Математичні Студії. Т.65, №2 Matematychni Studii. V.65, No.2
ISSN 1027-4634 (print) ISSN 2411-0620 (online)

A. I. BANDURA*, YA. V. BATSALA

GROWTH ESTIMATES FOR ANALYTIC FUNCTIONS IN THE UNIT POLYDISC WITH BOUNDED L-INDEX IN DIRECTION

A. I. Bandura, Ya. V. Batsala. *Growth estimates for analytic functions in the unit polydisc with bounded L-index in direction*, Mat. Stud. **65** (2026), 148–156.

This paper investigates the growth properties of analytic functions defined in the unit polydisc of $\mathbb{C}^n$ having bounded L-index in a direction. Special attention is devoted to the behavior of such functions under composition and to the relationships between their growth characteristics and the corresponding properties of their components. We establish new estimates for the maximum modulus and propose sufficient conditions under which the composition preserves given growth classes. The obtained results extend several known one-dimensional theorems to the multidimensional setting and refine existing bounds in the theory of entire and analytic functions. Furthermore, we analyze the interplay between the geometry of the polydisc and the growth indicators, revealing specific features that arise in higher dimensions. At the end, we consider one equation from the electromagnetism's theory written by directional derivative and study its analytic solutions.

1. Introduction. Let $\mathbf{0} = (0, \dots, 0)$, $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ be a given direction, $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{D}^n = \{z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n\colon |z_j| < 1, j \in \{1, 2, \dots, n\}\}$ be the unit polydisc, $L\colon \mathbb{D}^n \to \mathbb{R}_+$ be a continuous function such that for all $z = (z_1, z_2, \dots, z_n) \in \mathbb{D}^n$

$$L(z) > \beta \max_{1\le j\le n} \frac{|b_j|}{1-|z_j|}, \quad \beta = \text{const} > 1. \tag{1}$$

In recent papers, there was introduced ([1,2]) a notion of boundedness of L-index in direction for the class of analytic functions in the unit polydisc. These researches indicated necessary and sufficient conditions of belonging function to this class. This class is characterized by some directional logartihmic derivative estimates and some uniform distribution of zeros at the slice in the direction, and some regular local behavior.

An analytic function $F\colon \mathbb{D}^n \to \mathbb{C}$ is called a function of *bounded L-index in a direction* $\mathbf{b}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{D}^n$ the following inequality is valid

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m! L^m(z)} \le \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{k! L^k(z)} \colon\ 0 \le k \le m_0 \right\}, \tag{2}$$

where

$$\partial_{\mathbf{b}}^0 F(z) = F(z), \partial_{\mathbf{b}} F(z) = \sum_{j=1}^n \frac{\partial F(z)}{\partial z_j} b_j,\ \partial_{\mathbf{b}}^k F(z) = \partial_{\mathbf{b}}(\partial_{\mathbf{b}}^{k-1} F(z)),\ k \ge 2.$$

2020 *Mathematics Subject Classification:* 32A10, 32A17, 58C10.
Keywords: analytic function; bounded L-index in direction; maximum modulus; growth estimates; directional differential equation; directional wave equation; electromagnetism theory.
doi:10.30970/ms.65.2.148-156
*Corresponding author: A. I. Bandura

The definition was firstly introduced for entire functions of single variable in the classic paper of B. Lepson ([3]) if $L \equiv 1$, $\mathbf{b} = 1$, $n = 1$. The least integer $m_0 = m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b}$ *of an analytic function* F *and is denoted by* $N_{\mathbf{b}}(F, L, \mathbb{D}^n) = m_0$.

For a given $z \in \mathbb{D}^n$ we denote $D_z = \{t \in \mathbb{C}\colon z + t\mathbf{b} \in \mathbb{D}^n\}$. In other words, $D_z = \big\{t \in \mathbb{C}\colon |t| < \min\limits_{1\le j\le n} \frac{1-|z_j|}{|b_j|}\big\}$. Here, if $b_j = 0$ then we suppose $\frac{1-|z_j|}{|b_j|} = +\infty$. Denote

$$\lambda_{\mathbf{b}}(\eta) = \sup_{z\in\mathbb{D}^n} \sup_{t_1,t_2\in D_z} \left\{ \frac{L(z+t_1\mathbf{b})}{L(z+t_2\mathbf{b})}\colon\ |t_1 - t_2| \le \frac{\eta}{\min\{L(z+t_1\mathbf{b}), L(z+t_2\mathbf{b})\}} \right\}.$$

The notation $Q_{\mathbf{b}}(\mathbb{D}^n)$ stands for a class of positive continuous functions $L\colon \mathbb{D}^n \to \mathbb{R}_+$, satisfying (1) and ($\forall \eta \in [0, \beta]$) one has $\lambda_{\mathbf{b}}(\eta) < +\infty$.

2. Growth of analytic functions in $\mathbb{D}^n$ of bounded L-index in the direction. We denote $a^+ = \max\{a, 0\}$. Set $u(r) = u(z^0, \theta, r) = L(z^0 + re^{i\theta}\mathbf{b})$. The growth estimates of entire function of bounded l-index were obtained by A. Kuzyk and M. Sheremeta in [4]. They applied main results of Wiman-Valiron's theory to deduce these results for entire functions of one variable. But we do not know direct simple analogs of Wiman-Valiron's theory for analytic function in the unit polydisc. The known analogs assume additional restrictions by the analytic functions. Therefore, we used other condition by the functions L.

