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STOCHASTIC PARABOLIC EQUATIONS ON GRAPHS

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A biodegradable stent is a mesh that is used to treat the narrow or closed part of the artery to open and restore normal blood flow, which is made of a biodegradable material. However, since the struts in the stent are thin, a simple structural model (called the one-dimensional curved rod model) can be used to mathematically model the stent. So differential diffusion equations can describe the degradation of the stent. The movement of the blood causes microscopic trembling of the edges of the wall. Therefore, the perturbation of the corresponding differential equation by a white noise type term should correspond to the real situation.
We consider some linear parabolic equations on graphs with white noise terms. To investigate the initial-boundary value problem for these equations, we reduce it to the corresponding deterministic problem. First, we prove the existence and uniqueness of the weak solution to deterministic problem for parabolic equations on graphs. Finally, same results are obtained for linear stochastic parabolic equations on graphs.

Introduction. Let $M, n \in \mathbb{N}$ be some numbers, G be a simple connected directed graph with vertices P_j $(j = \overline{1, M})$ and edges Ω_i $(i = \overline{1, n})$. We parameterize every edge Ω_i on the interval $(0, \ell_i)$ (i.e., let $\Omega_i := (0, \ell_i)$, for convenience).

Let J_j^- and J_j^+ be sets of all numbers of edges that enter and leave the vertex P_j $(j = \overline{1, M})$, respectively (see [1]). Here, entering and leaving vertices means it is due to the parametrization of the edges. For example, for graph from Fig. 1 we get $J_1^- = \varnothing$, $J_1^+ = \{1,\ 3,\ 4\}$, etc.

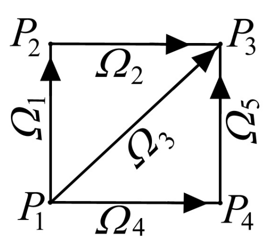

Fig 1: Example of graph.

By $\mathcal{G}$ we denote the set of all functions $\boldsymbol{z} = (z^1, \dots, z^n)$ such that $z^i \colon \overline{\Omega_i} \to \mathbb{R}$, $i = \overline{1, n}$, $\boldsymbol{z}$ is continuous in vertices, and the sum of all fluxes in each vertex equals zero, i.e.,

$$z^k(\ell_k) = z^d(\ell_d) = z^r(0) = z^s(0), \quad k, d \in J_j^-, \quad r, s \in J_j^+, \quad j = \overline{1, M}, \tag{1}$$

$$\sum_{k \in J_j^-} z_x^k(\ell_k) - \sum_{r \in J_j^+} z_x^r(0) = 0, \quad j = \overline{1, M}. \tag{2}$$

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We consider some problem for the stochastic parabolic equations on graphs. It solutions belong to $\mathcal{G}$ with respect to the spatial variables. Clearly, compatibility conditions (1)–(2) imply that we cannot observe the problem on each edge separately and have to solve it on the overall graph. Let us explain conditions (1)–(2).

Remark 1. Traditionally, for every $j \in \{1, \dots, M\}$, we have the following.

If the set $J_j^- \cup J_j^+$ consist of only one number, i.e., $J_j^- \cup J_j^+ = \{\alpha\}$, then all equalities in (1) that correspond this j are absent and equality (2) that corresponds this j has the form $z_x^\alpha(\ell_\alpha) = 0$ if $\alpha \in J_j^-$ and $z_x^\alpha(0) = 0$ if $\alpha \in J_j^+$.

If J_j^- has at least two elements and $J_j^+ = \varnothing$, then in equalities (1) that correspond this j the third and fourth terms are absent and in equality (2) that corresponds this j the second sum is absent.

If $J_j^- = \varnothing$ and J_j^+ has at least two elements, then in equalities (1) that correspond this j the first and second terms are absent and in equality (2) that corresponds this j the first sum is absent.

For the trivial case $n = 1$, $\Omega_1 := (0, \ell_1)$, $M = 2$ (see Fig. 2) we have the following: $J_1^- = \varnothing$, $J_1^+ = \{1\}$, $J_2^- = \{1\}$, and $J_2^+ = \varnothing$. So, in this trivial case, conditions (1)–(2) transform to $z_x^1(0) = 0,\ z_x^1(\ell_1) = 0$.

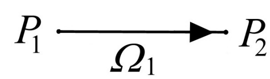


Fig. 2: Trivial graph.

Remark 2. The Kirchhoff-Neumann conditions (1)–(2) are the natural boundary conditions (see [2, p. 394]) for the differential operator $\frac{d^2}{dx^2}$ (for convenience, we write h_{xx} instead of $\frac{d^2h}{dx^2}$, etc.), because for smooth enough functions $\boldsymbol{z} = (z^1, \dots, z^n), \boldsymbol{v} = (v^1, \dots, v^n) \in \mathcal{G}$ we get

$$\sum_{i=1}^n \int_0^{\ell_i} z_{xx}^i(x) v^i(x)\, dx = L - \sum_{i=1}^n \int_0^{\ell_i} z_x^i(x) v_x^i(x)\, dx,$$

where

$$L = \sum_{i=1}^n (z_x^i(\ell_i)v^i(\ell_i) - z_x^i(0)v^i(0)) = \sum_{j=1}^M \left(\sum_{k \in J_j^-} z_x^k(\ell_k)v^k(\ell_k) - \sum_{r \in J_j^+} z_x^r(0)v^r(0) \right).$$

Conditions (1)–(2) imply that $L = 0$. Indeed, for this function $(v^1, \dots, v^n) \in \mathcal{G}$, corresponding conditions (1) imply that for every $j \in \{1, \dots, M\}$ there exists a constant p_j such that $v^k(\ell_k) = v^d(\ell_d) = v^r(0) = v^s(0) = p_j$, where $k, d \in J_j^-$, $r, s \in J_j^+$. Then (2) yields that

$$L = \sum_{j=1}^M p_j \left(\sum_{k \in J_j^-} z_x^k(\ell_k) - \sum_{r \in J_j^+} z_x^r(0) \right) = 0.$$

