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ON TRUNCATION ERROR BOUNDS OF BRANCHED CONTINUED FRACTION EXPANSIONS OF SOME RATIOS OF LAURICELLA–SARAN’S HYPERGEOMETRIC FUNCTIONS F_K

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The paper considers the problem of approximating Lauricella–Saran’s hypergeometric functions F_K in special cases by branched continued fractions as a special family of functions. Under the certain conditions on the elements of branched continued fraction expansions of some ratios of these functions, it is proven that every expansion converges to the function that is analytic in the domain of analytic continuation

$$\mathfrak{D}_\eta = \{\mathbf{z} \in \mathbb{R}^3 : z_1 \leq \eta, z_2 \leq \eta, z_3 \leq 0\}, \quad 0 < \eta < 1,$$

at least as fast as a geometric series with a ratio less than unity. For this purpose, the method based on the formula for the difference of two approximants of a branched continued fraction and the PF method (based on the so-called property of fork for a branched continued fraction with positive elements) was used.

1. Introduction. Branched continued fractions as a class of special functions are one of the most effective tools for representing the analytic functions that naturally arise in a variety of applied problems (see, for example, [16, 17, 18] and also [35, 42]). The expansions of some ratios of the Lauricella–Saran’s hypergeometric functions F_K into branched continued fractions were constructed and investigated in [2, 23]. This was done on the base of three-term recurrence relations ([2, Lemma 1]), using the idea of choosing certain identical values of the parameters of hypergeometric functions, proposed in [4]. The need to compute the values of the function F_K in special cases arises, in particular, in the problem of studying of the propagator seagull diagram [36]. Other applications of F_K can be found in [2, 15, 34] and also [39]. In the same way, the long-standing problem of studying the convergence of the branched continued fraction expansion of one ratio of Appel’s hypergeometric functions F_2 (see [14]) was partially solved in a special case in [1, 24, 26]. And recently, in [25, 40], two expansions of special ratios of the Lauricella–Saran’s hypergeometric functions F_M were constructed and investigated.

This paper establishes the estimates of convergence rate of the above-mentioned expansions for the ratios of Lauricella–Saran’s hypergeometric functions F_K in the special cases.

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Such estimates (also called truncation error bounds [25]) are important from a computational point view. Truncation error bounds were established for branched continued fraction expansions of some ratios of generalized hypergeometric functions ${}_3F_2$ in [5], Appell's hypergeometric functions F_1 in [27], F_2 in [26], F_4 in [30], Horn's hypergeometric functions H_3 in [3], H_4 in [21], Horn's confluent hypergeometric functions H_6 in [6], and Lauricella–Saran's hypergeometric functions F_M in [25]. Each of them is based on the formula for the difference of two approximants of a branched continued fraction (see [13, p. 28]), adapted to the constructed expansions (for more on branched continued fraction structures, see [9]). More about the convergence problem can be found in [10, 12] and also in [20, 32].

2. Expansions of some ratios of hypergeometric functions F_K . The function F_K is defined by triple power series (see [37, 41])

$$F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma_1, \gamma_2, \gamma_3; \mathbf{z}) = \sum_{p,q,r=0}^{+\infty} \frac{(\alpha_1)_p (\alpha_2)_{q+r} (\beta_1)_{p+r} (\beta_2)_q}{(\gamma_1)_p (\gamma_2)_q (\gamma_3)_r} \frac{z_1^p z_2^q z_3^r}{p! q! r!}, \quad (1)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$ herewith $\gamma_1, \gamma_2, \gamma_3 \notin \mathbb{Z}_{\leq}$, \mathbb{Z}_{\leq} denotes the nonpositive integers, $\mathbf{z} = (z_1, z_2, z_3) \in \mathfrak{D}_{F_K}$,

$$\mathfrak{D}_{F_K} = \{\mathbf{z} \in \mathbb{C}^3 : |z_1| < 1, |z_2| < 1, |z_3| < (1 - |z_1|)(1 - |z_2|)\},$$

$(\xi)_k = \xi(\xi + 1) \cdots (\xi + k - 1)$ is the Pochhammer symbol.

In [2, Theorem 3] the following is proved.