Theorem 1. *Let $L\colon \mathbb{D}^n \to \mathbb{R}_+$ be a continuous function satisfying* (1) *and for every $z^0 \in \mathbb{D}^n$ and every $\theta \in [0, 2\pi]$ the function $u(r, z^0, \theta)$ be a continuously differentiable function of real variable $r \in [0, R)$, where $R = R(z^0, \theta) = \min\{R \in \mathbb{R}_+\colon\ |z_j^0 + Re^{i\theta} b_j| = 1,\ j \in \{1, \ldots, n\}\}$. If an analytic function $F\colon \mathbb{D}^n \to \mathbb{C}$ is of bounded L-index in the direction $\mathbf{b} \in \mathbb{C}^n \setminus \{0\}$ then* $\forall z^0 \in \mathbb{D}^n$, $\forall \theta \in [0, 2\pi]$, $\forall r \in [0, R)$ $\forall p \in \mathbb{Z}_+$ *one has*

$$\ln \max_{|t|=r} \left(\frac{|\partial_{\mathbf{b}}^p F(z^0+t\mathbf{b})|}{p! L^p(z^0+t\mathbf{b})} \right) \le \ln\max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z^0)|}{k! L^k(z^0)}\colon 0 \le k \le N \right\} +$$
$$+ \max_{\theta\in[0,2\pi]} \int_0^r \left\{ (N+1) L(z^0 + te^{i\theta}\mathbf{b}) + N \frac{(-u_t'(z^0,\theta,t))^+}{L(z^0+te^{i\theta}\mathbf{b})} \right\} dt \tag{3}$$

If, in addition, for every $z^0 \in \mathbb{D}^n$, $\theta \in [0, 2\pi]$ one has $\left(-u_r'(z^0, \theta, r)\right)^+ / (L^2(z^0 + re^{i\theta}\mathbf{b})) \to 0$ as $\max\limits_{1\le j\le n} |z_j^0 + re^{i\theta} b_j| \to 1$ then for every $z^0 \in \mathbb{D}^n$ and $\theta \in [0, 2\pi]$ one has

$$\varlimsup_{\max\limits_{1\le j\le n} |z_j^0 + re^{i\theta}b_j| \to 1} \frac{\ln |F(z^0 + re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0 + te^{i\theta}\mathbf{b}) dt} \le N_{\mathbf{b}}(F, L, \mathbb{D}^n) + 1. \tag{4}$$

Proof. Our proof is similar to proof of one-dimensional result ([4]) for entire functions.

Denote $N = N_{\mathbf{b}}(F, L, \mathbb{D}^n)$. For fixed $z^0 \in \mathbb{D}^n$ and $\theta \in [0, 2\pi]$ we consider the function

$$g(r) = \max \left\{ \frac{|\partial_{\mathbf{b}}^k F(z^0 + re^{i\theta}\mathbf{b})|}{k! L^k(z^0 + re^{i\theta}\mathbf{b})}\colon 0 \le k \le N \right\}. \tag{5}$$

Since the function $\frac{|\partial_{\mathbf{b}}^k F(z^0+re^{i\theta}\mathbf{b})|}{k! L^k(z^0+re^{i\theta}\mathbf{b})}$ is continuously differentiable by real $r \in [0, R)$, the function g is continuously differentiable on $[0, R)$, apart from, possibly, a countable set and

$$g'(r) \le \max \left\{ \frac{d}{dr} \left(\frac{|\partial_{\mathbf{b}}^k F(z^0+re^{i\theta}\mathbf{b})|}{k! L^k(z^0+re^{i\theta}\mathbf{b})} \right) : 0 \le k \le N \right\} \le$$
$$\le \max_{0\le k\le N} \left\{ \frac{|\partial_{\mathbf{b}}^{k+1} F(z^0 + re^{i\theta}\mathbf{b})|}{k! L^k(z^0 + re^{i\theta}\mathbf{b})} - \frac{|\partial_{\mathbf{b}}^k F(z^0 + re^{i\theta}\mathbf{b})|}{k! L^k(z^0 + re^{i\theta}\mathbf{b})} \frac{k u'(r)}{L(z^0 + re^{i\theta}\mathbf{b})} \right\} \le$$

$$\leq \max\left\{\frac{|\partial_{\mathbf{b}}^{k+1}F(z^0+re^{i\theta}\mathbf{b})|}{(k+1)!L^{k+1}(z^0+re^{i\theta}\mathbf{b})}(k+1)L(z^0+re^{i\theta}\mathbf{b})+\frac{|\partial_{\mathbf{b}}^{k}F(z^0+re^{i\theta}\mathbf{b})|}{k!L^{k}(z^0+re^{i\theta}\mathbf{b})}k\times\right.$$
$$\left.\times\frac{(-u'(r))^+}{L(z^0+re^{i\theta}\mathbf{b})}\colon 0\leq k\leq N\right\}\leq g(r)\left((N+1)L(z^0+re^{i\theta}\mathbf{b})+N\frac{(-u'(r))^+}{L(z^0+re^{i\theta}\mathbf{b})}\right).$$

Thus, we have $\frac{d}{dr}\ln g(r)\leq (N+1)L(z^0+re^{i\theta}\mathbf{b})+N\frac{(-u'(r))^+}{L(z^0+re^{i\theta}\mathbf{b})}$. Since F is a function of bounded L-index in the direction $\mathbf{b}$ and N is the L-index in the direction $\mathbf{b}$ we have $g(r)\neq 0$ and $\ln g(r)$ is continuous function. Thus,

$$g(r)\leq g(0)\exp\left\{\int_0^r\left((N+1)L(z^0+te^{i\theta}\mathbf{b})+N\frac{(-u'(t))^+}{L(z^0+te^{i\theta}\mathbf{b})}\right)dt\right\}.$$

