathematical apparatus that has historically played an important role in number theory and function theory. In modern approximation theory, they are used for representing special functions and constructing their effective approximations. In particular, functional continued fractions and their generalizations allow for obtaining rational approximations that, in some cases, have better approximation properties than polynomial approximations.

The g–fractions

$$\cfrac{s_0}{1+\cfrac{g_1 z}{1+\cfrac{g_2(1-g_1)z}{1+\ddots}}}, \qquad s_0>0, \quad 0<g_k<1, \quad k\geq 0, \tag{1}$$

with $z\in\mathbb{C}$, are one of the key classes of functional continued fractions that find wide application in complex analysis, approximation theory, and computational mathematics. The g–fractions emerged as a subclass of the classical S-fractions (Stieltjes fractions). H. S. Wall first investigated the correspondence between a g–fraction and a formal power series, establishing the correspondence criterion ([1]). F. L. Bauer proposed the g–algorithm for computing the coefficients of the g–fraction (1) from the coefficients of a formal power series ([2, 3]). In [4], H. S. Wall described the class of all functions represented by g–fractions and proved the convergence of the g–fraction (1) to a holomorphic function in the region $G=\{z\in\mathbb{C}\colon |\arg(z+1)|<\pi\}$ and its uniform convergence on every compact subset of G.

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*Corresponding author: V. Hladun

For practical applications, estimates of the rate of convergence are important. In [5, 6], a priori error bounds for the approximants of g–fractions were established, and it was shown that the convergence is at least geometric. For this purpose, W. B. Gragg studied the π-fractions, showing that g–fractions are their even part ([7]).

A natural development of the analytic theory of g–fractions was its extension to functions of several variables, which led to multivariate g–fractions ([8, 9]). D. I. Bodnar and R. I. Dmytryshyn investigated the convergence of multivariate g–fractions in polydisc and multivariate parabolic domains and established estimates for the rate of convergence ([10, 11]), while K. Y. Kuchminska and S. M. Vozna established truncation error bounds for two-dimensional g–fractions ([12]).

R. I. Dmytryshyn constructed a complete theory of multidimensional g–fractions with independent variables ([13]). They allow for the approximation of a wide class of functions, including hypergeometric functions of several variables ([14]). He investigated the correspondence properties of multidimensional g–fractions with independent variables ([15, 16]) and developed constructive algorithms for expanding a formal power series into the corresponding multidimensional g–fraction, which is a generalization of Bauer's classical g–algorithm and a multidimensional analogue of Rutishauser's qd-algorithm for constructing a multidimensionally regular C-fraction ([17, 18, 19]). Convergence domains for branched continued fractions with independent variables were also constructed, and error estimates for their approximations in certain bounded domains of $\mathbb{C}^N$ were established ([20, 21, 22]).

The use of continued fractions as models for describing real processes raises the problem of their stability to perturbations. The coefficients and variables in the structure of a continued fraction can be system parameters obtained experimentally. Such quantities are not known with absolute precision and contain perturbations caused by various factors. The question of how these small perturbations in the model's parameters affect its output characteristics is crucial for its correctness and reliability. A model is considered stable if small perturbations in the input data lead to small changes in the result. Otherwise, the model is unstable and cannot be considered an adequate description of the real system. Therefore, the study of stability to perturbations of continued fractions and their generalizations is an important direction in modern analytic continued fraction theory. In particular, papers [23, 24, 25] investigate the stability to perturbations of continued fractions and branched continued fractions with real and complex elements, establishing sets of elements that guarantee their stability. These works provide a mathematical apparatus for assessing the reliability of models that use continued fractions and their generalizations, and determine the conditions under which such models remain stable. In [26], the stability to perturbations of continued fraction approximants with complex elements is investigated. Sufficient conditions for stability are obtained in the form of fundamental inequalities for partial numerators. The general results are applied to the g–fractions, and stability sets in $\mathbb{C}$ are constructed.

In the practical application of continued fractions, a fundamental problem of computational process stability arises. The coefficients of the fraction and its variable are the results of computations that contain rounding errors. The question of how these perturbations affect the result is important for assessing the reliability of the computations. A fraction that is unstable to perturbations can yield a result that differs significantly from the true value even for small errors in the coefficients, making it practically unusable. The issue of error accumulation in the computation of continued fractions became a subject of research in [27, 28] at the dawn of computational technology. A significant contribution to the analytical justification of computational stability was made by W. B. Jones and W. J. Thron, who established a connection between the stability of the backward recurrence algorithm for computing approximants of a continued fraction and its element sets ([29]). Rounding error analysis for forward recurrence algorithms was systematically reviewed in [30]. Modern research continues this theme, extending it to branched continued fractions. Papers [31, 32, 33] investigate the influence of coefficient perturbations on the accuracy of approximating special functions developed from Horn's hypergeometric functions H_4, H_6, H_7. The numerical stability of the backward recurrence algorithm for computing approximants of branched continued fraction expansions of the Lauricella–Saran hypergeometric function F_K ratios is studied in [34], where estimates of the relative errors of approximant computations are obtained and sufficient conditions for numerical stability sets are established.

Despite the significant number of works devoted to the convergence of g–fractions, the problem of obtaining a priori error bounds for their approximants to perturbations of coefficients and the variable, and establishing conditions for stability to perturbations, remains relevant ([26]). In the analytic theory of continued fractions, one of the approaches to studying the convergence of their approximant sequence is the method of element sets and corresponding value sets ([35, 36, 37]). In this paper, to study the stability to perturbations of g–fractions, we use the methodology of investigating the majorant of the relative error of the approximant and the sequence of value sets of the approximant tails.

The structure of the paper is as follows. In the second section, we derive recurrence formulas for the relative errors of the tails of the approximants and the approximants themselves of the g–fraction. In the third section, using the derived formulas, we construct a majorant continued fraction that bounds the perturbation error and obtain sufficient conditions for stability to perturbations, which require the divergence of the series formed by the majorant parameters. Conditions on the coefficients of the g–fraction that ensure stability at a given point in the complex plane are also established. In the fourth section, conditions on the coefficients are established under which the closed unit disk with its center at the origin of coordinates is a stability set for the g–fraction.

2. Formulas for the relative errors of approximants of a g–fraction. Let us consider the sequence of approximants $\{f_n(z)\}_{n\in\mathbb{N}}$ of the g–fraction

$$\cfrac{s_0}{1+\cfrac{g_1(1-g_0)z}{1+\cfrac{g_2(1-g_1)z}{1+\ddots}}}, \tag{2}$$

$$f_n(z)=\cfrac{s_0}{1+\cfrac{g_1(1-g_0)z}{1+\cfrac{g_2(1-g_1)z}{1+\ddots+\cfrac{g_n(1-g_{n-1})z}{1}}}},\quad n\geq 1. \tag{3}$$

Let $\hat{s}_0>0$, $0<\hat{g}_k<1$, $\hat{z}\in\mathbb{C}$ be the perturbed values of the coefficients s_0, g_k, $k\geq 0$, and the variable z of the g–fraction (2), respectively. The functional continued fraction

$$\cfrac{\hat{s}_0}{1+\cfrac{\hat{g}_1(1-\hat{g}_0)\hat{z}}{1+\cfrac{\hat{g}_2(1-\hat{g}_1)\hat{z}}{1+\ddots}}} \tag{4}$$

is called the *perturbed g–fraction of the g–fraction* (2). Let us denote by

$$\hat{f}_n(\hat{z}) = \cfrac{\hat{s}_0}{1 + \cfrac{\hat{g}_1(1-\hat{g}_0)\hat{z}}{1 + \cfrac{\hat{g}_2(1-\hat{g}_1)\hat{z}}{1 + \ddots + \cfrac{\hat{g}_n(1-\hat{g}_{n-1})\hat{z}}{1}}}}, \quad n \geq 1, \tag{5}$$

the approximants of the perturbed g–fraction (4).

Definition 1. The g–fraction (2) is called *stable to perturbations at a point* $z \in \mathbb{C}$ if for any $\varepsilon > 0$ there exists a $\delta_\varepsilon > 0$ such that for every $\hat{s}_0 > 0$ with $|\hat{s}_0 - s_0|/|s_0| < \delta_\varepsilon$, every $0 < \hat{g}_k < 1$, $k \geq 0$, with $|\hat{g}_k - g_k|/|g_k| < \delta_\varepsilon$, and every $\hat{z} \in \mathbb{C}$ with $|\hat{z} - z|/|z| < \delta_\varepsilon$, the inequalities $|\hat{f}_n - f_n|/|f_n| < \varepsilon$, $n \geq 1$, hold.