Theorem A. *The special ratio*

$$\frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} \quad (2)$$

has a formal branched continued fraction expansion

$$1 - z_1 - \frac{\nu_1 z_3}{1 - z_2 - \frac{\nu_2 z_3}{1 - z_1 - \frac{\nu_3 z_3}{1 - z_2 - \frac{\nu_4 z_3}{1 - \dots}}}}, \quad (3)$$

where $\alpha_1, \beta_2, \gamma_3 \notin \mathbb{Z}_{\leq}$, and

$$\nu_{2n-1} = \frac{(\alpha_2 + n - 1)(\gamma_3 + n - 1 - \beta_1)}{(\gamma_3 + 2n - 2)(\gamma_3 + 2n - 1)}, \quad \nu_{2n} = \frac{(\beta_1 + n)(\gamma_3 + n - \alpha_2)}{(\gamma_3 + 2n - 1)(\gamma_3 + 2n)}, \quad n \geq 1. \quad (4)$$

The following result can be proved in much the same way.

Theorem B. *The special ratio*

$$\frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2 + 1, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} \quad (5)$$

has a formal branched continued fraction expansion

$$1 - z_2 - \frac{\mu_1 z_3}{1 - z_1 - \frac{\mu_2 z_3}{1 - z_2 - \frac{\mu_3 z_3}{1 - z_1 - \frac{\mu_4 z_3}{1 - \dots}}}}, \quad (6)$$

where $\alpha_1, \beta_2, \gamma_3 \notin \mathbb{Z}_{\leq}$, and

$$\mu_{2n-1} = \frac{(\beta_1 + n - 1)(\gamma_3 + n - 1 - \alpha_2)}{(\gamma_3 + 2n - 2)(\gamma_3 + 2n - 1)}, \quad \mu_{2n} = \frac{(\alpha_2 + n)(\gamma_3 + n - \beta_1)}{(\gamma_3 + 2n - 1)(\gamma_3 + 2n)}, \quad n \geq 1. \quad (7)$$

The ratios (2) and (5) form the so-called complete group of ratios of the hypergeometric function $F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})$ (see [38, Definition 1]).

Let α_1, β_2 be complex numbers herewith $\alpha_1, \beta_2 \notin \mathbb{Z}_{\leq}$ and $\alpha_2, \beta_1, \gamma_3$ be real numbers. In [2, Theorem 6] it is shown that under the conditions

$$0 < \nu_n \leq \tau, \quad n \geq 1, \quad \tau > 0. \quad (8)$$

the branched continued fraction (3) converges uniformly on every compact subset of the domain

$$\mathfrak{D}_{\tau, \kappa} = \left\{ \mathbf{z} \in \mathbb{C}^3 : z_n \notin [1 - \kappa, +\infty), \quad n = 1, 2, \quad z_3 \notin \left[\frac{\kappa}{4\tau}, +\infty \right) \right\}, \quad 0 < \kappa < 1, \quad (9)$$

to a function $f(\mathbf{z})$ that is holomorphic in $\mathfrak{D}_{\tau, \kappa}$, and, in addition, the function $f(\mathbf{z})$ is an analytic continuation of the function (2) in the domain (8). A similar result is valid for the function (5) and its expansion (6) provided that

$$0 < \mu_n \leq \tau, \quad n \geq 1, \quad \tau > 0. \quad (10)$$

In [23, Theorem 2], under conditions that $\alpha_1, \alpha_2, \beta_1, \beta_2$, and γ_3 are complex constants such that $\alpha_1, \beta_2 \notin \mathbb{Z}_{\leq}$ and

$$|\nu_n| + \operatorname{Re}(\nu_n) \leq pq(1 - q), \quad n \geq 1, \quad p > 0, \quad 0 < q < 1,$$

the uniformly convergence of (3) on every compact subset of the domain

$$\mathfrak{D}_{p, q}^{\nu, \kappa} = \mathfrak{D}_{p, q} \bigcup \mathfrak{D}^{\nu, \kappa}, \quad (11)$$