Therefore, $\ln g(r)\leq \ln g(0)+\int_0^r\left((N+1)L(z^0+te^{i\theta}\mathbf{b})+N\frac{(-u'(t))^+}{L(z^0+te^{i\theta}\mathbf{b})}\right)dt$. Using (5), we obtain (3). If, in addition, for every $z^0\in\mathbb{D}^n$ and $\theta\in[0,2\pi]$ one has $(-u'_r(z^0,\theta,r))^+/(L^2(z^0+re^{i\theta}\mathbf{b}))\to 0$ as $\max_{1\leq j\leq n}|z_j^0+re^{i\theta}b_j|\to 1$, then

$$g(r)\leq g(0)\exp\left\{(N+1)\int_0^r\left(L(z^0+te^{i\theta}\mathbf{b})+\frac{(-u'_t(z^0,\theta,t))^+}{L(z^0+te^{i\theta}\mathbf{b})}\right)dt\right\}=$$
$$=g(0)\exp\left\{(N+1)(1+o(1))\int_0^r L(z^0+te^{i\theta}\mathbf{b})dt\right\}.$$

Thus, $|F(z^0+re^{i\theta}\mathbf{b})|\leq g(r)\leq g(0)\exp\left\{(N+1)(1+o(1))\int_0^r L(z^0+te^{i\theta}\mathbf{b})dt\right\}$ as $\max\limits_{1\leq j\leq n}|z_j^0+re^{i\theta}b_j|\to 1$ for every $\theta\in[0,2\pi]$, $z^0\in\mathbb{D}^n$, whence

$$\ln|F(z^0+re^{i\theta}\mathbf{b})|\leq g(0)+(N+1)(1+o(1))\int_0^r L(z^0+te^{i\theta}\mathbf{b})dt \text{ as } \max_{1\leq j\leq n}|z_j^0+re^{i\theta}b_j|\to 1. \quad (6)$$

Moreover, for every $z^0\in\mathbb{D}^n$ and $\theta\in[0,2\pi]$ we have

$$\varlimsup_{\max\limits_{1\leq j\leq n}|z_j^0+re^{i\theta}b_j|\to 1}\frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0+te^{i\theta}\mathbf{b})dt}\leq N_{\mathbf{b}}(F,L,\mathbb{D}^n)+1.$$

□

Remark 1. If we put $z^0=0$ then estimate (6) implies the following inequality

$$\varlimsup_{r\to 1/\max\limits_{1\leq j\leq n}|b_j|}\frac{\ln\max\{|F(t\mathbf{b})|\colon\ |t|=r\}}{\max\limits_{\theta\in[0,2\pi]}\int_0^r L(te^{i\theta}\mathbf{b})dt}\leq N_{\mathbf{b}}(F,L,\mathbb{D}^n)+1.$$

Denote $u(r,\theta)=l(re^{i\theta})$. For $n=1$ we deduce corollaries.

Corollary 1. *Let $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$, $l\colon\mathbb{D}\to\mathbb{R}_+$ be a continuous function such that for all $z\in\mathbb{D}$ $l(z)>\frac{\beta}{1-|z|}$ ($\beta>1$ is a some constant)and for every $\theta\in[0,2\pi]$ the function $l(re^{i\theta})$ be a continuously differentiable function of real variable $t\in[0,1)$. If $f(z)$ is an analytic function of bounded l-index then for every integer $p\geq 0$*

$$\ln\max_{|t|=r}\frac{|f^{(p)}(t)|}{p!l^p(t)}\leq\ln\max_{0\leq k\leq N}\frac{|f^{(k)}(0)|}{k!l^k(0)}+\max_{\theta\in[0,2\pi]}\int_0^r\left\{(N+1)l(te^{i\theta})+N\frac{(-u'_t(t,\theta))^+}{l(te^{i\theta})}\right\}dt. \quad (7)$$

If, in addition, $(-u'_r(r,\theta))^+/l^2(re^{i\theta})\to 0$ as $r\to 1$ uniformly with respect to $\theta\in[0,2\pi]$ then

$$\varlimsup_{r\to 1}\max_{\theta\in[0,2\pi]}\frac{\ln|f(re^{i\theta})|}{\int_0^r l(te^{i\theta})dt}\leq N(f,l)+1, \quad (8)$$

holds, where $N(f,l)$ is the l-index of the function f.

For $n=1$ corresponding corollary is an improvement of similar result of Sheremeta and Strochyk ([6]) because we do not assume that $l=l(|z|)$ and it matches with one-dimensional corrollary from growth estimates for analytic functions in the unit ball with bounded L-index in direction ([5]).

Corollary 2. *Let $F\colon \mathbb{D}^n\to\mathbb{C}$ be an analytic function of bounded L-index in the direction* $\mathbf{b}$, *$N=N_{\mathbf{b}}(F,L,\mathbb{D}^n)$ and z^0 be a fixed point in $\mathbb{D}^n$ such that $F(z^0)=1$. Then for every $r\in[0,\min\limits_{1\le j\le n}\frac{1-|z_j^0|}{|b_j|})$ the next inequality is valid*

$$\int_0^r \frac{n(t,z^0,1/F)}{t}dt\le \ln\max\{|F(z^0+t\mathbf{b})|\colon\ |t|=r\}\le \ln\max_{0\le k\le N}\frac{|\partial_{\mathbf{b}}^k F(z^0)|}{k!L^k(z^0)}+$$
$$+\max_{\theta\in[0,2\pi]}\int_0^r\left\{(N+1)L(z^0+te^{i\theta}\mathbf{b})+N\frac{(-u_t'(t,\theta))^+}{L(z^0+te^{i\theta}\mathbf{b})}\right\}dt.$$

Proof. We consider the function $F(z^0+t\mathbf{b})$ as a function of one variable t. Thus, the first inequality follows from the classical Jensen's Theorem. In addition, the second inequality follows from (3) for $p=0$. □

There are known similar results to Theorem 1 for entire functions of one variable [4] and for functions of one variable which are analytic in the unit disc [6].