Let n be an arbitrary natural number. The functional continued fractions

$$Q_k^{(n)}(z) = 1 + \cfrac{g_{k+1}(1-g_k)z}{1 + \cfrac{g_{k+2}(1-g_{k+1})z}{1 + \ddots + \cfrac{g_n(1-g_{n-1})z}{1}}}, \quad 0 \leq k \leq n-1, \quad Q_n^{(n)}(z) = 1, \tag{6}$$

are called the *tails of the nth approximant* (3).

Let us denote by $\varepsilon_{k,n}^{(Q)}$, $0 \leq k \leq n$, the relative errors of the tails $Q_k^{(n)}(z)$

$$\hat{Q}_k^{(n)}(\hat{z}) = 1 + \cfrac{\hat{g}_{k+1}(1-\hat{g}_k)\hat{z}}{1 + \cfrac{\hat{g}_{k+2}(1-\hat{g}_{k+1})\hat{z}}{1 + \ddots + \cfrac{\hat{g}_n(1-\hat{g}_{n-1})\hat{z}}{1}}} = Q_k^{(n)}(z)(1+\varepsilon_{k,n}^{(Q)}) \ (0 \leq k \leq n-1), \quad \hat{Q}_n^{(n)}(\hat{z}) = 1,$$

are the tails of the perturbed approximant (5).

Lemma 1. *The relative errors $\varepsilon_{k,n}^{(Q)}$ of the tails $Q_k^{(n)}(z)$ satisfy the recurrence relations*

$$\varepsilon_{k,n}^{(Q)} = -1 + \frac{1}{Q_k^{(n)}(z)} + \frac{(1 - 1/Q_k^{(n)}(z))(1+\varepsilon_{k+1}^{(g)})(1 - g_k/(1-g_k)\varepsilon_k^{(g)})(1+\varepsilon^{(z)})}{1+\varepsilon_{k+1,n}^{(Q)}} \tag{7}$$

for $0 \leq k \leq n-1$, where $\varepsilon_{n,n}^{(Q)} = 0$, and $\varepsilon_k^{(g)}$, $k \geq 0$, $\varepsilon^{(z)}$ are the relative errors of the coefficients g_k, $k \geq 0$, and the variable z, respectively, defined as $\hat{g}_k = g_k(1+\varepsilon_k^{(g)})$, $k \geq 0$, $\hat{z} = z(1+\varepsilon^{(z)})$.

Proof. Since $Q_n^{(n)}(z) = \hat{Q}_n^{(n)}(\hat{z}) = 1$, we have $\varepsilon_{n,n}^{(Q)} = 0$. For $0 \leq k \leq n-1$, we have

$$\varepsilon_{k,n}^{(Q)} = \frac{\hat{Q}_k^{(n)}(\hat{z}) - Q_k^{(n)}(z)}{Q_k^{(n)}(z)} = -1 + \frac{\hat{Q}_k^{(n)}(\hat{z})}{Q_k^{(n)}(z)} = -1 + \frac{1 + \hat{g}_{k+1}(1-\hat{g}_k)\hat{z}/\hat{Q}_{k+1}^{(n)}(\hat{z})}{Q_k^{(n)}(z)} =$$

$$= -1 + \frac{1}{Q_k^{(n)}(z)} + \frac{g_{k+1}(1+\varepsilon_{k+1}^{(g)})(1-g_k)(1 - g_k/(1-g_k)\varepsilon_k^{(g)})z(1+\varepsilon^{(z)})}{Q_k^{(n)}(z)Q_{k+1}^{(n)}(z)(1+\varepsilon_{k+1,n}^{(Q)})}, \quad 0 \leq k \leq n-1.$$

From the definition of the tail $Q_k^{(n)}(z)$ of the nth approximant (3), the recurrence formula follows

$$Q_k^{(n)}(z) = 1 + \frac{g_{k+1}(1-g_k)z}{Q_{k+1}^{(n)}(z)}, \quad 0 \le k \le n-1, \quad Q_n^{(n)}(z) = 1. \tag{8}$$

Then, by (8), we obtain

$$\varepsilon_{k,n}^{(Q)} = -1 + \frac{1}{Q_k^{(n)}(z)} + \frac{(Q_k^{(n)}(z)-1)(1+\varepsilon_{k+1}^{(g)})(1-g_k/(1-g_k)\varepsilon_k^{(g)})(1+\varepsilon^{(z)})}{Q_k^{(n)}(z)(1+\varepsilon_{k+1,n}^{(Q)})}, \quad 1 \le k \le n-1.$$

This implies formula (7). □

Since $f_n(z) = s_0/Q_0^{(n)}(z)$ and $\hat{f}_n(\hat{z}) = \hat{s}_0/\hat{Q}_0^{(n)}(\hat{z})$, by denoting $\varepsilon_0^{(s)}$, $\varepsilon_n^{(f)}$, $n \ge 1$, which are defined by the formulas $\hat{s}_0 = s_0(1+\varepsilon_0^{(s)})$, $\hat{f}_n(\hat{z}) = f_n(z)(1+\varepsilon_n^{(f)})$, $n \ge 1$, we obtain

$$\varepsilon_n^{(f)} = -1 + \frac{\hat{s}_0/\hat{Q}_0^{(n)}(\hat{z})}{s_0/Q_0^{(n)}(z)} = -1 + \frac{1+\varepsilon_0^{(s)}}{1+\varepsilon_{0,n}^{(Q)}}. \tag{9}$$

Then, from (7), the following formula $n \ge 1$ is derived:

$$\varepsilon_n^{(f)} = -1 + \\ + \cfrac{1+\varepsilon_0^{(s)}}{\cfrac{1}{Q_0^{(n)}} + \cfrac{\left(1-\frac{1}{Q_0^{(n)}(z)}\right)(1+\varepsilon_1^{(g)})(1-g_0/(1-g_0)\varepsilon_0^{(g)})(1+\varepsilon^{(z)})}{\cfrac{1}{Q_1^{(n)}} + \cfrac{\left(1-\frac{1}{Q_1^{(n)}(z)}\right)(1+\varepsilon_2^{(g)})(1-g_1/(1-g_1)\varepsilon_1^{(g)})(1+\varepsilon^{(z)})}{\cfrac{1}{Q_2^{(n)}} + \ddots + \cfrac{\left(1-\frac{1}{Q_{n-1}^{(n)}(z)}\right)(1+\varepsilon_n^{(g)})(1-g_{n-1}/(1-g_{n-1})\varepsilon_{n-1}^{(g)})(1+\varepsilon^{(z)})}{1}}}}. \tag{10}$$

3. Sufficient conditions for the stability to perturbations of g–fractions.

Theorem 1. *The g–fraction (2) is stable to perturbations if there exist constants α, β, $0 \le \alpha < 1$, $0 \le \beta < 1$, $\alpha + \beta \ne 0$, such that the relative errors of its coefficients s_0, g_k, $k \ge 0$, and the variable z satisfy the conditions*

$$|\varepsilon_0^{(s)}| \le \alpha, \quad |\varepsilon_k^{(g)}| \le \alpha, \quad k \ge 0, \quad |\varepsilon^{(z)}| \le \beta, \tag{11}$$

and there exist sequences of real constants $\{\eta_k\}_{k\in\mathbb{N}\cup\{0\}}$, $0 < \eta_k \le 1$, $\{\mu_k\}_{k\in\mathbb{N}\cup\{0\}}$, $0 \le \mu_k < 1$, such that

$$\eta_k + \mu_k + \eta_k\mu_k \le 1, \tag{12}$$

$$|1 - 1/Q_k^{(n)}(z)| \le \eta_k, \quad 0 \le k \le n-1, \quad n \ge 1, \tag{13}$$

where the tails $Q_k^{(n)}(z)$ are defined according to (6) and

$$\frac{4\eta_k(1+\alpha)(1+\alpha g_k/(1-g_k))(1+\beta)}{(1+\eta_k)(1+\eta_{k+1})} \le (1-\mu_k)(1+\mu_{k+1}), \quad k \ge 0, \tag{14}$$

and the series $\sum_{k=0}^{\infty}\mu_k$ *diverges. Moreover, for the relative errors of the approximants of the* g*–fraction* (2) *the following estimate holds*

$$|\varepsilon_n^{(f)}| \le \tilde{\varepsilon}_n^{(f)}(\alpha,\beta), \quad n \ge 1, \tag{15}$$

where

$$\tilde{\varepsilon}_n^{(f)}(\alpha,\beta) = \\ = -1 + \cfrac{1+\alpha}{1+\eta_0 - \cfrac{\eta_0(1+\alpha)(1+g_0/(1-g_0)\alpha)(1+\beta)}{1+\eta_1 - \cfrac{\eta_1(1+\alpha)(1+g_1/(1-g_1)\alpha)(1+\beta)}{1+\eta_2 - \ddots - \cfrac{\eta_{n-1}(1+\alpha)(1+g_{n-1}/(1-g_{n-1})\alpha)(1+\beta)}{1}}}}. \tag{16}$$

Proof. The proof is based on the majorant estimation technique. Using the recurrence relations for the relative errors established in Lemma 1, a majorant continued fraction $\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)$ is constructed that bounds the relative error of the approximant. It is proved that the sequence $\{\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)\}_{n\in\mathbb{N}}$ is strictly increasing and bounded on the set $I=[0,1)\times[0,1)$, and therefore converges to the function $\tilde{\varepsilon}(\alpha,\beta)$. The uniform convergence on I implies the continuity of $\tilde{\varepsilon}(\alpha,\beta)$ at $(0,0)$ with $\tilde{\varepsilon}(0,0)=0$, which together yield the stability condition.

