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OSCILLATORY PROCESSES UNDER IMPULSIVE PERTURBATIONS

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The problem of oscillatory processes under impulsive perturbations at fixed moments in time is considered. The mathematical model consists of a classical partial differential equation describing oscillatory processes, supplemented by boundary and initial conditions, along with additional conditions characterizing the impulsive effects at the specified time instances.

Using the method of separation of variables, the solution is constructed in the form of a Fourier series involving the eigenfunctions of Laplace operator. Conditions under which the resulting solution is classical are discussed. A special case is examined in which the domain is a rectangle, and in this case, an explicit analytic solution is obtained.

The results obtained in this paper may be especially useful in the analysis of mathematical models for various applied problems involving short-term effects, as well as in the further development of the theory of impulsive differential equations, including those involving partial derivatives.

1. Introduction. The theory of differential equations with impulse effects is a significant and rapidly advancing field of modern mathematics. The active development of this field began in the last century, with the works of Anatolii Myshkis and Anatolii Samoilenko ([13]), as well as A. Halanay and D. Veksler ([10]), playing a pivotal role in its advancement.

This field is closely linked to the broad application of differential equation theory in analyzing numerous mathematical models that describe various processes encountered in mechanics, biology, medicine, chemistry, economics, and many other scientific, technological, and practical domains. These areas often involve phenomena and processes of a brief or abrupt nature, such as sudden jumps. Examples of such phenomena include impacts, economic investments, sudden shifts in population size within biological communities, and others (see, [1], [3], [4], [5], [12], [24] and references therein).

Initially, impulse or jump phenomena were modeled mathematically using generalized functions, particularly the Dirac delta function. However, this approach did not always align well with the concept of classical solutions of differential equations, which are typically employed as models for these processes. Subsequently, differential equations with discontinuous right-hand sides were introduced to describe processes characterized by sudden changes in their behavior. Unfortunately, this approach often limited the effective use of classical techniques from modern differential equation theory. Nowadays, systems experiencing impulse effects are most commonly described by differential equations valid during intervals when no impulses occur, combined with additional conditions called impulse conditions. These conditions mathematically define how the system's trajectory changes abruptly from one path to another according to a specific rule.

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Impulse differential equations are divided into two categories ([16]): those with impulses occurring at predetermined fixed times, and those with impulses occurring at variable, or non-fixed, times. In the former, the impulse moments are known in advance, while in the latter, these moments are determined by extra conditions — for example, when the system reaches certain given values or extreme points of some of its characteristics.

Currently, the theory of systems with impulse effects is not only an important practical application of differential equations but also a source of fascinating mathematical problems. The additional impulse conditions give rise to unique properties that are characteristic only of impulse differential equations. In fact, even linear equations typically become nonlinear when impulse conditions are included ([19], [20]).

One of the main effects of impulse actions is the profound change they cause in the qualitative behavior of system solutions. Impulse conditions can impact the solvability of the problem, the continuation of solutions, their stability, and can lead to the existence of solutions within broader classes of functions, for example, allowing for quasi-periodic solutions that the original equations do not possess. Furthermore, even linear systems with impulses can display chaotic dynamics ([2], [18]). For instance, such systems can have multiple periodic solutions with different periods coexisting, in accordance with Sharkovsky's ordering ([21]).

Most research on impulsive differential equations has concentrated on ordinary differential equations ([6], [8], [11], [17]), while many real-world phenomena are modeled by partial differential equations ([9]). It is worth noting that investigating impulsive partial differential equations is a highly promising and important area for various practical applications.

Impulsive systems are characterized by discontinuous trajectories. A closely related class is that of differential inclusions ([23]), which have attracted considerable attention in recent years. Similar to impulsive systems, they capture dynamics with discontinuities, while also accounting for uncertainty in system behavior ([14], [15], [22]).

This study focuses on constructing a classical solution to an initial-boundary value problem for a wave equation subjected to impulsive disturbances, which describes the oscillatory processes in a sufficiently smooth and arbitrary bounded domain. By applying the method of separation of variables, a classical solution to the problem is constructed as a Fourier series involving the eigenfunctions of Laplace operator with the Dirichlet boundary condition. The case where the domain is a rectangle is considered, and the conditions under which the resulting solution is classical are examined.