to a function $f(\mathbf{z})$ holomorphic in $\mathfrak{D}_{p, q}^{\nu, \kappa}$ is proved, and, in addition, it is shown that $f(\mathbf{z})$ is an analytic continuation of (3) in $\mathfrak{D}_{p, q}^{\tau, \kappa}$, where

$$\mathfrak{D}_{p, q} = \left\{ \mathbf{z} \in \mathbb{C}^3 : z_n \neq \left[\frac{q}{2}, +\infty \right), \quad n = 1, 2, \quad |z_3| < \frac{1 + \cos(\arg(z_3))}{2p} \right\}$$

and

$$\mathfrak{D}^{\nu, \kappa} = \left\{ \mathbf{z} \in \mathbb{C}^3 : |z_n| < \frac{1 - \kappa}{2}, \quad n = 1, 2, \quad |z_3| < \frac{\kappa(1 - \kappa)}{2\nu} \right\}, \quad \nu = \sup_{n \in \mathbb{N}} \{|\nu_n|\}, \quad 0 < \kappa < 1.$$

Also, a similar result is valid for (5) and (6) provided that $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_3 \in \mathbb{C}$, $\alpha_1, \beta_2 \notin \mathbb{Z}_{\leq}$, and

$$|\mu_n| + \operatorname{Re}(\mu_n) \leq pq(1 - q), \quad n \geq 1, \quad p > 0, \quad 0 < q < 1,$$

where μ_n , $n \geq 1$, are defined by (7).

3. Main results. Let $\{f_n(\mathbf{z})\}$ be a sequence of approximants of branched continued fraction (3). We set

$$U_n^{(n)}(\mathbf{z}) = 1, \quad n \geq 1, \quad (12)$$

and, for $n \geq 1$, $1 \leq k \leq n$,

$$U_k^{(n)}(\mathbf{z}) = 1 - z_{1+\delta_1^{(-1)k+1}} - \frac{\nu_{k+1}z_3}{1 - z_{1+\delta_1^{(-1)k+2}} - \frac{\nu_{k+2}z_3}{1 - \dots - z_{1+\delta_1^{(-1)n-1}} - \frac{\nu_{n-1}z_3}{1 - z_{1+\delta_1^{(-1)n}} - \nu_n z_3}},$$

where δ_i^j is a Kronecker symbol, ν_n , $n \geq 1$, are defined by (4). Then, for $n \geq 2$, $1 \leq k \leq n-1$,

$$U_k^{(n)}(\mathbf{z}) = 1 - z_{1+\delta_1^{(-1)k+1}} - \frac{\nu_{k+1}z_3}{U_{k+1}^{(n)}(\mathbf{z})} \quad (13)$$

and $f_n(\mathbf{z}) = 1 - z_1 - \frac{\nu_1 z_3}{U_1^{(n)}(\mathbf{z})}$, $n \geq 1$.

Assume that $U_k^{(n)}(\mathbf{z}) \neq 0$ for all indices in some domain \mathfrak{D} , $\mathfrak{D} \subset \mathbb{C}^3$. Then for $n \geq 1$, $k \geq 1$ (see [13, p. 28])

$$f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) = -z_3^n \left(z_{1+\delta_1^{(-1)n+1}} + \frac{\nu_{n+1}z_3}{U_{n+1}^{(n+k)}(\mathbf{z})} \right) \prod_{r=1}^n \frac{\nu_r}{U_r^{(n+k)}(\mathbf{z})U_r^{(n)}(\mathbf{z})} \quad (14)$$

or for $n \geq 2$, $k \geq 2$

$$f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) = \frac{-\nu_1 z_3^n}{U_1^{(q)}(\mathbf{z})U_n^{(n+k)}(\mathbf{z})} \left(z_{1+\delta_1^{(-1)n+1}} + \frac{\nu_{n+1}z_3}{U_{n+1}^{(n+k)}(\mathbf{z})} \right) \prod_{r=1}^{[n/2]} \frac{\nu_{2r}}{U_{2r-1}^{(p)}(\mathbf{z})U_{2r}^{(p)}(\mathbf{z})} \prod_{r=1}^{[(n-1)/2]} \frac{\nu_{2r+1}}{U_{2r}^{(q)}(\mathbf{z})U_{2r+1}^{(q)}(\mathbf{z})},$$

where $[\cdot]$ denote integer part, $q = n+k$, $p = n$, if $n = 2s$, and $q = n$, $p = n+k$, if $n = 2s-1$, $s \geq 1$.