Let n be an arbitrary natural number. Let us denote by

$$\tilde{Q}_k^{(n)}(\alpha,\beta) = 1+\eta_k - \cfrac{\eta_k(1+\alpha)(1+g_k/(1-g_k)\alpha)(1+\beta)}{1+\eta_{k+1} - \cfrac{\eta_{k+1}(1+\alpha)(1+g_{k+1}/(1-g_{k+1})\alpha)(1+\beta)}{1+\eta_{k+2} - \ddots - \cfrac{\eta_{n-1}(1+\alpha)(1+g_{n-1}/(1-g_{n-1})\alpha)(1+\beta)}{1}}},$$

$0 \le k \le n-1$, the tails of the approximant (16).

Using the method of mathematical induction on k, for $k=n-1, n-2, \ldots, 0$, we will prove that if the inequalities (14) are satisfied, then the following estimates

$$\tilde{Q}_k^{(n)}(\alpha,\beta) > (1+\eta_k)(1+\mu_k)/2, \quad 0 \le k \le n-1, \tag{17}$$

hold. For $k=n-1$ we have

$$\begin{aligned}\tilde{Q}_{n-1}^{(n)}(\alpha,\beta) &= 1+\eta_{n-1} - \eta_{n-1}(1+\alpha)(1+g_{n-1}/(1-g_{n-1})\alpha)(1+\beta) \ge \\ &\ge 1+\eta_{n-1} - (1+\eta_{n-1})(1+\eta_n)(1-\mu_{n-1})(1+\mu_n)/4 > \\ &> (1+\eta_{n-1})(1-(1-\mu_{n-1})/2) = (1+\eta_{n-1})(1+\mu_{n-1})/2.\end{aligned}$$

Assuming that the estimates (17) hold for some $k=m+1$, $0 \le m \le n-2$,

$$\tilde{Q}_{m+1}^{(n)}(\alpha,\beta) > (1+\eta_{m+1})(1+\mu_{m+1})/2,$$

we prove them for $k=m$

$$\begin{aligned}\tilde{Q}_m^{(n)}(\alpha,\beta) &= 1+\eta_m - \eta_m(1+\alpha)(1+g_m/(1-g_m)\alpha)(1+\beta)/\tilde{Q}_{m+1}^{(n)}(\alpha,\beta) > \\ &> 1+\eta_m - \frac{2\eta_m(1+\alpha)(1+g_m/(1-g_m)\alpha)(1+\beta)}{(1+\eta_{m+1})(1+\mu_{m+1})} =\end{aligned}$$

$$= (1+\eta_m)\left(1 - \frac{2\eta_m(1+\alpha)(1+g_m/(1-g_m)\alpha)(1+\beta)}{(1+\eta_m)(1+\eta_{m+1})(1+\mu_{m+1})}\right) \geq$$
$$\geq (1+\eta_m)(1-(1-\mu_m)/2) = (1+\eta_m)(1+\mu_m)/2.$$

Let us denote $G_k^{(n)}(\alpha,\beta) = 1 - \tilde{Q}_k^{(n)}(\alpha,\beta)$. For the quantities $G_k^{(n)}(\alpha,\beta)$ the recurrence formula holds

$$G_k^{(n)}(\alpha,\beta) = -\eta_k + \frac{\eta_k(1+\alpha)(1+g_k/(1-g_k)\alpha)(1+\beta)}{1-G_{k+1}^{(n)}(\alpha,\beta)}, \quad 0 \leq k \leq n-1,$$

where $G_n^{(n)}(\alpha,\beta) = 0$.

If the inequalities (12), (17) are satisfied, then for the quantities $G_k^{(n)}$ the estimates hold

$$G_k^{(n)}(\alpha,\beta) < 1 - (1+\eta_k)(1+\mu_k)/2 \leq 1, \quad 0 \leq k \leq n-1. \tag{18}$$

Using the method of mathematical induction on k, for $k = n-1, n-2, \ldots, 0$, we will prove that, if the inequalities (13), (18) are satisfied, then for the relative errors of the tails $Q_k^{(n)}$ the following estimates hold

$$|\varepsilon_{k,n}^{(Q)}| \leq G_k^{(n)}(\alpha,\beta), \quad 0 \leq k \leq n-1. \tag{19}$$

For $k = n-1$ we have

$$|\varepsilon_{n-1,n}^{(Q)}| = |-1 + 1/Q_{n-1}^{(n)}(z) +$$
$$+(1-1/Q_{n-1}^{(n)}(z))(1+\varepsilon_n^{(g)})(1-g_{n-1}/(1-g_{n-1})\varepsilon_{n-1}^{(g)})(1+\varepsilon^{(z)})/(1+\varepsilon_{n,n}^{(Q)})| =$$
$$= |-1+1/Q_{n-1}^{(n)}(z) + (1-1/Q_{n-1}^{(n)})(1+\varepsilon_n^{(g)})(1-g_{n-1}/(1-g_{n-1})\varepsilon_{n-1}^{(g)})(1+\varepsilon^{(z)})| =$$
$$= |1-1/Q_{n-1}^{(n)}(z)||-1+(1+\varepsilon_n^{(g)})(1-g_{n-1}/(1-g_{n-1})\varepsilon_{n-1}^{(g)})(1+\varepsilon^{(z)})| \leq$$
$$\leq \eta_{n-1}(-1+(1+\alpha)(1+g_{n-1}/(1-g_{n-1})\alpha)(1+\beta)) =$$
$$= -\eta_{n-1} + \eta_{n-1}(1+\alpha)(1+g_{n-1}/(1-g_{n-1})\alpha)(1+\beta) = G_{n-1}^{(n)}(\alpha,\beta).$$

Assume that the estimates (19) hold for some $k = m+1$, $0 \leq m \leq n-2$. Then

$$|\varepsilon_{m,n}^{(Q)}| = |-1+1/Q_m^{(n)}(z) +$$
$$+(1-1/Q_m^{(n)}(z))(1+\varepsilon_{m+1}^{(g)})(1-g_m/(1-g_m)\varepsilon_m^{(g)})(1+\varepsilon^{(z)})/(1+\varepsilon_{m+1,n}^{(Q)})| =$$
$$= |1-1/Q_m^{(n)}(z)||-1+(1+\varepsilon_{m+1}^{(g)})(1-g_m/(1-g_m)\varepsilon_m^{(g)})(1+\varepsilon^{(z)})/(1+\varepsilon_{m+1,n}^{(Q)})| \leq$$
$$\leq \eta_m|(\varepsilon_{m+1}^{(g)} + \varepsilon^{(z)} - g_m/(1-g_m)\varepsilon_m^{(g)} + \varepsilon_{m+1}^{(g)}\varepsilon^{(z)} - g_m/(1-g_m)\varepsilon_m^{(g)}\varepsilon_{m+1}^{(g)} -$$
$$-g_m/(1-g_m)\varepsilon_m^{(g)}\varepsilon^{(z)} - g_m/(1-g_m)\varepsilon_m^{(g)}\varepsilon_{m+1}^{(g)}\varepsilon^{(z)} - \varepsilon_{m+1,n}^{(Q)}|/|1+\varepsilon_{m+1,n}^{(Q)}| \leq$$
$$\leq \eta_m(\alpha+\beta+g_m/(1-g_m)\alpha + \alpha\beta + g_m/(1-g_m)\alpha^2 + g_m/(1-g_m)\alpha\beta +$$
$$+g_m/(1-g_m)\alpha^2\beta + |\varepsilon_{m+1,n}^{(Q)}|)/(1-|\varepsilon_{m+1,n}^{(Q)}|) \leq$$
$$\leq \eta_m((1+\alpha)(1+g_m/(1-g_m)\alpha)(1+\beta)/(1-|\varepsilon_{m+1,n}^{(Q)}|) - 1) =$$
$$= -\eta_m + \eta_m(1+\alpha)(1+g_m/(1-g_m)\alpha)(1+\beta)/(1-|\varepsilon_{m+1,n}^{(Q)}|) \leq$$
$$\leq -\eta_m + \eta_m(1+\alpha)(1+g_m/(1-g_m)\alpha)(1+\beta)/(1-G_{m+1}^{(n)}(\alpha,\beta)) = G_m^{(n)}(\alpha,\beta).$$

Then, from formula (9) and estimate (19) for $k=0$, we obtain

$$|\varepsilon_n^{(f)}| = |-1 + (1+\varepsilon_0^{(s)})/(1+\varepsilon_{0,n}^{(Q)})| \le -1 + (1+\alpha)/(1-G_0^{(n)}(\alpha,\beta)).$$

The last estimate is equivalent to the estimate (15).