3. L-index in direction in a domain compactly embedded in the unit polydisc. Let D be an arbitrary bounded domain in $\mathbb{D}^n$ such that $\mathrm{dist}(D,\mathbb{B}^n)>0$. If inequality (2) holds for all $z\in D$ instead $\mathbb{D}^n$, then the analytic function $F\colon\mathbb{D}^n\to\mathbb{C}$ is called *a function of bounded L-index in the direction* $\mathbf{b}$ *in the domain D*. The least such integer m_0 is called the *L-index in the direction* $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ *in domain D* and is denoted by $N_{\mathbf{b}}(F,L,D)=m_0$. The notation $\overline{D}$ stands for a closure of the domain D.

Lemma 1. *Let D be an arbitrary bounded domain in $\mathbb{D}^n$ such that $d=\mathrm{dist}(D,\mathbb{D}^n)=\inf\limits_{z\in D}\min\limits_{1\le j\le n}(1-|z_j|)>0$, $\beta>1$, $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be an arbitrary direction. If $L\colon\mathbb{D}^n\to\mathbb{R}_+$ is continuous function such that $L(z)\ge\frac{\beta\max\limits_{1\le j\le n}|b_j|}{d}$, and $F\colon\mathbb{D}^n\to\mathbb{C}$ is analytic function such that $(\forall z^0\in\overline{D})\colon F(z^0+t\mathbf{b})\not\equiv 0$, then $N_{\mathbf{b}}(F,L,D)<\infty$.*

Proof. For every fixed $z^0\in\overline{D}$ we expand the analytic function $F(z^0+t\mathbf{b})$ in a power series by powers of t in the disc $\left\{t\in\mathbb{C}\colon |t|\le\frac{1}{L(z^0)}\right\}$

$$F(z^0+t\mathbf{b})=\sum_{m=0}^{\infty}\frac{\partial_{\mathbf{b}}^m F(z^0)}{m!}t^m. \tag{9}$$

The quantity $\frac{|\partial_{\mathbf{b}}^m F(z^0)|}{m!}$ is the modulus of a coefficient of the power series (9) at the point $t\in\mathbb{C}$ such that $|t|=\frac{1}{L(z^0)}$. Since $F(z)$ is analytic function, for every $z_0\in\overline{D}$ $\frac{|\partial_{\mathbf{b}}^m F(z^0)|}{m!L^m(z^0)}\to 0$ $(m\to\infty)$, i.e., there exists $m_0=m(z^0,\mathbf{b})$ such that inequality (2) holds at the point $z=z^0$ for all $m\in\mathbb{Z}_+$.

We prove that $\sup\{m_0\colon z^0\in\overline{D}\}<+\infty$. On the contrary we assume that the set of all values m_0 is unbounded in z^0, i.e., $\sup\{m_0\colon z^0\in\overline{D}\}=+\infty$. Hence, for every $m\in\mathbb{Z}_+$ there exists $z^{(m)}\in\overline{D}$ and $p_m>m$

$$\left|\frac{\partial_{\mathbf{b}}^{p_m}F(z^{(m)})}{p_m!L^{p_m}(z^{(m)})}\right|>\max\left\{\left|\frac{\partial_{\mathbf{b}}^{k}F(z^{(m)})}{k!L^{k}(z^{(m)})}\right|\colon\ 0\le k\le m\right\}. \tag{10}$$

Since $\{z^{(m)}\}\subset\overline{D}$, there exists subsequence $z'^{(m)}\to z'\in\overline{G}$ as $m\to+\infty$. By Cauchy's integral formula we have $\frac{\partial^p_{\mathbf{b}}F(z)}{p!}=\frac{1}{2\pi i}\int_{|t|=r}\frac{F(z+t\mathbf{b})}{t^{p+1}}dt$ for any $p\in\mathbb{N}$, $z\in D$. Rewrite (10) as following

$$\max_{0\le k\le m}\left|\frac{\partial^k F(z^{(m)})}{k!L^k(z^{(m)})}\right|<\frac{1}{L^{p_m}(z^{(m)})}\int_{|t|=r/L(z^{(m)})}\frac{|F(z^{(m)}+t\mathbf{b})|}{|t|^{p_m+1}}|dt|\le\frac{\max\{|F(z)|\colon z\in D_r\}}{r^{p_m}},\quad (11)$$

where $D_r=\bigcup_{z^*\in\overline{D}}\{z\in\mathbb{C}^n\colon |z-z^*|\le\frac{|\mathbf{b}|r}{L(z^*)}\}$. We can choose $r\in(1,\beta)$, because F is a function analytic in the unit polydisc . Evaluating the limit for every directional derivative of fixed order in (11) as $m\to\infty$ we obtain

$$(\forall k\in\mathbb{Z}_+)]\quad\left|\frac{\partial^k_{\mathbf{b}}F(z')}{k!L^k(z')}\right|\le\overline{\lim_{m\to\infty}}\frac{1}{r^{p_m}}\max\{|F(z)|\colon z\in D_r\}\le 0.$$

Thus, all derivatives in the direction $\mathbf{b}$ of the function F at the point z' equals 0 and $F(z')=0$. In view of (9) $F(z'+t\mathbf{b})\equiv 0$. It is a contradiction. □

4. Growth and boundedness of L-index in direction of analytic solutions of directional equations. From Hayman's Theorem for analytic functions in the unit polydisc from [2] and Lemma 1, we yield the following corollary.