The following result holds.

Theorem 1. Let α_1, β_2 be complex number herewith $\alpha_1, \beta_2 \notin \mathbb{Z}_{\leq}$ and α_2, β_1 , and γ_3 real numbers such that satisfy the conditions (8). Then:

(a) the branched continued fraction (3) converges to a finite value $f(\mathbf{z})$ for each $\mathbf{z} \in \mathfrak{D}_\eta$, where

$$\mathfrak{D}_\eta = \{\mathbf{z} \in \mathbb{C}^3: z_1 \leq \eta, z_2 \leq \eta, z_3 \leq 0\}, \quad 0 < \eta < 1; \quad (15)$$

(b) the convergence is uniform on every compact subset of the domain $\text{Int}(\mathfrak{D}_\eta)$, and the function $f(\mathbf{z})$ is analytic on $\text{Int}(\mathfrak{D}_\eta)$;

(c) if $f_n(\mathbf{z})$ denotes the n th approximant of (3), then for $n \geq 2$ and for each $\mathbf{z} \in \mathfrak{D}_\eta$

$$\begin{aligned} & |f(\mathbf{z}) - f_n(\mathbf{z})| \leq \\ & \leq \frac{(|z_{1+\delta_1^{(-1)n+1}}|(1 - z_{1+\delta_1^{(-1)n+2}}) + \tau|z_3|)(1 - z_{1+\delta_1^{(-1)n+1}} + \tau|z_3|)^{-1}(\tau|z_3|)^n}{(1 - z_1)(1 - z_2)(1 - z_1 + \tau|z_3|)^{[(n-1)/2]}(1 - z_2 + \tau|z_3|)^{[n/2]}}; \end{aligned} \quad (16)$$

(d) the function $f(\mathbf{z})$ is an analytic continuation of the function (2) in the domain $\text{Int}(\mathfrak{D}_\eta)$.

Proof. (a) Let \mathbf{z} be an arbitrary fixed point in (15). Using inequalities (8), from (13) for arbitrariness n and k such that $1 \leq k \leq n-1$ and $n \geq 2$, we have

$$U_k^{(n)}(\mathbf{z}) = 1 - z_{1+\delta_1^{(-1)k+1}} - \frac{\nu_{k+1}z_3}{U_{k+1}^{(n)}(\mathbf{z})} \geq 1 - z_{1+\delta_1^{(-1)k+1}} \geq 1 - \eta > 0. \quad (17)$$

Now, using (8), (15), and (17), for any $n \geq 2$ and $k \geq 2$, we have that

$$|-\nu_1 z_3|/|U_1^{(n)}(\mathbf{z})| \leq \tau|z_3|$$

and

$$\frac{1}{|U_n^{(n+k)}(\mathbf{z})|} \cdot \left| z_{1+\delta_1^{(-1)n+1}} + \frac{\nu_{n+1} z_3}{U_{n+1}^{(n+k)}(\mathbf{z})} \right| \leq \frac{1}{1 - z_{1+\delta_1^{(-1)n+1}}} \left(|z_{1+\delta_1^{(-1)n+1}}| + \frac{\tau|z_3|}{1 - z_{1+\delta_1^{(-1)n+2}}} \right).$$

Next, additionally using (13) and for $1 \leq k \leq n-1$ and $n \geq 2$, we obtain

$$\begin{aligned} \left| \frac{\nu_{k+1} z_3}{U_k^{(n+1)}(\mathbf{z}) U_{k+1}^{(n+1)}(\mathbf{z})} \right| &= \frac{\left| \frac{\nu_{k+1} z_3}{U_{k+1}^{(n+1)}(\mathbf{z})} \right|}{\left| 1 - z_{1+\delta_1^{(-1)k+1}} - \frac{\nu_{k+1} z_3}{U_{k+1}^{(n+1)}(\mathbf{z})} \right|} \leq \frac{\frac{\nu_{k+1} |z_3|}{U_{k+1}^{(n+1)}(\mathbf{z})}}{1 - z_{1+\delta_1^{(-1)k+1}} + \frac{\nu_{k+1} |z_3|}{U_{k+1}^{(n+1)}(\mathbf{z})}} \leq \\ &\leq \frac{\tau|z_3|}{1 - z_{1+\delta_1^{(-1)k+1}} + \tau|z_3|} \end{aligned}$$