Let us prove that the sequence $\{\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)\}_{n\in\mathbb{N}}$, where the approximants $\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)$ are defined according to (16), is strictly increasing.

For the difference of the tails $\tilde{Q}_k^{(n)}(\alpha,\beta)$ and $\tilde{Q}_k^{(n+p)}(\alpha,\beta)$, where the recurrence formula holds

$$\tilde{Q}_k^{(n)}(\alpha,\beta) - \tilde{Q}_k^{(n+p)}(\alpha,\beta) = \eta_k(1+\alpha)(1+g_k/(1-g_k)\alpha)(1+\beta)\frac{\tilde{Q}_{k+1}^{(n)}(\alpha,\beta) - \tilde{Q}_{k+1}^{(n+p)}(\alpha,\beta)}{\tilde{Q}_{k+1}^{(n)}(\alpha,\beta)\tilde{Q}_{k+1}^{(n+p)}(\alpha,\beta)},$$

$0 \le k \le n-1$, $p\in\mathbb{N}$, where

$$\tilde{Q}_n^{(n)}(\alpha,\beta) - \tilde{Q}_n^{(n+p)}(\alpha,\beta) = 1 - \tilde{Q}_n^{(n+p)}(\alpha,\beta) =$$
$$= \eta_n((1+\alpha)(1+g_n/(1-g_n)\alpha)(1+\beta)/\tilde{Q}_{n+1}^{(n+p)}(\alpha,\beta) - 1).$$

Then for the difference of the approximants $\tilde{\varepsilon}_{n+p}^{(f)}(\alpha,\beta)$ and $\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)$ we obtain the formula

$$\tilde{\varepsilon}_{n+p}^{(f)}(\alpha,\beta) - \tilde{\varepsilon}_n^{(f)}(\alpha,\beta) = (1+\alpha)\frac{\tilde{Q}_0^{(n)}(\alpha,\beta) - \tilde{Q}_0^{(n+p)}(\alpha,\beta)}{\tilde{Q}_0^{(n)}(\alpha,\beta)\tilde{Q}_0^{(n+p)}(\alpha,\beta)} = (1+\alpha)^{n+1}(1+\beta)^n \times$$
$$\times\Big(\frac{(1+\alpha)(1+g_n/(1-g_n)\alpha)(1+\beta)}{\tilde{Q}_{n+1}^{(n+p)}(\alpha,\beta)} - 1\Big)\frac{\prod_{k=0}^{n}\eta_k\prod_{k=0}^{n-1}(1+g_k/(1-g_k)\alpha)}{\prod_{k=0}^{n}\tilde{Q}_k^{(n)}(\alpha,\beta)\prod_{k=0}^{n}\tilde{Q}_k^{(n+p)}(\alpha,\beta)}. \tag{20}$$

Since $\alpha\ge 0$, $\beta\ge 0$, $\alpha+\beta\ne 0$, $\eta_k>0$, $0<g_k<1$ for all $k\ge 0$, and the estimates (17) hold, it follows from the formula (20) that $\{\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)\}_{n\in\mathbb{N}}$ is a strictly increasing sequence. Moreover, from (17) for $k=0$ it follows

$$\tilde{\varepsilon}_n^{(f)}(\alpha,\beta) = -1 + (1+\alpha)/\tilde{Q}_0^{(n)}(\alpha,\beta) < -1 + 2(1+\alpha)/((1+\eta_0)(1+\mu_0)), \quad n\ge 1,$$

which proves that the sequence $\{\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)\}_{n\in\mathbb{N}}$ is bounded. Therefore,

$$\lim_{n\to\infty}\tilde{\varepsilon}_n^{(f)}(\alpha,\beta) = \tilde{\varepsilon}(\alpha,\beta).$$

Let us prove that the sequence $\{\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)\}_{n\in\mathbb{N}}$ converges uniformly to the function $\tilde{\varepsilon}(\alpha,\beta)$ on the set $I=[0,1)\times[0,1)$. If the sequence $\{\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)\}_{n\in\mathbb{N}}$ is strictly increasing, then it converges uniformly on the set I, if for an arbitrary $\varepsilon>0$ there exists $n_\varepsilon\in\mathbb{N}$ such that for arbitrary $n>n_\varepsilon$, $p\in\mathbb{N}$ and for an arbitrary point $(\alpha,\beta)\in I$ the inequality $\tilde{\varepsilon}_{n+p}^{(f)}(\alpha,\beta) - \tilde{\varepsilon}_n^{(f)}(\alpha,\beta) < \varepsilon$ holds.

Let us transform the formula (20) to the form

$$\tilde{\varepsilon}_{n+p}^{(f)}(\alpha,\beta) - \tilde{\varepsilon}_n^{(f)}(\alpha,\beta) = \frac{(1+\alpha)\eta_n((1+\alpha)(1+g_n/(1-g_n)\alpha)(1+\beta)/\tilde{Q}_{n+1}^{(n+p)}(\alpha,\beta) - 1)}{\tilde{Q}_0^{(n+p(1+(-1)^n)/2)}(\alpha,\beta)}\times$$
$$\times\prod_{k=0}^{n-1}\frac{\eta_k(1+\alpha)(1+g_k/(1-g_k)\alpha)(1+\beta)}{\tilde{Q}_k^{(n+p(1+(-1)^{n+k+1})/2)}(\alpha,\beta)\tilde{Q}_{k+1}^{(n+p(1+(-1)^{n+k+1})/2)}(\alpha,\beta)}.$$

Let us estimate the multipliers written in the previous line. For $0 \le k \le n-1$ we obtain

$$\frac{\eta_k(1+\alpha)(1+g_k/(1-g_k)\alpha)(1+\beta)}{\tilde{Q}_k^{(n+p(1+(-1)^{n+k+1})/2)}(\alpha,\beta)\tilde{Q}_{k+1}^{(n+p(1+(-1)^{n+k+1})/2)}(\alpha,\beta)} = \frac{1+\eta_k-\tilde{Q}_k^{(n+(1+(-1)^{n+k+1})/2)}(\alpha,\beta)}{\tilde{Q}_k^{(n+p(1+(-1)^{n+k+1})/2)}(\alpha,\beta)} =$$
$$= \frac{1+\eta_k}{\tilde{Q}_k^{(n+p(1+(-1)^{n+k+1})/2)}(\alpha,\beta)} - 1 < \frac{2(1+\eta_k)}{(1+\eta_k)(1+\mu_k)} - 1 = \frac{1-\mu_k}{1+\mu_k}.$$

Then for the difference of the approximants $\tilde{\varepsilon}_{n+p}^{(f)}$ and $\tilde{\varepsilon}_n^{(f)}$ the estimate follows

$$\tilde{\varepsilon}_{n+p}^{(f)}(\alpha,\beta) - \tilde{\varepsilon}_n^{(f)}(\alpha,\beta) < \frac{2(1+\alpha)\eta_n\Big(\dfrac{2(1+\alpha)(1+g_n/(1-g_n)\alpha)(1+\beta)}{(1+\eta_{n+1})(1+\mu_{n+1})} - 1\Big)}{(1+\eta_0)(1+\mu_0)} \prod_{k=0}^{n-1} \frac{1-\mu_k}{1+\mu_k}.$$

From (14) we have:

$$\frac{2\eta_n(1+\alpha)(1+g_n/(1-g_n)\alpha)(1+\beta)}{(1+\eta_{n+1})(1+\mu_{n+1})} \le \frac{(1+\eta_n)(1-\mu_n)}{2},$$

then

$$\frac{2\eta_n(1+\alpha)(1+g_n/(1-g_n)\alpha)(1+\beta)}{(1+\eta_{n+1})(1+\mu_{n+1})} - \eta_n \le \frac{(1+\eta_n)(1-\mu_n)}{2} - \eta_n \le \frac{1-\eta_n}{2} < \frac{1}{2}.$$

Therefore

$$\tilde{\varepsilon}_{n+p}^{(f)}(\alpha,\beta) - \tilde{\varepsilon}_n^{(f)}(\alpha,\beta) < \frac{1+\alpha}{(1+\eta_0)(1+\mu_0)} \prod_{k=0}^{n-1} \frac{1-\mu_k}{1+\mu_k} < \frac{2}{(1+\eta_0)(1+\mu_0)} \prod_{k=0}^{n-1} \frac{1-\mu_k}{1+\mu_k} =: M_n,$$

where M_n is independent of $p \in \mathbb{N}$ and $(\alpha,\beta) \in I$. Since the series $\sum_{k=0}^{\infty} \mu_k$ diverges, we have $M_n \to 0$ as $n \to \infty$, and therefore for an arbitrary $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\tilde{\varepsilon}_{n+p}^{(f)}(\alpha,\beta) - \tilde{\varepsilon}_n^{(f)}(\alpha,\beta) < M_n < \varepsilon$ for arbitrary $n > n_\varepsilon$, $p \in \mathbb{N}$ and arbitrary $(\alpha,\beta) \in I$, which proves the uniform convergence of the sequence $\{\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)\}_{n\in\mathbb{N}}$ on the set I.