Corollary 3. *Let $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$, G be a domain compactly embedded in $\mathbb{D}^n$ such that $d=\operatorname{dist}(D,\mathbb{B}^n)=\inf_{z\in D}\min_{1\le j\le n}(1-|z_j|)>0$ and for all $z\in G$ $L(z)\ge\frac{\beta\max_{1\le j\le n}|b_j|}{d}$. An analytic function $F\colon\mathbb{D}^n\to\mathbb{C}$ has bounded L-index in the direction $\mathbf{b}$ if and only if there exist $p\in\mathbb{Z}_+$ and $C>0$ such that for all $z\in\mathbb{D}^n\setminus G$ the following relation holds*

$$\frac{|\partial^{p+1}_{\mathbf{b}}F(z)|}{L^{p+1}(z)}\le C\max\left\{\frac{|\partial^k_{\mathbf{b}}F(z)|}{L^k(z)}\colon\ 0\le k\le p\right\}.\quad (12)$$

We denote $a^+=\max\{a,0\}$. Set $u(r)=u(z^0,\theta,r)=L(z^0+re^{i\theta}\mathbf{b})$. Let $W_{\mathbf{b}}(\mathbb{D}^n)$ be a class of positive continuous function $L\colon\mathbb{D}^n\to\mathbb{R}_+$ satisfying all following conditions:

1) for all $z\in\mathbb{D}^n$ $L(z)>\beta\max_{1\le j\le n}\frac{|b_j|}{1-|z_j|}$, where $\beta=\mathrm{const}>1$;
2) for every $z^0\in\mathbb{D}^n$ and every $\theta\in[0,2\pi]$ the function $u(r,z^0,\theta)$ be a continuously differentiable function of real variable $r\in[0,r_0)$, where $r_0=\min_{1\le j\le n}\{s\in\mathbb{R}_+\colon\ |z_j^0+se^{i\theta}b_j|=1\}$;
3) for every $z^0\in\mathbb{D}^n$, $\theta\in[0,2\pi]$ one has $\left(\frac{d}{ds}\frac{1}{L(z^0+sre^{i\theta}\mathbf{b})}\Big|_{s=1}\right)^+\to 0$ as $\max_{1\le j\le n}|z_j^0+re^{i\theta}b_j|\to 1$, i.e. $r\to r_0$.

The conditions 2) and 3) together can be replaced by some strict condition $\partial_{\mathbf{b}}(1/L(Re^{i\Theta}))\to 0$ as $\max_{1\le j\le n}r_j\to 1$, where $Re^{i\Theta}=(r_1e^{i\theta_1},\dots,r_ne^{i\theta_n})$, $R=(r_1,\dots,r_n)$, $\theta_j\in[0,2\pi]$, and $L(Re^{i\Theta})$ is positive continuously differentiable function in each variable r_j, $j\in\{1,\dots,n\}$. Moreover, condition 3) is equivalent to $\frac{(-u'_r(z^0,\theta,r))^+}{L^2(z^0+re^{i\theta}\mathbf{b})}\to 0$ as $r\to r_0$. Beside, condition 1) yields that $\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx\to+\infty$ as $\max_{1\le j\le n}|z_j^0+re^{i\theta}b_j|\to 1$. First, we prove the following lemma.

Lemma 2. *Let $L \in W_{\mathbf{b}}(\mathbb{D}^n)$, $F\colon \mathbb{D}^n \to \mathbb{C}$ be an analytic function such that $\exists R \in [0,1)$ $\forall z \in \mathbb{D}^n$ $|z| < R$ one has $F(z+t\mathbf{b}) \not\equiv 0$. If there exist numbers $p \in \mathbb{Z}_+$, $C > 0$ such that for all $z \in \mathbb{D}^n$, $|z| \geq R$, the inequality is true*

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)} \leq C \max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^{k}(z)} : 0 \leq k \leq p\right\}, \tag{13}$$

then for every $z^0 \in \mathbb{D}^n$ and for every $\theta \in [0, 2\pi]$ one has

$$\varlimsup_{\max\limits_{1\le j\le n}|z_j^0+re^{i\theta}b_j|\to 1} \frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx} \leq \max\{1, C\}.$$

Proof. Let $\theta \in [0, 2\pi]$, $z^0 \in \mathbb{D}^n$ be fixed and $x \in [0, r_0)$ be such that $|z^0 + xe^{i\theta}\mathbf{b}| \geq R$. We define $\Omega_{z^0}(x) = \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z^0+xe^{i\theta}\mathbf{b})|}{L^k(z^0+xe^{i\theta}\mathbf{b})} : 0 \leq k \leq p\right\}$. The function $\frac{|\partial_{\mathbf{b}}^k F(z^0+xe^{i\theta}\mathbf{b})|}{L^k(z^0+xe^{i\theta}\mathbf{b})}$ is continuously differentiable by real $x \in [0, r^*]$, outside the zero set of function $|\partial_{\mathbf{b}}^k F(z^0 + xe^{i\theta}\mathbf{b})|$ because $L \in W_{\mathbf{b}}(\mathbb{D}^n)$. Thus, the function $\Omega_{z^0}(x)$ is a continuously differentiable function on $[0, r^*]$, apart from, possibly, a countable set. For absolutely continuous functions h_1, h_2, $\ldots$, h_k and $h(x) := \max\{h_j(z) \colon 1 \leq j \leq k\}$, $h'(x) \leq \max\{h_j'(x) \colon 1 \leq j \leq k\}$, $x \in [a, b]$ (see [7, Lemma 4.1, p. 81]). The function $\Omega_{z^0}(x)$ is absolutely continuous. Therefore,

$$\Omega_{z^0}'(x) \leq \max\left\{\frac{d}{dx}\left(\frac{1}{L^k(z^0+xe^{i\theta}\mathbf{b})} \cdot \left|\partial_{\mathbf{b}}^k F(z^0+xe^{i\theta}\mathbf{b})\right|\right) : 0 \leq k \leq p\right\}$$

except on a countable set of points.