2. Problem statement and main results. Let G be a bounded domain in $\mathbb{R}^N$, where $N \in \mathbb{N}$ is arbitrary fixed, and the boundary ∂G of G be a sufficiently smooth surface. Assume $\{t_k\}_{k\in\mathbb{N}\cup\{0\}}$ is a set of positive real numbers such that

$$0 =: t_0 < t_1 < t_2 < \dots < t_k < \dots, \quad \text{with } t_k \to +\infty \ \text{ as } \ k \to +\infty.$$

We consider a **problem**: to find a function $u(x,t)$, $(x,t) \in \overline{Q}$, which satisfies the wave equation

$$u_{tt} - a^2 \Delta u = 0 \quad \text{in} \quad G \times (\cup_{k\in\mathbb{N}} (t_{k-1}, t_k)), \tag{1}$$

Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial G \times [0, +\infty), \tag{2}$$

initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad x \in \overline{G}, \tag{3}$$

and impulsive conditions

$$u(x,t_k+0)-u(x,t_k-0)=0, \quad u_t(x,t_k+0)-u_t(x,t_k-0)=I_k(x), \quad x\in G, \tag{4}$$

where $\varphi, \psi, I_k, k \in \mathbb{N}$, are some continuous functions defined on $\overline{G}$, and $\Delta u = \sum_{i=1}^N u_{x_i x_i}$ is the value of the Laplace operator on u.

Assume that $\varphi, \psi, I_k, k \in \mathbb{N}$, are defined continuous functions on $\overline{G}$, while

$$\varphi(x)=0, \quad \psi(x)=0, \quad I_k(x)=0, \quad k\in\mathbb{N}, \quad \text{for all } x\in\partial G.$$

A classical solution of problem (1)–(4) is called a function

$$u\in C(\overline{G}\times[0,+\infty))\cap C^2(G\times(\cup_{k\in\mathbb{N}}(t_{k-1},t_k)))$$

such that there exist finite values $u_t(x,t_k+0)$, $u_t(x,t_k-0)$ for all $x\in G$, and it satisfies equation (1) and conditions (2)–(4) everywhere.

Let us present the scheme for finding the classical solution of problem (1)–(4) using the method of separation of variables.

First, let us consider the eigenvalue problem for the Laplace operator (Dirichlet problem): to find the values of the parameter $\lambda\in\mathbb{R}$ for which the problem

$$\Delta V+\lambda V=0 \text{ in } G, \quad V|_{\partial G}=0 \tag{5}$$

has non-trivial solutions (eigenfunctions) $V\in C^2(\overline{G})$. Such values of parameter λ are eigenvalues of Laplace operator.

It is known ([7]) that there exists a countable set $\{\lambda_n\}$ of eigenvalues in problem (5), and they form an increasing sequence:

$$0<\lambda_1<\lambda_n<\lambda_{n+1}\to+\infty \qquad (2\le n\ \to+\infty).$$

For each $n\in\mathbb{N}$, the set of solutions to problem (5) at $\lambda=\lambda_n$ can be written in the form

$$\left\{CV_n(x),\, x\in\overline{G}\mid C\in\mathbb{R}\right\},$$

where V_n is normalized by $\int_G |V_n(x)|^2\,dx=1$.

Notice, that

$$\int_G V_n(x)V_m(x)\,dx=0 \quad \text{for all } \ n,m\in\mathbb{N} \quad \text{such that } \ n\neq m.$$

Under certain additional conditions for the domain G, functions $\varphi, \psi, I_k, k\in\mathbb{N}$, it is possible to obtain (in the corresponding functional spaces) of this functions expantion in Fourier series

$$\varphi(x)=\sum_{n\in\mathbb{N}}\varphi_n V_n(x), \quad \psi(x)=\sum_{n\in\mathbb{N}}\psi_n V_n(x), \quad I_k(x)=\sum_{n\in\mathbb{N}}I_{k,n}V_n(x), \quad k\in\mathbb{N}, \quad x\in\overline{G},$$

where

$$\varphi_n:=\int_G\varphi(x)V_n(x)dx, \quad \psi_n:=\int_G\psi(x)V_n(x)dx, \quad I_{k,n}:=\int_G I_k(x)V_n(x)dx, \quad k,n\in\mathbb{N}. \tag{6}$$