Since the functions $\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)$, $n \ge 1$, for $0 < \eta_k \le 1$, $k \ge 0$, are continuous on the set I, and the sequence $\{\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)\}$ converges uniformly to the function $\tilde{\varepsilon}(\alpha,\beta)$ on the set I, the function $\tilde{\varepsilon}(\alpha,\beta)$ is also continuous on the set I.

From the uniform convergence of the sequence $\{\tilde{\varepsilon}_n^{(f)}(\alpha,\beta)\}_{n\in\mathbb{N}}$ on the set I it follows that

$$\lim_{\substack{\alpha\to+0\\ \beta\to+0}} \lim_{n\to\infty} \tilde{\varepsilon}_n^{(f)}(\alpha,\beta) = \lim_{n\to\infty} \lim_{\substack{\alpha\to+0\\ \beta\to+0}} \tilde{\varepsilon}_n^{(f)}(\alpha,\beta). \qquad (21)$$

Since $\lim\limits_{\substack{\alpha\to+0\\ \beta\to+0}} \tilde{\varepsilon}_n^{(f)}(\alpha,\beta) = \tilde{\varepsilon}_n^{(f)}(0,0) = 0$, then, by (21), we have $\tilde{\varepsilon}(0,0) = 0$.

Thus, for the relative error of the nth approximant of the g–fraction (2) the estimate holds

$$\varepsilon_n^{(f)} \le \tilde{\varepsilon}(\alpha,\beta), \quad n \ge 1,$$

where $\tilde{\varepsilon}(\alpha,\beta)$, is a function continuous at the point $(0,0)$ and $\tilde{\varepsilon}(0,0) = 0$. Then for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$, such that for all $(\alpha,\beta) \in I$, such that $\sqrt{\alpha^2+\beta^2} < \delta_\varepsilon$, the inequality $\tilde{\varepsilon}(\alpha,\beta) < \varepsilon$ holds. That is, for each $\hat{s}_0 > 0$, such that $|\hat{s}_0 - s_0|/|s_0| \le \alpha < \delta_\varepsilon/\sqrt{2}$, for each

$0 < \hat{g}_k < 1$, $k \geq 0$, such that $|\hat{g}_k - g_k|/|g_k| \leq \alpha < \delta_\varepsilon/\sqrt{2}$ and for each $\hat{z} \in \mathbb{C}$, such that $|\hat{z} - z|/|z| \leq \beta < \delta_\varepsilon/\sqrt{2}$, for the relative errors of the approximants of the g–fraction (2), the inequalities $|\varepsilon_n^{(f)}| \leq \tilde{\varepsilon}(\alpha, \beta) < \varepsilon, \quad n \geq 1$, hold. This proves that the conditions for stability of the g–fraction (2) to perturbations of its coefficients and variable are met. □

Corollary 1. *The estimate* (15) *for the relative error of the nth approximant of the g–fraction* (2) *can be represented as*

$$|\varepsilon_n^{(f)}| \leq C_{1,n}\alpha + C_{2,n}\beta + o(\sqrt{\alpha^2 + \beta^2}), \tag{22}$$

where the coefficients $C_{1,n}$ and $C_{2,n}$ are determined by the formulas

$$C_{1,n} = 1 + \sum_{k=0}^{n-1} \frac{1}{1 - g_k} \prod_{m=0}^{k} \eta_m, \qquad C_{2,n} = \sum_{k=0}^{n-1} \prod_{m=0}^{k} \eta_m.$$

Proof. Since the function $\tilde{\varepsilon}_n^{(f)}(\alpha, \beta)$ is continuously differentiable in the neighborhood of the point $(0, 0)$ for $\alpha \geq 0$, $\beta \geq 0$ $(\alpha + \beta \neq 0)$ and $\tilde{\varepsilon}_n^{(f)}(0, 0) = 0$, it can be written as

$$\tilde{\varepsilon}_n^{(f)}(\alpha, \beta) = \frac{\partial \tilde{\varepsilon}_n^{(f)}}{\partial \alpha}(0, 0)\alpha + \frac{\partial \tilde{\varepsilon}_n^{(f)}}{\partial \beta}(0, 0)\beta + o(\sqrt{\alpha^2 + \beta^2}).$$

Let us find the values of the partial derivatives $\frac{\partial \tilde{\varepsilon}_n^{(f)}}{\partial \alpha}(0, 0)$ and $\frac{\partial \tilde{\varepsilon}_n^{(f)}}{\partial \beta}(0, 0)$.

According to formula (16), the majorant of the relative error $\tilde{\varepsilon}_n^{(f)}$ can be expressed as

$$\tilde{\varepsilon}_n^{(f)}(\alpha, \beta) = -1 + \frac{1 + \alpha}{\tilde{Q}_0^{(n)}(\alpha, \beta)},$$

where the tails $\tilde{Q}_k^{(n)}$ are defined recursively by

$$\tilde{Q}_k^{(n)}(\alpha, \beta) = 1 + \eta_k - \frac{\eta_k(1 + \alpha)(1 + \frac{g_k}{1 - g_k}\alpha)(1 + \beta)}{\tilde{Q}_{k+1}^{(n)}(\alpha, \beta)}, \quad 0 \leq k \leq n - 1, \quad \tilde{Q}_n^{(n)} = 1. \tag{23}$$

Then

$$\frac{\partial \tilde{\varepsilon}_n^{(f)}}{\partial \alpha} = \frac{\tilde{Q}_0^{(n)}(\alpha, \beta) - (1 + \alpha)\frac{\partial \tilde{Q}_0^{(n)}}{\partial \alpha}}{\left(\tilde{Q}_0^{(n)}(\alpha, \beta)\right)^2}, \qquad \frac{\partial \tilde{\varepsilon}_n^{(f)}}{\partial \beta} = -\frac{(1 + \alpha)\frac{\partial \tilde{Q}_0^{(n)}}{\partial \beta}}{\left(\tilde{Q}_0^{(n)}(\alpha, \beta)\right)^2}.$$

Given that $\tilde{Q}_k^{(n)}(0, 0) = 1$ for all $0 \leq k \leq n$, we obtain

$$C_{1,n} = \frac{\partial \tilde{\varepsilon}_n^{(f)}}{\partial \alpha}(0, 0) = 1 - \frac{\partial \tilde{Q}_0^{(n)}}{\partial \alpha}(0, 0), \quad C_{2,n} = \frac{\partial \tilde{\varepsilon}_n^{(f)}}{\partial \beta}(0, 0) = -\frac{\partial \tilde{Q}_0^{(n)}}{\partial \beta}(0, 0).$$

To find the derivatives of the tails $\tilde{Q}_k^{(n)}$, we use the method of mathematical induction. Let $q_{k,n} = \frac{\partial \tilde{Q}_k^{(n)}}{\partial \alpha}(0, 0)$. We shall prove that

$$q_{k,n} = -\sum_{j=k}^{n-1} \frac{1}{1 - g_j} \prod_{m=k}^{j} \eta_m, \quad 0 \leq k \leq n - 1. \tag{24}$$

For $k = n-1$, we have $\tilde{Q}_{n-1}^{(n)} = 1 + \eta_{n-1} - \eta_{n-1}(1+\alpha)(1+\frac{g_{n-1}}{1-g_{n-1}}\alpha)(1+\beta)$. Then

$$q_{n-1,n} = -\eta_{n-1}\Big(1 + \frac{g_{n-1}}{1-g_{n-1}}\Big) = -\frac{\eta_{n-1}}{1-g_{n-1}}.$$

Assuming the validity of formula (24) for $k = p+1$, for $k = p$ from the recurrence relation (23), we have

$$q_{p,n} = -\eta_p\Big(\frac{1}{1-g_p} - q_{p+1,n}\Big) = -\frac{\eta_p}{1-g_p} + \eta_p q_{p+1,n} =$$
$$= -\frac{\eta_p}{1-g_p} - \eta_p \sum_{j=p+1}^{n-1} \frac{1}{1-g_j} \prod_{m=p+1}^{j} \eta_m = -\sum_{j=p}^{n-1} \frac{1}{1-g_j} \prod_{m=p}^{j} \eta_m.$$

For $k = 0$, we obtain $C_{1,n} = 1 - q_{0,n} = 1 + \sum_{k=0}^{n-1} \frac{1}{1-g_k} \prod_{m=0}^{k} \eta_m$.