Using the inequality $\frac{d}{dx}|\varphi(x)| \leq \left|\frac{d}{dx}\varphi(x)\right|$, which holds for complex-valued function of real argument except at the points $x = t$ such that $\varphi(t) = 0$, in view of (13) we obtain

$$\Omega_{z^0}'(x) \leq \max_{0\le k\le p}\left\{|e^{i\theta}|\frac{\left|\partial_{\mathbf{b}}^{k+1}F(z^0+xe^{i\theta}\mathbf{b})\right|}{L^k(z^0+xe^{i\theta}\mathbf{b})} - \left|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})\right|\frac{k\cdot u_x'(z^0,\theta,x)}{L^{k+1}(z^0+xe^{i\theta}\mathbf{b})}\right\} \leq$$
$$\leq \max_{0\le k\le p}\left\{\frac{\left|\partial_{\mathbf{b}}^{k+1}F(z^0+xe^{i\theta}\mathbf{b})\right|}{L^{k+1}(z^0+xe^{i\theta}\mathbf{b})}L(z^0+xe^{i\theta}\mathbf{b}) - \frac{\left|\partial_{\mathbf{b}}^{k}F(z^0+xe^{i\theta}\mathbf{b})\right|}{L^{k}(z^0+xe^{i\theta}\mathbf{b})}\frac{ku_x'(z^0,\theta,x)}{L(z^0+xe^{i\theta}\mathbf{b})}\right\} \leq$$
$$\leq \Omega_{z^0}(x)\left(CL(z^0+xe^{i\theta}\mathbf{b}) + p\frac{(-u_x'(z^0,\theta,x))^+}{L(z^0+xe^{i\theta}\mathbf{b})}\right).$$

From condition 3) in the definition of the class $W_{\mathbf{b}}(\mathbb{D}^n)$ we have $u_x'(z^0,\theta,x) = o(L^2(z^0 + xe^{i\theta}\mathbf{b}))$ as $x \to r_0$, then

$$\Omega_{z^0}'(x) \leq \Omega_{z^0}(x)(\max\{1,C\}L(z^0+xe^{i\theta}\mathbf{b}) + p\varepsilon L(z^0+xe^{i\theta}\mathbf{b})) \leq \Omega_{z^0}(x)\times$$
$$\times L(z^0+xe^{i\theta}\mathbf{b})(\max\{1,C\}+p\varepsilon) \leq \Omega_{z^0}(x)L(z^0+xe^{i\theta}\mathbf{b})\max\{1,C\}(1+p\varepsilon)$$

for all $\varepsilon > 0$ and for all $x \in [x_0(z^0,\theta,\varepsilon), r_0)$ outside a countable set of points for given $z^0 \in \mathbb{D}^n$ and $\theta \in [0, 2\pi]$ Hence, there exists $r_1 \geq x_0(z^0, \theta, \varepsilon)$ such that

$$\Omega_{z^0}(r) \leq \Omega_{z^0}(r_1) \cdot \exp\left\{(1+p\varepsilon)\max\{1,C\}\int_{r_1}^r L(z^0+xe^{i\theta}\mathbf{b})dx\right\}$$

for every $r \in [r_1, r_0)$. From the definition of $\Omega_{z^0}(x)$ for $k = 0$ we obtain that

$$|F(z^0+re^{i\theta}\mathbf{b})| \leq \Omega_{z^0}(r_1) \cdot \exp\left\{(1+p\varepsilon)\max\{1,C\}\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx\right\},$$

$$\ln|F(z^0+re^{i\theta}\mathbf{b})| \le \ln\Omega_{z^0}(r_1) + (1+p\varepsilon)\max\{1,C\}\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx,$$

$$\frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx} \le \frac{\ln\Omega_{z^0}(r_1)}{\int_0^r L(z^0+xe^{i\theta}\mathbf{b})dx} + (1+p\varepsilon)\max\{1,C\}.$$

From this inequality for all $z^0 \in \mathbb{D}^n$ and $\theta\in[0,2\pi]$ we obtain that

$$\overline{\lim_{\max\limits_{1\le j\le n}|z_j^0+re^{i\theta}b_j|\to 1}} \frac{\ln|F(z^0+re^{i\theta}\mathbf{b})|}{\int\limits_0^r L(z^0+xe^{i\theta}\mathbf{b})dx} \le \max\{1,C\}.$$

□

Using proved lemma we formulate and prove proposition providing growth estimates of analytic solutions of the partial differential equation

$$g_0(z)\partial_{\mathbf{b}}^p F(z) + g_1(z)\frac{\partial^{p-1}F(z)}{\partial\mathbf{b}^{p-1}} + \cdots + g_p(z)F(z) = h(z). \tag{14}$$

Let us denote $QW_{\mathbf{b}}(\mathbb{D}^n) = Q_{\mathbf{b}}(\mathbb{D}^n)\cap W_{\mathbf{b}}(\mathbb{D}^n)$.

Theorem 2. *Let $L\in QW_{\mathbf{b}}(\mathbb{D}^n)$, functions g_0, g_1, .., g_p, and h be analytic in the unit polydisc and there exists $R\in[0,1)$ such that for all $z\in\mathbb{D}^n$, $|z|\ge R$, the following conditions hold*

1) $|g_j(z)| \le m_j L^j(z)|g_0(z)|$ for $1\le j\le p$;

2) $|\partial_{\mathbf{b}} g_j(z)| < M_j\cdot L^{j+1}(z)|g_0(z)|$ for $0\le j\le p$;

3) $|\partial_{\mathbf{b}} h(z)| \le M\cdot L(z)\cdot|h(z)|$,

where m_j and M are nonnegative constants and M_j are positive constants. If an analytic function $F\colon\mathbb{D}^n\to\mathbb{C}$ satisfies equation (14) and $\forall z\in\mathbb{D}^n$, $|z|<R$, $F(z+t\mathbf{b})\not\equiv 0$ then F has bounded L-index in the direction $\mathbf{b}$ and for all $z^0\in\mathbb{D}^n$, $\theta\in[0,2\pi]$

$$\overline{\lim_{\max\limits_{1\le j\le n}|z_j^0+re^{i\theta}b_j|\to 1}} \frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int\limits_0^r L(z^0+te^{i\theta}\mathbf{b})dt} \le \max\{1,C\}, \tag{15}$$

where $C=\sum_{j=1}^p M_j + (M+1)\sum_{j=1}^p m_j + M$.