and, furthermore,

$$\left| \frac{\nu_{n+1} z_3}{U_n^{(n+1)}(\mathbf{z}) U_{n+1}^{(n+1)}(\mathbf{z})} \right| = \frac{\nu_{n+1} z_3}{1 - z_{1+\delta_1^{(-1)n+1}} - \nu_{n+1} z_3} \leq \frac{\tau|z_3|}{1 - z_{1+\delta_1^{(-1)n+1}} + \tau|z_3|}.$$

Due to the above, for $n \geq 2$ and $k \geq 2$, we have the following

$$\begin{aligned} &|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| \leq \\ &\leq \frac{(|z_{1+\delta_1^{(-1)n+1}}|(1 - z_{1+\delta_1^{(-1)n+2}}) + \tau|z_3|)(1 - z_{1+\delta_1^{(-1)n+1}} + \tau|z_3|)^{-1}(\tau|z_3|)^n}{(1 - z_{1+\delta_1^{(-1)n+1}})(1 - z_{1+\delta_1^{(-1)n+2}})(1 - z_1 + \tau|z_3|)^{[(n-1)/2]}(1 - z_2 + \tau|z_3|)^{[n/2]}} = \\ &= \frac{(|z_{1+\delta_1^{(-1)n+1}}|(1 - z_{1+\delta_1^{(-1)n+2}}) + \tau|z_3|)(\tau|z_3|)^n}{(1 - z_1)(1 - z_2)(1 - z_{1+\delta_1^{(-1)n+1}} + \tau|z_3|)(1 - z_1 + \tau|z_3|)^{[(n-1)/2]}(1 - z_2 + \tau|z_3|)^{[n/2]}}. \quad (18) \end{aligned}$$

Obviously, for an arbitrary fixed $\mathbf{z} \in \mathfrak{D}_\eta$,

$$\frac{(|z_{1+\delta_1^{(-1)n+1}}|(1 - z_{1+\delta_1^{(-1)n+2}}) + \tau|z_3|)(\tau|z_3|)^n}{(1 - z_1)(1 - z_2)(1 - z_{1+\delta_1^{(-1)n+1}} + \tau|z_3|)(1 - z_1 + \tau|z_3|)^{[(n-1)/2]}(1 - z_2 + \tau|z_3|)^{[n/2]}} \rightarrow 0$$

as $n \rightarrow +\infty$. Thus, due to the arbitrariness of k , we get (a).

(b) Let \mathfrak{K} be an arbitrary compact subset of $\text{Int}(\mathfrak{D}_\eta)$. Then there exists a constant $C > 0$ such that for $n \geq 2$, $k \geq 2$, and for all $\mathbf{z} \in \mathfrak{K}$ from (18) we obtain

$$|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| < \frac{(C(1 - \eta) + \tau C)(\tau C)^n}{(1 - \eta)^3(1 - \eta + \tau C)^{n-1}}.$$

Now, if m, n are arbitrary integer numbers such that $m \geq 2$, $n + 2 \geq k \geq 2$, then, for all $\mathbf{z} \in \mathfrak{K}$,

$$|f_{n+m}(\mathbf{z}) - f_n(\mathbf{z})| \leq |f_{n+m}(\mathbf{z}) - f_k(\mathbf{z})| + |f_n(\mathbf{z}) - f_k(\mathbf{z})|.$$

Due to

$$\frac{(C(1 - \eta) + \tau C)(\tau C)^n}{(1 - \eta)^3(1 - \eta + \tau C)^{n-1}} \rightarrow 0 \quad (n \rightarrow +\infty),$$

(b) follows.

(c) follows directly from (18) as $k \rightarrow +\infty$.