Rewriting the expressions above, we also get for $\boldsymbol{z}, \boldsymbol{v} \in \mathcal{G}$

$$-\sum_{i=1}^n \int_0^{\ell_i} z_x^i(x) v_x^i(x)\, dx = -L + \sum_{i=1}^n \int_0^{\ell_i} z_{xx}^i(x) v^i(x)\, dx = \sum_{i=1}^n \int_0^{\ell_i} z_{xx}^i(x) v^i(x)\, dx.$$

Differential equations on graph model many practical problems. For example, these equations describe the biodegradable stents. A biodegradable stent is a mesh that is used to treat the narrow or closed part of the artery to open and restore normal blood flow, that is made of biodegradable material (see [1], [3], [4] for more details). However, since struts in the stent are thin, a simple structure model, called the one-dimensional curved rod model, can be used (see [1], [5]). The movement of the blood causes microscopic trembling of the edges of the wall and losing of the stent material. Therefore, the perturbation of the corresponding differential equation by a white noise type term should correspond to the real situation.

Differential equations on graph and its application are considered in [2], [6], [7], [8], [9], [10], etc. Stochastic parabolic equations and its application are considered in [11], [12], [13], [14], [15], [16], etc. Differential equations on graph with the random parameter in the main part of the equations are considered in [17], [18], [19], [20], [21], etc. Here we found the sufficient conditions of the existence and uniqueness of the weak solution to problem (57)–(59) below for the stochastic parabolic equations perturbed by the white noise term. The corresponding problem without white noise term is considered in paper [1].

The paper is organized as follows. In Section 1 we consider an auxiliary deterministic (i.e., not random) problem. Some facts from stochastic calculus is in Section 2. The main results are in Section 3.

1. Deterministic problem. Let $T > 0$, $\ell_i > 0$ be numbers, $\Omega_i = (0, \ell_i)$, $Q^i_{0,T} = \Omega_i \times (0, T)$, $i = \overline{1, n}$. In this section we seek a function $\widetilde{\boldsymbol{u}} = (\widetilde{u}^{\,1}, \ldots, \widetilde{u}^{\,n})$ such that $\widetilde{u}^i \colon Q^i_{0,T} \to \mathbb{R}$,

$$\widetilde{u}^i_t - a\, \widetilde{u}^i_{xx} = \widetilde{f}^i(x, t), \quad (x, t) \in Q^i_{0,T}, \quad i = \overline{1, n}, \tag{3}$$

$$\begin{cases} \widetilde{u}^k(\ell_k, t) = \widetilde{u}^d(\ell_d, t) = \widetilde{u}^r(0, t) = \widetilde{u}^s(0, t), \ k, d \in J^-_j, \quad r, s \in J^+_j, \\ \sum\limits_{k \in J^-_j} \widetilde{u}^k_x(\ell_k, t) - \sum\limits_{r \in J^+_j} \widetilde{u}^r_x(0, t) = 0, \end{cases} \quad j = \overline{1, M}, \ t \in (0, T), \tag{4}$$

$$\widetilde{u}^i(x, 0) = u^i_0(x), \quad x \in \Omega_i, \quad i = \overline{1, n}, \tag{5}$$

where $a > 0$, $\widetilde{f}^i \colon Q^i_{0,T} \to \mathbb{R}$, and $u^i_0 \colon \Omega_i \to \mathbb{R}$ are given, $i = \overline{1, n}$.

Let us introduce notation. Let $\|\cdot\|_x \equiv \|\cdot ; X\|$ be a norm in some Banach space X, X^* be a dual space, $\langle \cdot, \cdot \rangle_X$ be a scalar product between X^* and X, $(\cdot, \cdot)_Y$ be an inner product in some Hilbert space Y, and $|\cdot|_Y := \sqrt{(\cdot, \cdot)_Y}$ be a norm of Y. Let $\boldsymbol{X} := X^1 \times \ldots \times X^n$ be the Cartesian product of some Banach spaces $X^1, \ldots, X^n$. If $\boldsymbol{v} = (v^1, \ldots, v^n) \in \boldsymbol{X}$, then we set $\|v; \boldsymbol{X}\| := (\|v^1; X^1\|^2 + \ldots + \|v^n; X^n\|^2)^{1/2}$. If $\boldsymbol{Y} := Y^1 \times \ldots \times Y^n$ be the Cartesian product of some Hilbert spaces $Y^1, \ldots, Y^n$ with the inner products $(\cdot, \cdot)_{Y^i}$, $i = \overline{1, n}$, respectively, then $\boldsymbol{Y}$ is the Hilbert space with the inner product $(\boldsymbol{v}, \boldsymbol{y})_{\boldsymbol{Y}} := (v^1, y^1)_{Y^1} + \ldots + (v^n, y^n)_{Y^n}$ and with the norm $|\boldsymbol{y}|_{\boldsymbol{Y}} := \sqrt{(\boldsymbol{y}, \boldsymbol{y})_{\boldsymbol{Y}}}$, where $\boldsymbol{v} = (v^1, \ldots, v^n)$, $\boldsymbol{y} = (y^1, \ldots, y^n) \in \boldsymbol{Y}$.