Proof. First, we note that the second condition of the theorem with $j=0$ implies that $g_0(z)\ne 0$ for $z\in\mathbb{D}^n$, $|z|\ge R$. Taking into account that the function $F(z)$ satisfies equation (14), we calculate the derivative in the direction $\mathbf{b}$ for this equation

$$g_0(z)\partial_{\mathbf{b}}^{p+1}F(z)+\sum_{j=0}^p \partial_{\mathbf{b}} g_j(z)\partial_{\mathbf{b}}^{p-j}F(z)+\sum_{j=1}^p g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z)+\sum_{j=1}^p g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z)=\partial_{\mathbf{b}} h(z). \tag{16}$$

By condition 3), we obtain $|\partial_{\mathbf{b}} h(z)| \le ML(z)h(z) \le ML(z)\sum_{j=0}^p |g_j(z)|\left|\partial_{\mathbf{b}}^{p-j}F(z)\right|$. By (16) $\partial_{\mathbf{b}}^{p+1}F(z) = \frac{1}{g_0(z)}\Big(\partial_{\mathbf{b}} h(z) - \sum_{j=0}^p \partial_{\mathbf{b}} g_j(z)\cdot\partial_{\mathbf{b}}^{p-j}F(z) - \sum_{j=1}^p g_j(z)\partial_{\mathbf{b}}^{p-j+1}F(z)\Big)$. Putting in the first condition of the theorem $m_0=1$, in view of the second condition we obtain

$$\left|\partial_{\mathbf{b}}^{p+1}F(z)\right| \le \frac{1}{g_0(z)}\Big(ML(z)\sum_{j=0}^p |g_j(z)|\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| + \sum_{j=0}^p |\partial_{\mathbf{b}} g_j(z)|\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| +$$

$$+\sum_{j=1}^{p}|g_j(z)|\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|\Big) \leq ML(z)\sum_{j=0}^{p} m_j L^j(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| +$$
$$+\sum_{j=0}^{p} M_j L^{j+1}(z)\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| + \sum_{j=1}^{p} m_j L^j(z)\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|.$$

Dividing this inequality by $L^{p+1}(z)$, we obtain

$$\frac{1}{L^{p+1}(z)}\left|\partial_{\mathbf{b}}^{p+1}F(z)\right| \leq M\sum_{j=0}^{p} m_j \frac{1}{L^{p-j}(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| + \sum_{j=0}^{p} M_j \frac{1}{L^{p-j}(z)} \times \left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right| +$$
$$+\sum_{j=1}^{p} m_j \frac{\left|\partial_{\mathbf{b}}^{p-j+1}F(z)\right|}{L^{p-j+1}(z)} \leq \Big(M\sum_{j=0}^{p} m_j + \sum_{j=0}^{p} M_j + \sum_{j=1}^{p} m_j\Big)\max\left\{\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^k(z)} : 0\leq k\leq p\right\} =$$
$$= \Big((M+1)\sum_{j=1}^{p} m_j + \sum_{j=0}^{p} M_j + M\Big)\max\left\{\frac{1}{L^k(z)}\left|\partial_{\mathbf{b}}^{k}F(z)\right| : 0\leq k\leq p\right\}$$

for all $z\in\mathbb{D}^n$, $|z|\geq R$. Thus, by Lemma 2 estimate (15) holds, and by Corollary 3 the analytic function $F(z)$ is of bounded L-index in the direction b. □

In the case when equation (14) is homogeneous, the previous theorem can be simplified.

Theorem 3. *Let $L\in QW_{\mathbf{b}}(\mathbb{D}^n)$, functions $g_0, g_1, \ldots, g_p$ be analytic in the unit polydisc and there exists $R\in[0,1)$ such that for all $z\in\mathbb{D}^n$, $|z|\geq R$, one has $|g_j(z)|\leq m_j L^j(z)|g_0(z)|$ for $1\leq j\leq p$, where m_j are some nonnegative constants. If an analytic function $F\colon \mathbb{D}^n\to\mathbb{C}$ satisfies equation (14) with $h(z)\equiv 0$ and $\forall z\in\mathbb{D}^n$, $|z|<R$, $F(z+t\mathbf{b})\not\equiv 0$ then $F(z)$ is of bounded L-index in the direction $\mathbf{b}$ and for all $z^0\in\mathbb{D}^n$, $\theta\in[0,2\pi]$*

$$\overline{\lim_{\max\limits_{1\leq j\leq n}|z_j^0+re^{i\theta}b_j|\to 1}} \frac{\ln|F(z^0+e^{i\theta}r\mathbf{b})|}{\int_0^r L(z^0+te^{i\theta}\mathbf{b})dt} \leq \max\left\{1, \sum_{j=1}^{p} m_j\right\}. \tag{17}$$

Proof. Equation (14) implies $g_0(z)\partial_{\mathbf{b}}^{p}F(z) = -\sum_{j=1}^{p} g_j(z)\partial_{\mathbf{b}}^{p-j}F(z)$. Then $|g_0(z)|\ |\partial_{\mathbf{b}}^{p}F(z)| \leq \sum_{j=1}^{p}|g_j(z)|\ \left|\partial_{\mathbf{b}}^{p-j}F(z)\right|$. Dividing the obtained inequality by $g_0(z)L^p(z)$ and using assumptions of the theorem by the functions $g_j(z)$, we obtain

$$\frac{1}{L^p(z)}\left|\partial_{\mathbf{b}}^{p}F(z)\right| \leq \sum_{j=1}^{p}\left|\frac{g_j(z)}{g_0(z)}\right|\frac{|\partial_{\mathbf{b}}^{p-j}F(z)|}{L^p(z)} \leq \sum_{j=1}^{p}\frac{m_j}{L^{p-j}(z)}\left|\partial_{\mathbf{b}}^{p-j}F(z)\right| \leq \sum_{j=1}^{p} m_j \max_{0\leq k\leq p-1}\frac{|\partial_{\mathbf{b}}^{k}F(z)|}{L^k(z)}.$$