Let $r_{k,n} = \frac{\partial \tilde{Q}_k^{(n)}}{\partial \beta}(0,0)$. Similarly to the previous case, it can be shown by induction that

$$r_{k,n} = -\sum_{j=k}^{n-1} \prod_{m=k}^{j} \eta_m, \quad 0 \le k \le n-1.$$

Thus, for $k = 0$, we have $C_{2,n} = -r_{0,n} = \sum_{k=0}^{n-1} \prod_{m=0}^{k} \eta_m$. □

Corollary 2. *The periodic g–fraction*

$$\cfrac{s_0}{1 + \cfrac{g(1-g)z}{1 + \cfrac{g(1-g)z}{1 + \ddots}}}, \tag{25}$$

where $s_0 > 0$, $0 < g < 1$, is stable to perturbations if there exist constants α, β, $0 \le \alpha < 1$, $0 \le \beta < 1$, $\alpha + \beta \ne 0$, such that the relative errors $\varepsilon^{(s)}$, $\varepsilon^{(g)}$, $\varepsilon^{(z)}$ of its coefficients s_0, g and the variable z satisfy conditions

$$|\varepsilon^{(s)}| \le \alpha, \quad |\varepsilon^{(g)}| \le \alpha, \quad |\varepsilon^{(z)}| \le \beta, \tag{26}$$

and there exists a real constant $0 < \eta < 1$ such that

$$|1 - 1/Q_k^{(n)}(z)| \le \eta, \quad 0 \le k \le n-1, \quad n \ge 1,$$

where the tails $Q_k^{(n)}(z)$ are defined according to (6) and

$$4\eta(1+\alpha)(1+\alpha g/(1-g))(1+\beta) < (1+\eta)^2. \tag{27}$$

Moreover, for the relative errors of the approximants of the g–fraction (25), the estimate (22) holds, where

$$C_{1,n} = 1 + \frac{\eta(1-\eta^n)}{(1-g)(1-\eta)}, \quad C_{2,n} = \frac{\eta(1-\eta^n)}{1-\eta}. \tag{28}$$

Proof. Dividing both sides of inequality (27) by $(1+\eta)^2$, we obtain

$$\frac{4\eta(1+\alpha)(1+\alpha g/(1-g))(1+\beta)}{(1+\eta)^2} < 1. \tag{29}$$

Let us define the quantity μ^* as

$$\mu^* = \sqrt{1 - \frac{4\eta(1+\alpha)(1+\alpha g/(1-g))(1+\beta)}{(1+\eta)^2}}. \tag{30}$$

It follows from inequality (29) that $\mu^* \in \mathbb{R}$ and $\mu^* > 0$. Furthermore, since the conditions of the corollary specify $\alpha \geq 0$, $\beta \geq 0$, and $\alpha+\beta \neq 0$, it follows that $(1+\alpha)(1+\alpha g/(1-g))\times$ $\times(1+\beta) > 1$. Then

$$\mu^* < \sqrt{1 - \frac{4\eta}{(1+\eta)^2}} = \frac{1-\eta}{1+\eta}. \tag{31}$$

We choose the constant μ such that $0 < \mu \leq \mu^*$. From (31), it follows that $\mu < (1-\eta)/(1+\eta)$, which is equivalent to condition (12)

$$\eta + \mu + \eta\mu < 1.$$

From the condition $\mu \leq \mu^*$ and formula (30), by means of algebraic transformations, we obtain inequality (14)

$$\frac{4\eta(1+\alpha)(1+\alpha g/(1-g))(1+\beta)}{(1+\eta)^2} \leq (1-\mu^2).$$

Thus, for the chosen value of μ, conditions (12) and (14) of Theorem 1 are satisfied.

The coefficients $C_{1,n}$ and $C_{2,n}$ in the case of $g_k = g$ and $\eta_k = \eta$ are computed as partial sums of a geometric series with common ratio $0 < \eta < 1$. □

Remark 1. Inequality (27) has a non-trivial solution $\alpha \geq 0, \beta \geq 0, \alpha+\beta \neq 0$ if and only if $\frac{(1+\eta)^2}{4\eta} > 1$. For the quantity $\frac{(1+\eta)^2}{4\eta}$ with $\eta \in (0,1)$, the following estimate holds:

$$\frac{(1+\eta)^2}{4\eta} = 1 + \frac{(1-\eta)^2}{4\eta} > 1,$$

which guarantees the existence of constants $\alpha \geq 0$, $\beta \geq 0$, $\alpha+\beta \neq 0$, and, consequently, a non-empty set of admissible perturbations for the coefficients s_0, g, and/or the variable z.

Let us consider the continued fraction

$$\cfrac{a_0}{1+\cfrac{a_1}{1+\cfrac{a_2}{1+\ddots}}} \tag{32}$$

Definition 2. A sequence $\{W_k\}_{k\in\mathbb{N}\cup\{0\}}$, $\varnothing \neq W_k \subset \mathbb{C}$, $k \geq 0$, is called a *sequence of value sets for the tails $Q_k^{(n)}$ of the approximant of the continued fraction* (32) if $1 \in W_k$, $k \geq 0$, for every $w \in W_k$,

$$1 + a_k/w \in W_{k-1}, \quad k \geq 1. \tag{33}$$

Theorem 2. *The g–fraction* (2) *is stable to perturbations at the point* $z = re^{2i\varphi}$, $r \geq 0$, $-\pi/2 < \varphi < \pi/2$, *if the relative errors of its coefficients* s_0, g_k, $k \geq 0$, *and the variable* z *satisfy conditions (11), there exists a sequence of positive constants* $\{\mu_k\}_{k\in\mathbb{N}\cup\{0\}}$, $0 \leq \mu_k < 1$, *such that for* $k \geq 0$

$$g_{k+1}(1-g_k)r(1+\mu_k) \leq (1+g_{k+1}(1-g_k)r)\cos\varphi(1-\mu_k), \tag{34}$$

$$4g_{k+1}(1-g_k)r(1+g_{k+2}(1-g_{k+1})r)\cos\varphi(1+\alpha)(1+\alpha g_k/(1-g_k))(1+\beta) \leq (\cos\varphi+ \\ +g_{k+1}(1-g_k)r(1+\cos\varphi))(\cos\varphi+g_{k+2}(1-g_{k+1})r(1+\cos\varphi))(1-\mu_k)(1+\mu_{k+1}), \tag{35}$$

and the series $\sum_{k=0}^{\infty}\mu_k$ *diverges. Moreover, for the relative errors of the approximants of the* g*–fraction* (2), *the estimate* (15) *holds, where*

$$\eta_k = \frac{g_{k+1}(1-g_k)r}{(1+g_{k+1}(1-g_k)r)\cos\varphi}, \quad k \geq 0. \tag{36}$$

Proof. The proof is based on the method of value sets for the approximant tails. The half-plane $W_\varphi = \{w \in \mathbb{C}\colon \operatorname{Re}((w-1)e^{-i\varphi}) \geq 0\}$, $-\pi/2 < \varphi < \pi/2$, is shown to be a value set for the tails of the approximant of the g–fraction at a given point z in the complex plane. By means of the properties of linear fractional transformations, upper bounds for the quantities $|1-1/Q_k^{(n)}|$ are derived, which enable the explicit determination of the sequence $\{\eta_k\}$. It is then shown that conditions (34) and (35) guarantee the fulfillment of the requirements of Theorem 1, ensuring the stability to perturbations of the g–fraction at the point z.

We will prove that the set W_φ is a set of tails $Q_k^{(n)}$ of the approximant of the continued fraction (32), where $a_0 = s_0$, $a_k = g_k(1-g_{k-1})re^{2i\varphi}$, $k \geq 1$. The set W_φ is a closed half-plane whose boundary $\partial W_\varphi = \{w \in \mathbb{C}\colon \operatorname{Re}((w-1)e^{-i\varphi}) = 0\}$ contains the point $w = 1$, which ensures that condition $1 \in W_k$, $k \geq 0$, is satisfied. Since $-\pi/2 < \varphi < \pi/2$, then $\cos\varphi > 0$ and $0 \notin W_\varphi$. Then the inversion $1/w$ maps the set W_φ into the disk

$$1/W_\varphi = \{w \in \mathbb{C}\colon |w - e^{-i\varphi}/(2\cos\varphi)| \leq 1/(2\cos\varphi)\}.$$

Then

$$V_{k-1} = 1 + a_k/W_\varphi = \\ = \{w \in \mathbb{C}\colon |w - (1+g_k(1-g_{k-1})re^{i\varphi}/(2\cos\varphi))| \leq g_k(1-g_{k-1})r/(2\cos\varphi)\}. \tag{37}$$

The set $V_{k-1} \subset W_\varphi$ if the following conditions are met:

$$c_{k-1} \in W_\varphi, \tag{38}$$

$$\min\{|c_{k-1}-w|, w \in \partial W_\varphi\} \geq g_k(1-g_{k-1})r/(2\cos\varphi), \tag{39}$$

where $c_{k-1} = 1 + g_k(1-g_{k-1})re^{i\varphi}/(2\cos\varphi)$ is the center of the disk (37). Since $\cos\varphi > 0$, then $\operatorname{Re}((c_{k-1}-1)e^{-i\varphi}) = g_k(1-g_{k-1})r/(2\cos\varphi) \geq 0$, which proves that (38) holds. Since $\min\{|c_{k-1}-w|, w \in \partial W_\varphi\} = g_k(1-g_{k-1})r/(2\cos\varphi)$, the set is tangent to the boundary ∂W_φ, which proves that inequality (39) holds.