(d) It is obvious that

$$\frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{0})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{0})} = 1.$$

Then, there exists $0 < \varepsilon < 1/4$ such that function (2) is analytic in domain

$$\mathfrak{D}_\varepsilon = \{\mathbf{z} \in \mathbb{R}^3: -\varepsilon < z_1 < 0, -\varepsilon < z_2 < 0, -\varepsilon < z_3 < 0\},$$

and $\mathfrak{D}_\varepsilon \subset (\mathfrak{D}_{F_K} \cap \text{Int}(\mathfrak{D}_\eta))$, in particular, $\mathfrak{D}_{1/8} \subset (\mathfrak{D}_{F_K} \cap \text{Int}(\mathfrak{D}_\eta))$.

Let \mathbf{z} be an arbitrary fixed point in \mathfrak{D}_ε . Then it is obvious that the elements of expansion (3) are positive, which gives us the following (see, [13, Proposition 1.4])

$$f_{2n}(\mathbf{z}) < f_{2n+2}(\mathbf{z}) < f_{2n+1}(\mathbf{z}) < f_{2n-1}(\mathbf{z}), \quad n \geq 1.$$

This, together with (a), for each $\mathbf{z} \in \mathfrak{D}_\varepsilon$ ensures the convergence of the sequences of even and odd approximants of branched continued fraction (3) to a finite value $f(\mathbf{z})$.

Now, we consider

$$\frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} - f_n(\mathbf{z}), \quad n \geq 1,$$

where

$$\begin{aligned} & \frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} = \\ & = 1 - z_2 - \frac{\nu_1 z_3}{1 - z_1 - \frac{\nu_2 z_3}{1 - \dots - z_{1+\delta_1^{(-1)n}} - \frac{\nu_n z_3}{1 - z_{1+\delta_1^{(-1)n+1}} - \frac{\nu_{n+1} z_3}{V_{n+1}^{(n+1)}(\mathbf{z})}}}}}, \end{aligned}$$

and

$$V_{n+1}^{(n+1)}(\mathbf{z}) = \frac{F_K\left(\alpha_1, \alpha_2 + \sum_{r=0}^n \delta_1^{(-1)^r}, \beta_1 + \sum_{r=0}^{n-1} \delta_1^{(-1)^r}, \beta_2; \alpha_1, \beta_2, \gamma_3 + n + 1; \mathbf{z}\right)}{F_K\left(\alpha_1, \alpha_2 + \sum_{r=0}^{n+1} \delta_1^{(-1)^r}, \beta_1 + \sum_{r=0}^n \delta_1^{(-1)^r}, \beta_2; \alpha_1, \beta_2, \gamma_3 + n + 2; \mathbf{z}\right)}.$$

Let

$$\begin{aligned} & V_k^{(n+1)}(\mathbf{z}) = \\ & = 1 - z_{1+\delta_1^{(-1)^{k+1}}} - \frac{\nu_{k+1} z_3}{1 - z_{1+\delta_1^{(-1)^{k+2}}} - \frac{\nu_{k+2} z_3}{1 - \dots - z_{1+\delta_1^{(-1)^n}} - \frac{\nu_n z_3}{1 - z_{1+\delta_1^{(-1)^{n+1}}} - \frac{\nu_{n+1} z_3}{V_{n+1}^{(n+1)}(\mathbf{z})}}}}}. \end{aligned}$$

Then, similarly to (13), for $n \geq 2$, $1 \leq k \leq n-1$ we have

$$V_k^{(n)}(\mathbf{z}) = 1 - z_{1+\delta_1^{(-1)^{k+1}}} - \frac{\nu_{k+1} z_3}{V_{k+1}^{(n)}(\mathbf{z})}.$$

Obviously, $U_k^{(n)}(\mathbf{z}) \neq 0$, $V_k^{(n)}(\mathbf{z}) \neq 0$ for all indices and for all $\mathbf{z} \in \mathfrak{D}_\varepsilon$. Then, taking (14) into account for $n \geq 1$, we obtain

$$\begin{aligned} & \frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} - f_n(\mathbf{z}) = \\ & = -z_3^n \left(z_{1+\delta_1^{(-1)^{n+1}}} + \frac{\nu_{n+1} z_3}{V_{n+1}^{(n+1)}(\mathbf{z})} \right) \prod_{r=1}^n \frac{\nu_r}{V_r^{(n+1)}(\mathbf{z}) U_r^{(n)}(\mathbf{z})}. \end{aligned}$$