Suppose that $\ell > 0$, $p \in [1, \infty]$, $m \in \mathbb{N}$, $L^p(0, \ell)$ is the Lebesgue space (see [22, p. 22-24]), $W^{m,p}(0, \ell)$ and $W^{m,p}_0(0, \ell)$ are the Sobolev spaces (see [22, p. 45]), $H^m(0, \ell) := W^{m,2}(0, \ell)$, and $H^m_0(0, \ell) := W^{m,2}_0(0, \ell)$. Let $C([0, T]; X)$ and $C^1([0, T]; X)$ be spaces of the X-valued smooth functions (see [14, p. 15], [23, p. 190]), $L^p(0, T; X)$ be the Lebesgue-Bochner space (see [14, p. 7]), $L^1_{\mathrm{loc}}(0, T; X)$ be the space of the locally integrable functions (see [14, p. 7]), $W^{m,p}(0, T; X)$ be the Sobolev-Bochner space (see [14, p. 15]), and $H^m(0, T; X) := W^{m,2}(0, T; X)$, where X is some Banach space. By definition, put

$$V := \Big\{ \boldsymbol{v} \in H^1(\Omega_1) \times \ldots \times H^1(\Omega_n) \colon$$
$$v^k(\ell_k) = v^d(\ell_d) = v^r(0) = v^s(0), \quad k, d \in J^-_j, \quad r, s \in J^+_j, \quad j = \overline{1, M} \Big\}, \tag{6}$$

$$V_2 := H^2(\Omega_1) \times \ldots \times H^2(\Omega_n), \qquad H := L^2(\Omega_1) \times \ldots \times L^2(\Omega_n). \tag{7}$$

Here V and H are Hilbert spaces and

$$\|\boldsymbol{v}\| \equiv \|\boldsymbol{v}\|_V := \left(\sum_{i=1}^{n} \|v^i\|^2_{H^1(\Omega_i)}\right)^{1/2}, \qquad |\boldsymbol{v}|_H \equiv \|\boldsymbol{v}\|_H := \left(\sum_{i=1}^{n} \|v^i\|^2_{L^2(\Omega_i)}\right)^{1/2}, \tag{8}$$

are norms on them, respectively. Moreover, $V \overset{-}{\circlearrowleft} H$, i.e., the space V is densely and continuously embedded in H (see [1, p. 3]). Thus, we have the Helfand triple $V \overset{-}{\circlearrowleft} H \cong H^* \overset{-}{\circlearrowleft} V^*$ (see [23, p. 232-233] for more details).

From [24, p. 188] we have that

$$H^1(0,T;V,V^*) := \{\boldsymbol{u} \in L^2(0,T;V) \mid \boldsymbol{u}_t \in L^2(0,T;V^*)\} \tag{9}$$

is a Hilbert space and $\|\boldsymbol{u}; H^1(0,T;V,V^*)\| = (\|\boldsymbol{u}; L^2(0,T;V)\|^2 + \|\boldsymbol{u}_t; L^2(0,T;V^*)\|^2)^{1/2}$ is the norm on it. Moreover, $C^1([0,T];V)$ is dense in $H^1(0,T;V,V^*)$, $H^1(0,T;V,V^*) \subset C([0,T];H)$, and the following integration by parts formula it holds

$$\int_{t_1}^{t_2} \langle \boldsymbol{u}_t(t), \boldsymbol{v}(t)\rangle_V \, dt = (\boldsymbol{u}(t_2), \boldsymbol{v}(t_2))_H - (\boldsymbol{u}(t_1), \boldsymbol{v}(t_1))_H - \int_{t_1}^{t_2} \langle \boldsymbol{v}_t(t), \boldsymbol{u}(t)\rangle_V \, dt, \tag{10}$$

where $0 \le t_1 < t_2 \le T$ and $\boldsymbol{u}, \boldsymbol{v} \in H^1(0,T;V,V^*)$ (see [24, p. 190-191]). Clearly, if $\boldsymbol{v} = \boldsymbol{u}$, then (10) implies that

$$\int_{t_1}^{t_2} \langle \boldsymbol{u}_t(t), \boldsymbol{u}(t)\rangle_V \, dt = \frac{1}{2}|\boldsymbol{u}(t_2)|^2_H - \frac{1}{2}|\boldsymbol{u}(t_1)|^2_H. \tag{11}$$

By $\mathfrak{a}\colon V \times V \to \mathbb{R}$, we define the bilinear form

$$\mathfrak{a}(\boldsymbol{z}, \boldsymbol{v}) := a \sum_{i=1}^{n} \int_0^{\ell_i} z^i_x(x) v^i_x(x) \, dx \equiv a(\boldsymbol{z}_x, \boldsymbol{v}_x)_H, \quad \boldsymbol{z}, \boldsymbol{v} \in V. \tag{12}$$

Note that (see [1, p. 3]) the constant functions are in the kernel of $\mathfrak{a}$ and it holds

$$\mathfrak{a}(\boldsymbol{v}, \boldsymbol{v}) + a|\boldsymbol{v}|^2_H = a\|\boldsymbol{v}\|^2, \quad \boldsymbol{v} \in V. \tag{13}$$

Suppose that the following conditions are satisfied:

(A): $a > 0$;

(FD): $\widetilde{\boldsymbol{f}} := (\widetilde{f}^1, \ldots, \widetilde{f}^n) \in L^2(0,T;H)$, where H is taken from (7);

(UD): $\boldsymbol{u}_0 := (u^1_0, \ldots, u^n_0) \in H$.

Definition 1. A vector-valued function $\widetilde{\boldsymbol{u}} \in H^1(0,T;V,V^*)$, where V is taken from (6), is called a *weak solution* of problem (3)–(5) if for all $\boldsymbol{v} \in L^2(0,T;V)$ we have that

$$\int_0^T \langle \widetilde{\boldsymbol{u}}_t(t), \boldsymbol{v}(t)\rangle_V \, dt + \int_0^T \mathfrak{a}(\widetilde{\boldsymbol{u}}(t), \boldsymbol{v}(t)) \, dt = \int_0^T (\widetilde{\boldsymbol{f}}(t), \boldsymbol{v}(t))_H \, dt \tag{14}$$

holds and

$$\widetilde{\boldsymbol{u}}(0) = \boldsymbol{u}_0. \tag{15}$$

Since $H^1(0,T;V,V^*) \subset C([0,T];H)$, then initial condition (15) is understood in sense of the space $C([0,T];H)$.