Let us estimate the quantities $1-1/Q_{k-1}^{(n)}$, $1 \leq k \leq n$, from above, by first transforming them

$$1-1/Q_{k-1}^{(n)} = a_k/(Q_{k-1}^{(n)}Q_k^{(n)}) = g_k(1-g_{k-1})re^{2i\varphi}/(Q_{k-1}^{(n)}Q_k^{(n)}) = \\ = \frac{g_k(1-g_{k-1})re^{2i\varphi}/Q_k^{(n)}}{1+g_k(1-g_{k-1})re^{2i\varphi}/Q_k^{(n)}} = 1 - \frac{1}{1+g_k(1-g_{k-1})re^{2i\varphi}/Q_k^{(n)}}, \quad 1 \leq k \leq n.$$

Since $Q_k^{(n)} \in W_\varphi$, $0 \leq k \leq n$, then

$$|1-1/Q_{k-1}^{(n)}| \leq \max\{|1-1/(1+g_k(1-g_{k-1})re^{2i\varphi}/w)|, w \in W_\varphi\} = \max\{|1-1/w|, w \in V_{k-1}\}.$$

From the inequality $|c_{k-1}| \geq g_k(1-g_{k-1})r/(2\cos\varphi)$, it follows that $0 \notin V_{k-1}$. Then the inversion $1/w$ maps the disk V_{k-1} into the disk

$$1/V_{k-1} = \{w \in \mathbb{C}\colon |w - (1 + g_k(1-g_{k-1})re^{-i\varphi}/(2\cos\varphi))/(1 + g_k(1-g_{k-1})r)| \leq \\ \leq g_k(1-g_{k-1})r/(2\cos\varphi(1 + g_k(1-g_{k-1})r))\},$$

and the set $1 - 1/V_{k-1}$ has the form

$$1 - 1/V_{k-1} = \{w \in \mathbb{C}\colon |w - g_k(1-g_{k-1})r(1 - e^{-i\varphi}/(2\cos\varphi))/(1 + g_k(1-g_{k-1})r)| \leq \\ \leq g_k(1-g_{k-1})r/(2\cos\varphi(1 + g_k(1-g_{k-1})r))\}.$$

Then

$$\max\{|1 - 1/w|, w \in V_{k-1}\} = |g_k(1-g_{k-1})r(1 - e^{-i\varphi}/(2\cos\varphi))/(1 + g_k(1-g_{k-1})r)| + \\ + g_k(1-g_{k-1})r/(2\cos\varphi(1 + g_k(1-g_{k-1})r)) = g_k(1-g_{k-1})r/((1 + g_k(1-g_{k-1})r)\cos\varphi),$$

and $|1 - 1/Q_{k-1}^{(n)}| \leq g_k(1-g_{k-1})r/((1 + g_k(1-g_{k-1})r)\cos\varphi)$, $1 \leq k \leq n$.

Let the quantities η_k, $k \geq 0$, be defined according to (36). Then condition (34) is equivalent to condition (12). From conditions (1), (34) and the inequalities $-\pi/2 < \varphi < \pi/2$, it follows that $0 < \eta_k \leq (1-\mu_k)/(1+\mu_k) \leq 1$, $k \geq 0$. If for the relative errors $\varepsilon_0^{(s)}$, $\varepsilon_k^{(g)}$, $k \geq 0$, $\varepsilon^{(z)}$ inequalities (11) hold, and for the quantities α, β, r, φ, g_k, μ_k, $k \geq 0$ condition (35) is met, which is equivalent to condition (14), then, according to Theorem 1, the g–fraction (2) is stable to perturbations at the point $z = re^{2i\varphi}$, if the series $\sum_{k=0}^{\infty} \mu_k$ diverges. Moreover, for the relative errors of its approximants, the estimate (15) holds, where the quantities η_k, $k \geq 0$, are defined according to (36). □

From Theorem 2 and Corollary 2, for the periodic g–fraction (25) it follows that

Corollary 3. *The periodic g–fraction* (25) *is stable to perturbations at the point $z = re^{2i\varphi}$, $r \geq 0$, $-\pi/2 < \varphi < \pi/2$, if the relative errors of its coefficients s_0, g, and the variable z satisfy conditions* (26), *and*

$$\frac{g(1-g)r}{1 + g(1-g)r} < \cos\varphi, \tag{40}$$

$$4g(1-g)r(1 + g(1-g)r)\cos\varphi(1+\alpha)(1 + \alpha g/(1-g))(1+\beta) < \\ < (\cos\varphi + g(1-g)r(1 + \cos\varphi))^2, \tag{41}$$

Moreover, for the relative errors of the approximants of the g–fraction (25), *the estimate* (22) *holds, where $C_{1,n}$, $C_{2,n}$ are defined by* (28), *and* $\eta = \frac{g(1-g)r}{(1+g(1-g)r)\cos\varphi}$.

Example 1. Let us describe the following numerical experiment we conducted.Consider the periodic g–fraction (25) with $g = 1/2$

$$\cfrac{s_0}{1 + \cfrac{x/4}{1 + \cfrac{x/4}{1 + \ddots}}}, \tag{42}$$

where $x \in (0, +\infty)$. According to Corollary 3, the g–fraction (42) is stable to perturbations at the point x if

$$(1+\alpha)^2(1+\beta) < 1 + \frac{4}{4x + x^2}, \tag{43}$$

where α and β are constants that bound the relative errors of the coefficients s_0, $g = 1/2$, and the variable x, and for the relative errors of its approximants the estimate (22) holds, where $C_{1,n}$, $C_{2,n}$ are defined by (28), and $\eta = x/(4+x)$.

It is worth noting that as $x \to 0^+$, the right-hand side of inequality (43), given by $1 + \frac{4}{4x+x^2}$, tends to $+\infty$. Consequently, the bounds α and β for the admissible relative errors of the coefficients and the variable of the g–fraction (42) can also tend to infinity. Conversely, as $x \to +\infty$, the right-hand side of inequality (43) tends to 1. Since the left-hand side of (43) is strictly greater than 1 for any $\alpha \geq 0, \beta \geq 0$ such that $\alpha + \beta \neq 0$, the stability conditions for the g–fraction (42) necessitate that $\alpha \to 0^+$ and $\beta \to 0^+$ as x tends to $+\infty$.

To illustrate the theoretical result, a numerical experiment was conducted to investigate the relative error of the 50th approximant of the periodic g–fraction (42) to perturbations of its coefficients $s_0 = 1$, $g = 1/2$, and the variable $x \in [10^{-8}, 1]$ with relative errors $\varepsilon_0^{(s)} = \varepsilon^{(g)} = \varepsilon^{(z)} = 10^{-2}$. The experimental results demonstrated that the actual relative error of the approximant is majorized by both the nonlinear majorant $\hat{\varepsilon}_n^{(f)}(\alpha, \beta)$ and its linear approximation $C_{1,n}\alpha + C_{2,n}\beta$ across the entire studied range of the variable $x \in [10^{-8}, 1]$. At small values of x, all curves attain values close to 10^{-2}, indicating that in this case, the approximant error is determined by the perturbations of the coefficients and the variable, while the influence of the fraction structure is negligible. As x increases, an increasing trend is observed in both the theoretical majorants, which approach a value of $1.8 \cdot 10^{-2}$ as $x \to 1$, and the actual relative error of the approximant, although the latter grows much more slowly. This confirms that the relative error of the approximant depends not only on the magnitude of the input perturbations but also on the values of the fraction's coefficients and variable. The proximity of the nonlinear majorant to its linear part suggests that the linear approximation is sufficiently accurate and can be effectively used for error estimation without computing the nonlinear majorant.

4. Stability of the g–fraction in the closed unit disk. Let $\{E_k\}_{k\in\mathbb{N}\cup\{0\}}$, $\varnothing \neq E_k \subset \mathbb{C}$, $k \geq 0$, be a sequence of element sets of the continued fraction (32), i.e., $a_k \in E_k$, $k \geq 0$.