Then classical solution of the problem (1)–(4) can be found in the form

$$u(x,t)=\sum_{n\in\mathbb{N}}T_n(t)V_n(x), \quad (x,t)\in\overline{G}\times[0,+\infty), \tag{7}$$

where for each $n \in \mathbb{N}$ function $T_n \in C([0,+\infty)) \cap C^2(\cup_{k\in\mathbb{N}}(t_{k-1},t_k))$ such that

$$T_n''(t) + (a\mu_n)^2 T_n(t) = 0, \quad t \in \cup_{k\in\mathbb{N}}(t_{k-1},t_k), \tag{8}$$

$$T_n(0) = \varphi_n, \quad T_n'(0) = \psi_n, \tag{9}$$

$$T_n(t_k+0) = T_n(t_k-0), \quad T_n'(t_k+0) = T_n'(t_k-0) + I_{k,n}, \quad k \in \mathbb{N}, \tag{10}$$

where $\mu_n := \sqrt{\lambda_n}$.

We found the value of $T_n(t)$, $t \in [0,+\infty)$, consecutively at numerical intervals

$$[t_0,t_1], [t_1,t_2], \ \ldots\ , [t_{k-1},t_k], \ \ldots\ .$$

First we find the expression T_n on the interval $[t_0,t_1]$. For this from (8), (9) we have

$$T_n''(t) + (a\mu_n)^2 T_n(t) = 0, \quad t \in [t_0,t_1],$$
$$T_n(0) = \varphi_n, \quad T_n'(0) = \psi_n.$$

From here we find

$$T_n(t) = \varphi_n \cos a\mu_n t + \frac{\psi_n}{a\mu_n}\sin a\mu_n t, \quad t \in [t_0,t_1]. \tag{11}$$

Now we find the expression T_n on the interval $[t_1,t_2]$. From (8)–(11) we have

$$T_n''(t) + (a\mu_n)^2 T_n(t) = 0, \quad t \in (t_1,t_2],$$
$$T_n(t_1) = \varphi_n \cos a\mu_n t_1 + \frac{\psi_n}{a\mu_n}\sin a\mu_n t_1, \quad T_n'(t_1) = -a\mu_n\varphi_n \sin a\mu_n t_1 + \psi_n \cos a\mu_n t_1 + I_{1,n}.$$

From here we have

$$T_n(t) = \varphi_n \cos a\mu_n t + \frac{\psi_n}{a\mu_n}\sin a\mu_n t + \frac{I_{1,n}}{a\mu_n}\sin a\mu_n(t-t_1), \quad t \in (t_1,t_2]. \tag{12}$$

From (11) and (12) we get

$$T_n(t) = \varphi_n \cos a\mu_n t + \frac{\psi_n}{a\mu_n}\sin a\mu_n t + \frac{I_{1,n}}{a\mu_n}\theta(t-t_1)\sin a\mu_n(t-t_1), \quad t \in [t_0,t_2], \tag{13}$$

where $\theta = \begin{cases} 1, \text{if} & s \geq 0, \\ 0, \text{if} & s < 0, \end{cases}$ is the Heaviside function.

Similarly, we find the expression T_n on the following intervals (see (13)):

$$T_n(t) = \varphi_n \cos a\mu_n t + \frac{\psi_n}{a\mu_n}\sin a\mu_n t + \sum_{k\in\mathbb{N}} \frac{I_{k,n}}{a\mu_n}\theta(t-t_k)\sin a\mu_n(t-t_k), \quad t \in [0,+\infty). \tag{14}$$

From (7) and on the basis of (14), we obtain a formal solution to problem (1)–(4):

$$u(x,t) = \sum_{n\in\mathbb{N}} \Big[\varphi_n \cos a\mu_n t + \frac{\psi_n}{a\mu_n}\sin a\mu_n t + \sum_{k\in\mathbb{N}} \frac{I_{k,n}}{a\mu_n}\theta(t-t_k)\sin a\mu_n(t-t_k)\Big] V_n(x), \tag{15}$$

for $(x,t) \in \overline{G} \times [0,+\infty)$.

Let us give examples of such cases when problem (1)–(4) can be solved completely.