Therefore, for $n \geq 1$ and for all $\mathbf{z} \in \mathfrak{D}_\varepsilon$,

$$f_{2n}(\mathbf{z}) < \frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})} < f_{2n-1}(\mathbf{z}).$$

Now, due to the above, including (a), we get $\lim_{n \rightarrow +\infty} f_{2n}(\mathbf{z}) = \lim_{n \rightarrow +\infty} f_{2n-1}(\mathbf{z}) = f(\mathbf{z})$. Then also for all $\mathbf{z} \in \mathfrak{D}_\varepsilon$,

$$f(\mathbf{z}) = \frac{F_K(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z})}{F_K(\alpha_1, \alpha_2, \beta_1 + 1, \beta_2; \alpha_1, \beta_2, \gamma_3 + 1; \mathbf{z})}.$$

Finally, applying [2, Theorem 2], we obtain (d). \square

By setting $\beta_1 = 0$ and replacing γ_3 with $\gamma_3 - 1$ in Theorem 1, we obtain a similar result for the branched continued fraction

$$\frac{1}{1 - z_1 - \frac{u_1 z_3}{1 - z_2 - \frac{u_2 z_3}{1 - z_1 - \frac{u_3 z_3}{1 - z_2 - \frac{u_4 z_3}{1 - \dots}}}}}, \quad (19)$$

where

$$u_{2n-1} = \frac{(\alpha_2 + n - 1)(\gamma_3 + n - 2)}{(\gamma_3 + 2n - 3)(\gamma_3 + 2n - 2)}, \quad u_{2n} = \frac{n(\gamma_3 + n - 1 - \alpha_2)}{(\gamma_3 + 2n - 2)(\gamma_3 + 2n - 1)}, \quad n \geq 1.$$

Corollary 1. *If α_1 and β_2 be complex numbers herewith $\alpha_1, \beta_2 \notin \mathbb{Z}_\leq$, α_2 and γ_3 be real numbers such that $0 < u_n \leq \varkappa$, $n \geq 1$, $\varkappa > 0$, then:*

- (a) *the branched continued fraction (19) converges to a finite value $g(\mathbf{z})$ for each $\mathbf{z} \in \mathfrak{D}_\eta$, where \mathfrak{D}_η is defined by (15);*
- (b) *the convergence is uniform on every compact subset of (15) to the function $g(\mathbf{z})$ analytic in $\text{Int}(\mathfrak{D}_\eta)$;*
- (c) *if $g_n(\mathbf{z})$ denotes the n th approximant of (19), then for $n \geq 3$ and for each $\mathbf{z} \in \mathfrak{D}_\eta$*

$$\begin{aligned} & |g(\mathbf{z}) - g_n(\mathbf{z})| \leq \\ & \leq \frac{(|z_{1+\delta_1^{(-1)^{n+1}}}|(1 - z_{1+\delta_1^{(-1)^{n+2}}}) + \varkappa|z_3|)(1 - z_{1+\delta_1^{(-1)^{n+1}}} + \varkappa|z_3|)^{-1}(\varkappa|z_3|)^{n-1}}{(1 - z_1)^3(1 - z_2)(1 - z_1 + \varkappa|z_3|)^{[(n-2)/2]}(1 - z_2 + \varkappa|z_3|)^{[(n-1)/2]}}; \end{aligned} \quad (20)$$

- d) *the function $g(\mathbf{z})$ is an analytic continuation of $F_K(a_1, a_2, 1, b_2; a_1, b_2, c_3; \mathbf{z})$ in $\text{Int}(\mathfrak{D}_\eta)$.*

The following theorem can be proved in much the same way as Theorem 1.