Note that every classical (i.e., smooth enough) solution $\widetilde{\boldsymbol{u}}$ to problem (3)–(5) is a weak solution of this problem because if we multiply (3) by the element v^i of the test function $\boldsymbol{v} \in V$, integrating over x and integrating by parts, similarly as in Remark 2, we get

$$\Big(-\widetilde{\boldsymbol{u}}_{xx}(t), \boldsymbol{v}(t)\Big)_H = -\sum_{i=1}^{n} \int_0^{\ell_i} \widetilde{u}^i_{xx}(x,t) v^i(x,t)\, dx = \Big(\widetilde{\boldsymbol{u}}_x(t), \boldsymbol{v}_x(t)\Big)_H, \quad t \in [0,T].$$

We will use the following statement.

Proposition 1 (orthonormal basis on graphs, see Lemma 2 [1], p. 5-6)**.** *There exist sequences* $\{\lambda_\mu\}_{\mu=0}^{\infty}$ *and* $\{\boldsymbol{w}^\mu\}_{\mu=0}^{\infty}$ *such that*

(i) $\{\boldsymbol{w}^\mu\}_{\mu=0}^{\infty} \subset V \cap V_2$ *is an orthonormal basis for the space* H*;*

(ii) $\lambda_0 = 0$, $\boldsymbol{w}^0 = \Big(\Big(\sum\limits_{i=1}^{n} \ell_i\Big)^{-1/2}, \dots, \Big(\sum\limits_{i=1}^{n} \ell_i\Big)^{-1/2}\Big)$ *(then* $|\boldsymbol{w}^0|_H = 1$*);*

(iii) every function $\boldsymbol{w}^\mu = (w^{\mu,1}, \dots, w^{\mu,n})$ *satisfies the following*

$$-w^{\mu,i}_{xx} = \lambda_\mu w^{\mu,i}, \quad i = \overline{1,n}. \tag{16}$$

Let us prove the following results.

Theorem 1. *Suppose that conditions* ***(A)***, ***(FD)***, ***(UD)*** *hold. Then the following statements are true:*

1) (Uniqueness) Problem (3)–(5) has at most one weak solution.

2) (A priori estimate) Every weak solution of problem (3)–(5) satisfies the following inequality

$$\|\widetilde{\boldsymbol{u}}; C([0,T];H)\| + \|\widetilde{\boldsymbol{u}}; L^2(0,T;V)\| \le C_1\Big(|\boldsymbol{u}_0|_H + \|\widetilde{\boldsymbol{f}}; L^2(0,T;H)\|\Big), \tag{17}$$

where the constant $C_1 > 0$ *depends only on the constants* T *and* a *from (3).*

3) (Existence) There exists a weak solution $\widetilde{\boldsymbol{u}}$ *of problem (3)–(5).*

For the sake of convenience, let us denote by $SP(\boldsymbol{u}_0, \widetilde{\boldsymbol{f}})$ the set of all weak solutions of problem (3)–(5). Theorem 1 yields that $SP(\boldsymbol{u}_0, \widetilde{\boldsymbol{f}}) \neq \varnothing$ and if $\widetilde{\boldsymbol{u}} \in SP(\boldsymbol{u}_0, \widetilde{\boldsymbol{f}})$, then $\widetilde{\boldsymbol{u}}$ is an unique weak solution of problem (3)–(5).

Proof of Theorem 1. Uniqueness. Suppose that $\widetilde{\boldsymbol{u}}^1, \widetilde{\boldsymbol{u}}^2 \in SP(\boldsymbol{u}_0, \widetilde{\boldsymbol{f}})$ and $\widehat{\boldsymbol{u}} := \widetilde{\boldsymbol{u}}^1 - \widetilde{\boldsymbol{u}}^2$. Then $\widehat{\boldsymbol{u}}(0) = 0$. Let $\tau \in (0,T]$ and

$$\chi_{0,\tau}(t) = \begin{cases} 1, & \text{if} \quad t \in [0,\tau]; \\ 0, & \text{if} \quad t \notin [0,\tau]. \end{cases} \tag{18}$$

From (14) with $\boldsymbol{v} = \chi_{0,\tau}\widehat{\boldsymbol{u}}$, we easily get

$$\int_0^\tau \langle \widehat{\boldsymbol{u}}_t(t), \widehat{\boldsymbol{u}}(t)\rangle_V \, dt + \int_0^\tau \mathfrak{a}(\widehat{\boldsymbol{u}}(t), \widehat{\boldsymbol{u}}(t))\, dt = 0, \quad \tau \in (0,T]. \tag{19}$$

Since $\widehat{\boldsymbol{u}} \in H^1(0,T;V,V^*)$, using integration by parts formula of type (11), from (19) we obtain that $\frac{1}{2}|\widehat{\boldsymbol{u}}(\tau)|_H^2 \le \frac{1}{2}|\widehat{\boldsymbol{u}}(0)|_H^2 = 0$. So, $\widehat{\boldsymbol{u}} = 0$ and $\widetilde{\boldsymbol{u}}^1 = \widetilde{\boldsymbol{u}}^2$.