Example 1. Let $N = 1$, $G = (0,l)$, where $l > 0$ is some number. It is easy to verify that

$$\mu_n = \frac{\pi n}{l}, \quad V_n(x) = \sqrt{\frac{2}{l}}\sin\frac{\pi n}{l}x, \quad n \in \mathbb{N}, \quad x \in [0,l].$$

Based on the classical theory of trigonometric Fourier series, we obtain the following result:

Proposition 1. *Let*

$$\varphi \in C^3[0,l], \quad \psi \in C^2[0,l], \quad I_k \in C^2[0,l], \quad k \in \mathbb{N},$$
$$\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = 0, \quad \psi(0) = \psi(l) = 0, \quad I_k(0) = I_k(l) = 0, \quad k \in \mathbb{N}.$$

Then problem (1)–(4) *has a unique classical solution. It can be represented as the sum of the Fourier series* (15)*, where*

$$\varphi_n := \sqrt{\frac{2}{l}} \int_0^l \varphi(x) \sin \frac{\pi n}{l} x dx, \;\; \psi_n := \sqrt{\frac{2}{l}} \int_0^l \psi(x) \sin \frac{\pi n}{l} x dx, \;\; I_{k,n} = \sqrt{\frac{2}{l}} \int_0^l I_k(x) \sin \frac{\pi n}{l} x dx,$$

for $k, n \in \mathbb{N}$. *The series converges uniformly on the set* $\overline{G} \times [0, \infty)$ *together with the series obtained from it by termwise differentiation with respect to the spatial variables up to and including second order. Moreover, the series obtained by termwise differentiation of* (15) *with respect to* t *up to and including second order converge uniformly on set* $\overline{G} \times (\cup_{k\in\mathbb{N}}(t_{k-1}, t_k))$.

Example 2. Let $N \geq 2$ — arbitrary number, $G = (0, l_1) \times \cdots \times (0, l_N) \equiv \prod_{i=1}^N (0, l_i)$, $\overline{G} = \prod_{i=1}^N [0, l_i]$. Then we have

$$\mu_n := \pi \left(\sum_{i=1}^N \left(\frac{n_i}{l_i} \right)^2 \right)^{1/2}, \quad V_n(x) = \left(2^N / \prod_{i=1}^N l_i \right)^{1/2} \cdot \prod_{i=1}^N \sin \frac{\pi n_i}{l_i} x_i,$$
$$n = (n_1, \ldots, n_N) \in \mathbb{N}^N, \; x = (x_1, \ldots, x_N) \in \overline{G}.$$

Proposition 2. *Let*

$$\varphi(x) = \prod_{i=1}^N \varphi_{\langle i \rangle}(x_i), \; \psi(x) = \prod_{i=1}^N \psi_{\langle i \rangle}(x_i), \; I_k(x) = \prod_{i=1}^N I_{\langle i \rangle, k}(x_i), \; x = (x_1, \ldots, x_N) \in \overline{G},$$

where, for each $i \in \{1, \ldots, N\}$, *we have* $\varphi_{\langle i \rangle} \in C^3[0, l_i], \psi_{\langle i \rangle} \in C^2[0, l_i], \; I_{\langle i \rangle, k} \in C^2[0, l_i]$,

$$\varphi_{\langle i \rangle}(0) = \varphi_{\langle i \rangle}(l_i) = \varphi''_{\langle i \rangle}(0) = \varphi''_{\langle i \rangle}(l_i) = 0, \; \psi_{\langle i \rangle}(0) = \psi_{\langle i \rangle}(l_i) = 0, I_{\langle i \rangle, k}(0) = I_{\langle i \rangle, k}(l_i) = 0,$$

$k \in \mathbb{N}$. *Then problem* (1)–(4) *has one and only one classical solution. It is represented by formula* (15)*, and the series converges uniformly on the set* $\overline{G} \times [0, \infty)$*, together with the series obtained from it by termwise differentiation with respect to the spatial variables up to and including second order. Moreover, the series obtained by termwise differentiation of* (15) *with respect to* t *up to and including second order converge uniformly on set* $\overline{G} \times (\cup_{k\in\mathbb{N}}(t_{k-1}, t_k))$.

Open problem. By imposing additional conditions on $\varphi, \psi, I_k, k \in \mathbb{N}$, it remains to prove that the function u given by formula (15) is a classical solution to problem (1)–(4) in general. This reduces to establishing the convergence of the series on the right-hand side of (15), as well as the convergence of the series obtained from it by termwise differentiation with respect to the variables t and $x_1, \ldots, x_N$ up to second order, in an appropriate sense.

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