Theorem 2. *Let $\alpha_1, \beta_2 \in \mathbb{C}$ herewith $\alpha_1, \beta_2 \notin \mathbb{Z}_\leq$, $\alpha_2, \beta_1, \gamma_3 \in \mathbb{R}$ and the inequalities (10) hold. Then:*

- (a) *the branched continued fraction (6) converges to a finite value $f(\mathbf{z})$ for each $\mathbf{z} \in \mathfrak{D}_\eta$, where \mathfrak{D}_η is defined by (15);*
- (b) *the convergence is uniform on every compact subset of the domain $\text{Int}(\mathfrak{D}_\eta)$, and the function $f(\mathbf{z})$ is analytic on $\text{Int}(\mathfrak{D}_\eta)$;*
- (c) *if $f_n(\mathbf{z})$ denotes the n th approximant of (6), then for $n \geq 2$ and for each $\mathbf{z} \in \mathfrak{D}_\eta$ the truncation error bound (16) holds;*
- (d) *the function $f(\mathbf{z})$ is an analytic continuation of the function (5) in the domain $\text{Int}(\mathfrak{D}_\eta)$.*

Corollary 2. Let $\alpha_1, \beta_2 \in \mathbb{C}$ herewith $\alpha_1, \beta_2 \notin \mathbb{Z}_{\leq}$, $\beta_1, \gamma_3 \in \mathbb{R}$ and $0 < v_n \leq \varkappa$, $n \geq 1$, $\varkappa > 0$, where

$$v_{2n-1} = \frac{(\beta_1 + n - 1)(\gamma_3 + n - 2)}{(\gamma_3 + 2n - 3)(\gamma_3 + 2n - 2)}, \quad v_{2n} = \frac{n(\gamma_3 + n - 1 - \beta_1)}{(\gamma_3 + 2n - 2)(\gamma_3 + 2n - 1)}, \quad n \geq 1.$$

Then:

(a) the branched continued fraction

$$\frac{1}{1 - z_2 - \frac{v_1 z_3}{1 - z_1 - \frac{v_2 z_3}{1 - z_2 - \frac{v_3 z_3}{1 - z_1 - \frac{v_4 z_3}{1 - \dots}}}}}, \quad (21)$$

converges to a finite value $g(\mathbf{z})$ for each $\mathbf{z} \in \mathfrak{D}_\eta$, where \mathfrak{D}_η is defined by (15);

(b) the convergence is uniform on every compact subset of (15) to the function $g(\mathbf{z})$ analytic in $\text{Int}(\mathfrak{D}_\eta)$;

(c) if $g_n(\mathbf{z})$ denotes the n th approximant of (21), then for $n \geq 3$ and for each $\mathbf{z} \in \mathfrak{D}_\eta$ the truncation error bound (20) holds;

(d) the function $g(\mathbf{z})$ is an analytic continuation of $F_K(a_1, 1, b_1, b_2; a_1, b_2, c_3; \mathbf{z})$ in $\text{Int}(\mathfrak{D}_\eta)$.

4. Conclusions. For each $\mathbf{z} \in \text{Int}(\mathfrak{D}_\eta)$, where \mathfrak{D}_η is defined by (15), we have established that under conditions (8) the branched continued fraction (3) converges to the function $f(\mathbf{z})$ which is analytic in $\text{Int}(\mathfrak{D}_\eta)$, at least as fast as geometric series with ratio

$$\rho(\mathbf{z}) = \tau|z_3|/(\min\{1 - z_1, 1 - z_2\} + \tau|z_3|). \text{ Thus, } \overline{\lim}_{n \rightarrow +\infty} |f(\mathbf{z}) - f_n(\mathbf{z})|^{1/n} \leq \rho(\mathbf{z}),$$

where $f_n(\mathbf{z})$ is n th approximant of (3). Using the PF method (based on the so-called property of fork for a branched continued fraction with positive elements) [2], it is also established that under conditions (8), the function $f(\mathbf{z})$, to which the expansion (3) converges, is an analytic extension of the function (2) in the domain (15). A similar result is valid for the function (5) and its expansion (6) provided that the inequalities (10) hold.

Further research is to establish truncation error bounds for expansions (3) and (5) in the domains of (9) and (11). For this, we can try to use the ideas of obtaining estimates in [7, 8, 11]. Another direction is to establish sets of numerical stability of these expansions. Here, we can use methods for both continued fractions [19, 31] and branched continued fractions [28, 29, 33].

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