A priori estimate. For $\widetilde{\boldsymbol{u}} \in SP(\boldsymbol{u}_0, \widetilde{\boldsymbol{f}})$ and $\boldsymbol{v} = \chi_{0,\tau}\widetilde{\boldsymbol{u}}$, from (14), using integration by parts formula of type (11), we obtain

$$\frac{1}{2}|\widetilde{\boldsymbol{u}}(\tau)|_H^2 - \frac{1}{2}|\widetilde{\boldsymbol{u}}(0)|_H^2 + \int_0^\tau \mathfrak{a}(\widetilde{\boldsymbol{u}}(t), \widetilde{\boldsymbol{u}}(t))\, dt = \int_0^\tau (\widetilde{\boldsymbol{f}}(t), \widetilde{\boldsymbol{u}}(t))_H \, dt, \quad \tau \in (0,T]. \tag{20}$$

The Cauchy inequality

$$|\alpha\beta| \le \frac{\alpha^2}{2} + \frac{\beta^2}{2}, \quad \alpha, \beta \in \mathbb{R}, \tag{21}$$

implies that

$$|(\widetilde{\boldsymbol{f}}, \widetilde{\boldsymbol{u}})_H| \le |\widetilde{\boldsymbol{f}}|_H \cdot |\widetilde{\boldsymbol{u}}|_H \le \frac{1}{2}|\widetilde{\boldsymbol{f}}|_H^2 + \frac{1}{2}|\widetilde{\boldsymbol{u}}|_H^2.$$

Then from (13) and (20), we obtain

$$\frac{1}{2}|\widetilde{\boldsymbol{u}}(\tau)|_H^2 + a\int_0^\tau \|\widetilde{\boldsymbol{u}}(t)\|^2\,dt \le \frac{1}{2}|\boldsymbol{u}_0|_H^2 + \frac{1}{2}\int_0^\tau |\widetilde{\boldsymbol{f}}(t)|_H^2\,dt + \left(\frac{1}{2}+a\right)\int_0^\tau |\widetilde{\boldsymbol{u}}(t)|_H^2\,dt. \tag{22}$$

Taking into account the Gronwall-Bellman Lemma (see, for example, Proposition 3.23 [25, p. 872]), from (22), we get (17).

Existence. For example, from Theorem 11.7 ([24, p. 192]), we easily get that linear problem (3)–(5) has a weak solution $\widetilde{\boldsymbol{u}}$. We present only sketch of the proof for convenience.

Let us use the Faedo-Galerkin method. Take sequences $\{\lambda_\mu\}_{\mu=0}^\infty$ and $\{\boldsymbol{w}^\mu\}_{\mu=0}^\infty$ from Proposition 1. Let us define the function $\widetilde{\boldsymbol{u}}^m\colon [0,T] \to V$ by the rule

$$\widetilde{\boldsymbol{u}}^m(t) = \sum_{\mu=0}^m \varphi_\mu^m(t)\boldsymbol{w}^\mu, \quad t\in[0,T], \quad m \in \mathbb{N}\cup\{0\}, \tag{23}$$

where the real-valued functions $\varphi_0^m, \dots, \varphi_m^m$ satisfy the following problem

$$(\widetilde{\boldsymbol{u}}_t^m(t), \boldsymbol{w}^\mu)_H + \mathfrak{a}(\widetilde{\boldsymbol{u}}^m(t), \boldsymbol{w}^\mu) = (\widetilde{\boldsymbol{f}}(t), \boldsymbol{w}^\mu)_H, \quad t \in [0,T], \tag{24}$$

$$\varphi_\mu^m(0) = (\boldsymbol{u}_0, \boldsymbol{w}^\mu)_H, \quad \mu = \overline{0,m}. \tag{25}$$

Note that the function $\boldsymbol{u}_0^m := \sum_{\mu=0}^m \varphi_\mu^m(0)\boldsymbol{w}^\mu$ satisfies the following

$$\boldsymbol{u}_0^m \underset{m\to\infty}{\longrightarrow} \boldsymbol{u}_0 \quad \text{strongly in} \quad H. \tag{26}$$

Moreover, the orthonormality of $\{\boldsymbol{w}^\mu\}_{\mu=0}^\infty$ in the space H, and the Bessel inequality for the Fourier series (see Theorem 1 [26, p. 323]) imply that, for every $m \in \mathbb{N}\cup\{0\}$, it holds

$$|\boldsymbol{u}_0^m|_H^2 = \left|\sum_{\mu=0}^m \varphi_\mu^m(0)\boldsymbol{w}^\mu\right|_H^2 = \sum_{\mu=0}^m |\varphi_\mu^m(0)|^2 \le |\boldsymbol{u}_0|_H^2. \tag{27}$$

Let $H_m = \mathrm{Lin}\{\boldsymbol{w}^0, \dots, \boldsymbol{w}^m\}$ be a linear span of the set $\{\boldsymbol{w}^0, \dots, \boldsymbol{w}^m\}$ (see [26, p. 138] for more details). Problem (24)–(25) is the initial-value problem for a linear system of the ordinary differential equations. From [1], we have that there exists an unique solution to problem (24)–(25) such that $\widetilde{\boldsymbol{u}}^m \in H^1([0,T]; H_m)$.