Definition 3. A sequence $\{W_k\}_{k\in\mathbb{N}\cup\{0\}}$, $\varnothing \neq W_k \subset \mathbb{C}$, $k \geq 0$, is called a *sequence of value sets for the tails $Q_k^{(n)}$ of the approximant of the continued fraction* (32), which corresponds to the sequence of element sets $\{E_k\}_{k\in\mathbb{N}\cup\{0\}}$, if condition $1 \in W_k$, $k \geq 0$, is satisfied and for every $a_k \in E_k$, and for every $w \in W_k$, condition (33) is satisfied.

Theorem 3. *The set $\Omega = \{w \in \mathbb{C}\colon |w| \leq 1\}$ is a set of stability for the g–fraction* (2) *if the relative errors of its coefficients s_0, g_k, $k \geq 0$, and variable z satisfy conditions* (11), *and there exists a sequence of positive constants $\{\mu_k\}_{k\in\mathbb{N}\cup\{0\}}$, $0 \leq \mu_k < 1$, such that*

$$g_k \geq (1+\mu_k)/2, \quad k \geq 0, \tag{44}$$

$$4g_{k+1}(1-g_k)(1+\alpha)(1+\alpha g_k/(1-g_k))(1+\beta) \leq (1-\mu_k)(1+\mu_{k+1}), \quad k \geq 0, \tag{45}$$

and the series $\sum_{k=0}^{\infty} \mu_k$ diverges. Under these conditions, the relative errors of the approximants of the g–fraction (2) *satisfy the estimate* (15), *where $\eta_k = (1-g_k)/g_k$, $k \geq 0$.*

Proof. The proof is based on the method of element sets and value sets for the approximant tails. The collection of sets

$$W_k = \{w \in \mathbb{C}\colon |w| \geq g_k\}, \quad k \geq 0, \tag{46}$$

is shown to be a sequence of value sets for the tails of the approximant of the continued fraction (32), corresponding to the sequence of element sets

$$E_0 = \mathbb{R}_+, \quad E_k = \{w \in \mathbb{C} \colon |w| \le g_k(1 - g_{k-1})\}, \quad k \ge 1. \tag{47}$$

From the inclusion $Q_k^{(n)} \in W_k$, upper bounds for the quantities $|1-1/Q_k^{(n)}|$ are derived, which determine the sequence $\{\eta_k\}$ via $\eta_k = (1-g_k)/g_k$. It is then verified that conditions (44) and (45) guarantee the fulfillment of the requirements of Theorem 1 for all $z \in \Omega$, establishing the closed unit disk as a stability set for the g–fraction (2).

If the coefficients of the g–fraction (2) satisfy conditions (1) and $z \in \Omega = \{w \in \mathbb{C} \colon |w| \le 1\}$, then $|g_k(1 - g_{k-1})z| \le g_k(1 - g_{k-1})$, $k \ge 1$. Let z be a fixed point of the set Ω. Let us denote $a_0 = s_0$, $a_k = g_k(1 - g_{k-1})z$, $k \ge 1$. Then the collection of sets (47), is a sequence of element sets of the continued fraction (32).

Let us prove that the collection of sets (46) is a sequence of value sets for the approximant tails of the continued fraction (32), which corresponds to the sequence of element sets (47).

From inequalities (1), the fulfillment of condition $1 \in W_k$, $k \ge 0$, follows. Let us prove that if $a_k \in E_k$, $k \ge 0$, then condition (33) is satisfied.

Since $g_k > 0$ for $k \ge 0$, then $0 \notin W_k$ and the function $t(w) = 1/w$ maps the set W_k to the set $1/W_k = \{w \in \mathbb{C} \colon |w| \le 1/g_k\}$. Then

$$V_k = 1 + a_k/W_k = \{w \in \mathbb{C} \colon |w - 1| \le |a_k|/g_k\} \tag{48}$$

and $V_k \subset W_{k-1}$ if $|a_k|/g_k + g_{k-1} \le 1$. The latter inequality is equivalent to the inequality which defines the sets (47).

Since $Q_k^{(n)} \in W_k$, $0 \le k \le n$, then $|Q_k^{(n)}| \ge g_k$, $0 \le k \le n$, and for the quantities $1-1/Q_k^{(n)}$, the estimates $|1-1/Q_k^{(n)}| = |g_{k+1}(1-g_k)z/(Q_k^{(n)}Q_{k+1}^{(n)})| \le g_{k+1}(1-g_k)/(g_k g_{k+1}) = (1-g_k)/g_k$, $0 \le k \le n-1$, hold.

Let $\eta_k = (1 - g_k)/g_k$, $k \ge 0$. Then condition (44) is equivalent to condition (12). From conditions (1), (44), it follows that $0 < \eta_k \le (1 - \mu_k)/(1 + \mu_k) \le 1$, $k \ge 0$. If for the relative errors $\varepsilon_0^{(s)}$, $\varepsilon_k^{(g)}$, $k \ge 0$, $\varepsilon^{(z)}$ inequalities (11) hold, and for the quantities $\alpha, \beta, g_k, \mu_k, k \ge 0$ condition (45) is met, which is equivalent to condition (14), then according to Theorem 1, the set $\Omega = \{w \in \mathbb{C} \colon |w| \le 1\}$ is a set of stability for the g–fraction (2). Moreover, by substituting $\eta_k = (1 - g_k)/g_k$ into (16), we obtain an estimate for the relative errors of its approximants. □

Corollary 4. *The periodic g–fraction* (25) *with $1/2 < g < 1$, is stable to perturbations for all $z \in \Omega$, where $\Omega = \{w \in \mathbb{C} \colon |w| \le 1\}$, if the relative errors of its coefficients s_0, g, and the variable z satisfy conditions* (26)*, and*

$$4g(1 - g)(1 + \alpha)(1 + \alpha g/(1 - g))(1 + \beta) < 1. \tag{49}$$

Moreover, for the relative errors of the approximants of the g–fraction (25)*, the estimate* (22) *holds, where* $C_{1,n} = \frac{2g^{n+1}-(1-g)^n}{g^n(2g-1)}$, $C_{2,n} = \frac{(1-g)(g^n-(1-g)^n)}{g^n(2g-1)}$.

Proof. Setting $g_k = g$ for all $k \ge 0$ in Theorem 3, we obtain $\eta_k = \eta = (1 - g)/g$ for all $k \ge 0$. Define $\mu^* = \sqrt{1 - 4g(1 - g)(1 + \alpha)(1 + \alpha g/(1 - g))(1 + \beta)}$. From (49) it follows that $\mu^* \in \mathbb{R}$ and $\mu^* > 0$. Furthermore, since $\alpha \ge 0$, $\beta \ge 0$ with $\alpha + \beta \ne 0$, it follows that $(1+\alpha)(1+\alpha g/(1-g))(1+\beta) > 1$, and therefore $\mu^* < \sqrt{1 - 4g(1 - g)} = 2g - 1$. We choose μ such that $0 < \mu \le \mu^*$. From the inequality $\mu < 2g - 1$ it follows that condition (44) is satisfied. From the condition $\mu \le \mu^*$, we obtain condition (45). Thus, for the chosen value of μ, conditions (44) and (45) of Theorem 3 are satisfied with constant sequences $\mu_k = \mu$ and $\eta_k = \eta$. Therefore, by Theorem 3, the g–fraction (2) is stable to perturbations for all $z \in \Omega$.

The coefficients $C_{1,n}$ and $C_{2,n}$ in the case $\eta = (1 - g)/g$ are computed as partial sums of a geometric series with common ratio $0 < (1 - g)/g < 1$. □

5. Conclusions. In this work, a recurrence formula for the relative error of the approximant tails of a g–fraction has been established. Based on this, a representation of the relative error of the approximant in the form of a finite continued fraction is obtained, whose coefficients depend on the exact values of the fraction's elements and their relative errors. Sufficient conditions for the stability of the g–fraction have been established, which link the stability of the fraction to the behavior of the majorant of the relative error of the approximant. Using the method of value sets for the approximant tails, the general stability conditions (Theorem 1) were applied to find constraints on the coefficients of the g–fraction that ensure stability at a specific point in the complex plane (Theorem 2), and stability in the unit disk (Theorem 3) was proven. These results have significant practical importance as they define the constraints on the coefficients of the fraction and its variable that ensure stability to perturbations.

The results of this work provide a mathematical toolkit for a priori estimation of the reliability of models described by g–fractions, and for constraining its coefficients and variable, from which the model's behavior is predictable. Specifically, the results can be used for controlling the accuracy of calculations in the development and study of algorithms that use continued fraction approximations. The results can be extended to other classes of continued fractions and to multivariate generalizations, which is a promising direction for further research.

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Lviv Polytechnic National University
Lviv, Ukraine
volodymyr.r.hladun@lpnu.ua

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