Thus, all conditions of Corollary 3 are obeyed. Hence, the function F is of bounded L-index in the direction $\mathbf{b}$ and by Lemma 2 estimate (17) is true. □

Below there is presented an example of a directional derivative equation from electromagnetism theory (all formnulas and notation are based on [8]). For electric field $\mathbf{E}(\mathbf{r},t)$ one has

$$(\mathbf{s}\cdot\nabla)\mathbf{E} + \mu\sigma\mathbf{E} + \mu\varepsilon\frac{\partial\mathbf{E}}{\partial t} = 0, \tag{18}$$

where $\mathbf{s}\cdot\nabla$ is the derivative in the direction of wave propagation, μ is the magnetic permeability, ε is the dielectric permittivity, σ is the media conductivity.

This equation can describe a damped wave in a conductive medium (for example, in a transformer or a cable). It can be interpreted as: the projection of the wave equation onto the direction of propagation (or along a line), that is, as a simplified (directional) form of Maxwell's equations in a conducting medium. It corresponds to waves in cables, waves in transformers, and signal attenuation in conductors. The equation with the term $\mu\sigma\mathbf{E}$ describes losses (heating), the term $\mu\varepsilon\frac{\partial\mathbf{E}}{\partial t}$ describes the wave nature, the operator ∇ describes the spatial propagation. Equation (18) can be deduced from the differential form of Maxwell's equation for conducting medium without free charges (i.e. $\nabla\cdot\mathbf{E}\approx 0$): $\begin{cases}\nabla\times\mathbf{E}=-\mu\frac{\partial\mathbf{H}}{\partial t};\\ \nabla\times\mathbf{H}=\sigma\mathbf{E}+\varepsilon\frac{\partial\mathbf{E}}{\partial t},\end{cases}$ where $\sigma\mathbf{E}=\mathbf{J}$ is the conductive current (Ohm's law), $\varepsilon\frac{\partial\mathbf{E}}{\partial t}$ is the bias current. Using our notations from Introduction, equation (18) can be rewritten as the following $\partial_{\mathbf{b}}\mathbf{E}+\mu\sigma\mathbf{E}=0$, where $\mathbf{b}=(\mathbf{s},\mu\varepsilon)$. So, by analog of Theorem 3 for entire functions Equation (18) has entire solutions $\mathbf{E}$ every of which is of bounded index in the direction $\mathbf{b}$ and their growth does not exceed $\exp\{m\max\{1,\mu\sigma\}\}$ as $m\to+\infty$ and $\mathbf{E}=\mathbf{E}(\mathbf{r}+e^{i\theta}m\mathbf{s},t+e^{i\theta}m\mu\sigma)$. But in a general case $\mu=\mu(\mathbf{r})$, $\varepsilon=\varepsilon(\mathbf{r})$, $\sigma=\sigma(\mathbf{r})$. This means that the medium is inhomogeneous, with properties that vary spatially (for example, multilayer cables (insulation, conductor, shield), soil with varying conductivity, electric motor windings, waveguides with dielectric inserts). If μ, σ are constants and $\sigma(r)$ is the analytic function in the polydisc, we can apply Theorem 3 and obtain conlusion about boundedness of L-index in the direction $\mathbf{b}$ for some L and all analytic solutions $\mathbf{E}$.

Author Contributions. A. Bandura have contributed in the conceptualization, methodology, investigation, writing, review and editing. Ya. Batsala has suggested and described the example from electromagnetism theory.

Acknowledgments. This research (A. Bandura) was funded by the National Research Foundation of Ukraine (project 2025.07/0427, "Newest complex probabilistic methods for studying asymptotic properties of analytical solutions of differential equations represented by multiple random series and integrals and their potential applications", 0126U002547).

References

1. A. Bandura, P. Kurliak, O. Skaskiv, *Some Results on Composition of Analytic Functions in a Unit Polydisc*, Univers. J. Math. Appl. **7** (3) (2024), 121–128. doi:10.32323/ujma.1444221
2. A. Bandura, T. Salo, *Analytic in a Unit Polydisc Functions of Bounded L-Index in Direction*, Mat. Stud. **60** (1) (2023), 55–78. doi:10.30970/ms.60.1.55-78
3. B. Lepson, *Differential Equations of Infinite Order, Hyperdirichlet Series and Entire Functions of Bounded Index*, in: Entire Functions and Related Parts of Analysis, J. Korevaar (ed.), Proceedings of Symposia in Pure Math., 1968, 11, 298–307.
4. A.D. Kuzyk, M.N. Sheremeta, *Entire Functions of Bounded l-Distribution of Values*, Math. Notes **39** (1) (1986), 3–8. doi:10.1007/BF01647624
5. A. Bandura, O. Skaskiv, *Linear Directional Differential Equations in the Unit Ball: Solutions of Bounded L-Index*, Math. Slovaca **69** (5) (2019), 1089–1098. doi:10.1515/ms-2017-0292
6. S.N. Strochyk, M.M. Sheremeta, *Analytic in the Unit Disc Functions of Bounded Index*, Dopov. Akad. Nauk Ukr. (1) (1993), 19–22. (in Ukrainian)
7. M. Sheremeta, Analytic Functions of Bounded Index, VNTL Publishers, Lviv, 1999.
8. J.A. Stratton, Electromagnetic Theory, Wiley-IEEE Press, 2015.

Ivano-Frankivsk National Technical University of Oil and Gas
Ivano-Frankivsk, Ukraine

andriy.bandura@nung.edu.ua
yaroslav.batsala@nung.edu.ua

Received 04.03.2026
Revised 10.06.2026