Multiplying both sides of μ-th equality of (24) by $\varphi_\mu^m(t)$, summing, and integrating the obtained equalities, we get

$$\int_0^\tau \Big[(\widetilde{\boldsymbol{u}}_t^m(t), \widetilde{\boldsymbol{u}}^m(t))_H + \mathfrak{a}(\widetilde{\boldsymbol{u}}^m(t), \widetilde{\boldsymbol{u}}^m(t))\Big]\,dt = \int_0^\tau (\widetilde{\boldsymbol{f}}(t), \widetilde{\boldsymbol{u}}^m(t))_H\,dt, \quad \tau \in (0,T]. \tag{28}$$

By the Cauchy inequality (21), we obtain

$$|(\widetilde{\boldsymbol{f}}, \widetilde{\boldsymbol{u}}^m)_H| \le |\widetilde{\boldsymbol{f}}|_H \cdot |\widetilde{\boldsymbol{u}}^m|_H \le \frac{1}{2}|\widetilde{\boldsymbol{f}}|_H^2 + \frac{1}{2}|\widetilde{\boldsymbol{u}}^m|_H^2.$$

Then, using an integration by parts formula, (13), and (27), from (28), we obtain

$$\frac{1}{2}|\widetilde{\boldsymbol{u}}^m(\tau)|_H + a\int_0^\tau \|\widetilde{\boldsymbol{u}}^m(t)\|^2\,dt \le \frac{1}{2}|\boldsymbol{u}_0|_H^2 + \frac{1}{2}\int_0^T |\widetilde{\boldsymbol{f}}(t)|_H^2\,dt + \Big(\frac{1}{2}+a\Big)\int_0^\tau |\widetilde{\boldsymbol{u}}^m(t)|_H^2\,dt. \quad (29)$$

Taking into account the Gronwall-Bellman Lemma (see, for example, Proposition 3.23 [25, p. 872]), from (29), we get

$$\max_{0\le t\le T} |\widetilde{\boldsymbol{u}}^m(t)|_H^2 + \int_0^T \|\widetilde{\boldsymbol{u}}^m(t)\|^2\,dt \le C_2\Big[|\boldsymbol{u}_0|_H^2 + \int_0^T |\widetilde{\boldsymbol{f}}(t)|_H^2\,dt\Big], \quad (30)$$

where the constant $C_2 > 0$ depends only on T and a.

From (30), the convergence in (24) on a subsequence now follows, in corresponding topologies. The limit function $\widetilde{\boldsymbol{u}}$ of the sequence $\{\widetilde{\boldsymbol{u}}^m\}_{m=0}^\infty$ satisfies the conditions of Definition 1 and it is a solution of problem (3)–(5). □

Theorem 2 (continuous dependence)**.** *Suppose conditions **(A)**, **(FD)**, **(UD)** hold. Then the solution $\widetilde{\boldsymbol{u}}$ of problem (3)–(5) continuously depends on input data $\boldsymbol{u}_0, \widetilde{\boldsymbol{f}}$, i.e., if $\widetilde{\boldsymbol{u}}^1 \in SP(\boldsymbol{u}_0^1, \widetilde{\boldsymbol{f}}^1)$ and $\widetilde{\boldsymbol{u}}^2 \in SP(\boldsymbol{u}_0^2, \widetilde{\boldsymbol{f}}^2)$, then*

$$\max_{0\le t\le T} |\widetilde{\boldsymbol{u}}^1(t) - \widetilde{\boldsymbol{u}}^2(t)|_H^2 + \int_0^T \|\widetilde{\boldsymbol{u}}^1(t) - \widetilde{\boldsymbol{u}}^2(t)\|^2\,dt \le C_3\Big[|\boldsymbol{u}_0^1 - \boldsymbol{u}_0^2|_H^2 + \int_0^T |\widetilde{\boldsymbol{f}}^1(t) - \widetilde{\boldsymbol{f}}^2(t)|_H^2\,dt\Big], \quad (31)$$

where the constant $C_3 > 0$ depends only on the constants T and a from (3).

Proof. Let $\widetilde{\boldsymbol{u}}^1 \in SP(\boldsymbol{u}_0^1, \widetilde{\boldsymbol{f}}^1)$ and $\widetilde{\boldsymbol{u}}^2 \in SP(\boldsymbol{u}_0^2, \widetilde{\boldsymbol{f}}^2)$. By definition, put $\widehat{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}^1 - \widetilde{\boldsymbol{u}}^2$, $\widehat{\boldsymbol{u}}_0 = \boldsymbol{u}_0^1 - \boldsymbol{u}_0^2$, and $\widehat{\boldsymbol{f}} = \widetilde{\boldsymbol{f}}^1 - \widetilde{\boldsymbol{f}}^2$. Let $\tau \in (0, T]$ and $\chi_{0,\tau}$ is taken from (18). Then from (14) with $\boldsymbol{v} = \chi_{0,\tau}\widehat{\boldsymbol{u}}$, we easily get

$$\int_0^\tau \langle \widehat{\boldsymbol{u}}_t(t), \widehat{\boldsymbol{u}}(t)\rangle_V\,dt + \int_0^\tau \mathfrak{a}(\widehat{\boldsymbol{u}}(t), \widehat{\boldsymbol{u}}(t))\,dt = \int_0^\tau (\widehat{\boldsymbol{f}}(t), \widehat{\boldsymbol{u}}(t))_H\,dt, \quad \tau \in (0, T]. \quad (32)$$

Since $\widehat{\boldsymbol{u}} \in H^1(0, T; V, V^*)$, using integration by parts formula of type (11), similarly as (30), from (32) we obtain (31). □

Now, suppose that the function $\widetilde{\boldsymbol{f}} = (\widetilde{f}^1, \dots, \widetilde{f}^n)$ has the form

$$\widetilde{\boldsymbol{f}} = \boldsymbol{f} + a\,\boldsymbol{b}_{xx}, \quad \text{i.e.,} \quad \widetilde{f}^i(x,t) = f^i(x,t) + a\,b^i_{xx}(x,t), \quad (x,t) \in Q^i_{0,T}, \quad i = \overline{1,n}, \quad (33)$$

where $a > 0$ is taken from (3) and the following conditions are hold

(FDD): $\boldsymbol{f} = (f^1, \dots, f^n)$, $f^i \in L^2(Q^i_{0,T})$, $i = \overline{1,n}$;

(B): $\boldsymbol{b} = (b^1, \dots, b^n)$, $b^i \in L^2(0, T; H_0^2(\Omega_i))$, $i = \overline{1,n}$.

Let $SP(\boldsymbol{u}_0, \boldsymbol{f}, \boldsymbol{b}) := SP(\boldsymbol{u}_0, \widetilde{\boldsymbol{f}})$ be a set of the weak solutions $\widetilde{\boldsymbol{u}}$ of problem (3)–(5) in sense of Definition 1 (we indicate the dependence on $\boldsymbol{f}$ and $\boldsymbol{b}$ in the notation for convenience).

Remark 3. Clearly, if conditions **(FDD)** and **(B)** holds, then the function $\widetilde{\boldsymbol{f}}$ from (33) satisfies condition **(FD)** and the corresponding solution of problem (3)–(5) exists (so, we have that $SP(\boldsymbol{u}_0, \boldsymbol{f}, \boldsymbol{b}) \ne \varnothing$), it is unique and continuously depends on input data.

We will need some modification of Theorem 2. Let

$$Z_{01} = H_0^1(\Omega_1) \times \ldots \times H_0^1(\Omega_n), \qquad Z_{02} = H_0^2(\Omega_1) \times \ldots \times H_0^2(\Omega_n). \tag{34}$$

We consider these spaces with the norms $\|\boldsymbol{z}\|_{Z_{01}} = |\boldsymbol{z}_x|_H$ and $\|\boldsymbol{z}\|_{Z_{02}} = |\boldsymbol{z}_{xx}|_H$ (see notation (8)), respectively. Clearly, if condition **(B)** holds, then $\boldsymbol{b}(t) \in Z_{02} \subset Z_{01}$ for a.e. $t \in (0, T)$.

Theorem 3 (continuous dependence in special case of data-in). *Suppose that the function $\widetilde{\boldsymbol{f}}$ satisfies (33) and conditions **(A)**, **(FDD)**, **(B)**, **(UD)** hold. Then the solution $\widetilde{\boldsymbol{u}}$ of problem (3)–(5) continuously depends on input data $\boldsymbol{u}_0, \boldsymbol{f}, \boldsymbol{b}$, i.e., if $\widetilde{\boldsymbol{u}}^1 \in SP(\boldsymbol{u}_0^1, \boldsymbol{f}^1, \boldsymbol{b}^1)$ and $\widetilde{\boldsymbol{u}}^2 \in SP(\boldsymbol{u}_0^2, \boldsymbol{f}^2, \boldsymbol{b}^2)$, then*

$$\max_{0\le t\le T} |\widetilde{\boldsymbol{u}}^1(t) - \widetilde{\boldsymbol{u}}^2(t)|_H^2 + \int_0^T \|\widetilde{\boldsymbol{u}}^1(t) - \widetilde{\boldsymbol{u}}^2(t)\|^2\, dt \le$$

$$\le C_4\Big\{|\boldsymbol{u}_0^1 - \boldsymbol{u}_0^2|_H^2 + \int_0^T \Big[|\boldsymbol{f}^1(t) - \boldsymbol{f}^2(t)|_H^2 + |\boldsymbol{b}_x^1(t) - \boldsymbol{b}_x^2(t)|_H^2\Big]\, dt\Big\}, \tag{35}$$

where the constant $C_4 > 0$ depends only on the constants T and a from (3).

Proof. Let $\widetilde{\boldsymbol{u}}^s \in SP(\boldsymbol{u}_0^s, \boldsymbol{f}^s, \boldsymbol{b}^s)$, $\widetilde{\boldsymbol{f}}^s = \boldsymbol{f}^s + a\, \boldsymbol{b}_{xx}^s$, $s = 1, 2$. By definition, put $\widehat{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}^1 - \widetilde{\boldsymbol{u}}^2$, $\widehat{\boldsymbol{u}}_0 = \boldsymbol{u}_0^1 - \boldsymbol{u}_0^2$, $\widehat{\boldsymbol{f}} = \widetilde{\boldsymbol{f}}^1 - \widetilde{\boldsymbol{f}}^2$, $\boldsymbol{f} = \boldsymbol{f}^1 - \boldsymbol{f}^2$, $\widehat{\boldsymbol{b}} = \boldsymbol{b}^1 - \boldsymbol{b}^2$, and (32) holds. Integrating by parts (notice that $b_x^{s,i}|_{x=0} = b_x^{s,i}|_{x=\ell_i} = 0$, $s = 1, 2$, $i = \overline{1, n}$), we get

$$\Big(\widehat{\boldsymbol{f}}(t), \widehat{\boldsymbol{u}}(t)\Big)_H = \Big(\boldsymbol{f}(t) + a\, \widehat{\boldsymbol{b}}_{xx}(t), \widehat{\boldsymbol{u}}(t)\Big)_H = \Big(\boldsymbol{f}(t), \widehat{\boldsymbol{u}}(t)\Big)_H - \mathfrak{a}\Big(\widehat{\boldsymbol{b}}(t), \widehat{\boldsymbol{u}}(t)\Big).$$

Then from (32) we obtain

$$\int_0^\tau \langle \widehat{\boldsymbol{u}}_t(t), \widehat{\boldsymbol{u}}(t)\rangle_V\, dt + \int_0^\tau \mathfrak{a}(\widehat{\boldsymbol{u}}(t), \widehat{\boldsymbol{u}}(t))\, dt = \int_0^\tau \Big[(\boldsymbol{f}(t), \widehat{\boldsymbol{u}}(t))_H - \mathfrak{a}(\widehat{\boldsymbol{b}}(t), \widehat{\boldsymbol{u}}(t))\Big]\, dt$$

Using the estimate $|-\mathfrak{a}(\widehat{\boldsymbol{b}}, \widehat{\boldsymbol{u}})| = a|(\widehat{\boldsymbol{b}}_x, \widehat{\boldsymbol{u}}_x)_H| \le \frac{a}{2}|\widehat{\boldsymbol{b}}_x|_H^2 + \frac{a}{2}|\widehat{\boldsymbol{u}}_x|_H^2$, similarly as (31), we obtain (35). □

2. Properties of white noise. Let us recall some facts from stochastic calculus. Let $(\mathbb{S}, \mathcal{F}, \mathbb{P})$ be a complete probability space (i.e., $\mathbb{S}$ is a sample space, $\mathcal{F}$ is a σ-algebra of the subsets of $\mathbb{S}$, and $\mathbb{P}$ is a probability measure).

For a random variable $\xi\colon \mathbb{S} \to \mathbb{R}$ (i.e., measurable function with respect to $\mathcal{F}$) let us denote its mathematical expectation by $\mathbb{E}[\xi] := \int_{\mathbb{S}} \xi(\omega)\, \mathbb{P}(d\omega)$.

Let X be some Banach space and $q \in [1, \infty]$. We will use the notation $L_q(\mathbb{S})$ and $L_q(\mathbb{S}; X)$ from [27, p. 54] for denoting the *random Lebesgue space* and *random Lebesgue-Bochner space*, respectively. Recall theirs definitions and write L_q instead of $L_q(\mathbb{S})$ for convenience.

Definition 2. If $q \in [1, \infty)$, then L_q consists of all real-valued random variables $\eta\colon \mathbb{S} \to \mathbb{R}$ such that

$$\|\eta\|_{L_q} := \Big(\mathbb{E}\Big[|\eta|^q\Big]\Big)^{1/q} \equiv \Big(\int_{\mathbb{S}} |\eta(\omega)|^q\, \mathbb{P}(d\omega)\Big)^{1/q} < \infty.$$

If $q = \infty$, then L_∞ consists of all $\mathbb{P}$-essentially bounded real-valued random variables (see [28, p. 324-325] for more details), i.e., the random variables $\eta\colon \mathbb{S} \to \mathbb{R}$ such that

$$\|\eta\|_{L_\infty} := \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega\in\mathbb{S}} |\eta(\omega)| \equiv \inf\{K > 0\colon |\eta(\omega)| \le K \ \ \mathbb{P}\text{-almost every (a.e.) } \omega \in \mathbb{S}\} < \infty.$$

Note that $\eta_1 = \eta_2$ in L_q iff $\eta_1(\omega) = \eta_2(\omega)$ $\mathbb{P}$-a.e. $\omega \in \mathbb{S}$, i.e., almost surely (a.s.)

The following Proposition is needed for the sequel.

Proposition 2. *1) (see* [28, p. 325]*) The linear space* $\{L_q, \|\cdot\|_{L_q}\}$ *is Banach space;*
2) (see. [29, p. 17]*)* $L_p \subset L_q$ *if* $p \geq q$*;*
3) (see. [29, p. 17]*) if* $q = 2$*, then* L_2 *is Hilbert space with respect to the inner product*

$$(\eta_1, \eta_2)_{L_2} := \mathbb{E}[\eta_1 \cdot \eta_2].$$

We will write $\eta = \underset{k\to\infty}{\text{l.i.m.}}\ \eta_k$ (limit in mean) if $\eta_k \underset{k\to\infty}{\longrightarrow} \eta$ strongly in L_2.

Definition 3. $L_q(\mathbb{S}; X)$ consists of all X-valued random variables $z\colon \mathbb{S} \to X$ such that

$$\|z; L_q(\mathbb{S}; X)\| := \left(\mathbb{E}\Big[\|z\|_X^q\Big]\right)^{1/q} < +\infty \quad \text{if} \quad q \in [1, \infty),$$
$$\|z; L_\infty(\mathbb{S}; X)\| := \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega\in\mathbb{S}} \|z(\omega)\|_X < +\infty \quad \text{if} \quad q = \infty.$$

Moreover, $z_1 = z_2$ in $L_q(\mathbb{S}; X)$ iff $z_1(\omega) = z_2(\omega)$ in X a.s.

Similarly to Proposition 2, we have that the linear space $\{L_q(\mathbb{S}; X), \|\cdot; L_q(\mathbb{S}; X)\|\}$ is Banach space (see [30, p. 17–18]). Some properties of the random Lebesgue-Bochner spaces are investigated in [31], [32].

Let us take $\ell, T > 0$ and denote $\Omega = (0, \ell)$, $Q_{0,T} = \Omega \times (0, T)$, $\Pi_{0,T} = \Omega \times (0, T) \times \mathbb{S}$, and $\Theta_{0,T} = (0, T) \times \mathbb{S}$.

Remark 4. Using the products of the probability measure $\mathbb{P}$ with the Lebesgue measures on $\mathbb{R}$ and $\mathbb{R}^2$, respectively, we can (as in Definition 2) introduce the random Lebesgue spaces $L_q(\Theta_{0,T})$, $L_q(\Omega \times \mathbb{S})$, $L_q(\Pi_{0,T})$, etc. Similarly as in [11, p. 211] and Theorem 8.28 [23, p. 218], from the Fubini theorem, we obtain the following equalities: $L_q(\Theta_{0,T}) = L^q(0, T; L_q) = L_q(\mathbb{S}; L^q(0, T))$, $L_q(\Pi_{0,T}) = L^q(0, T; L_q(\Omega \times \mathbb{S})) = L_q(\mathbb{S}; L^q(Q_{0,T}))$, etc.

Definition 4 (see [33], p. 38)**.** A function (*stochastic process*) $W = W(t, \omega)\colon [0, +\infty) \times \mathbb{S} \to \mathbb{R}$ is called the *standard Wiener process* if the following conditions are satisfied:

1) $W \in C([0, +\infty); L_2)$;
2) $W(0) = W(0, \omega) = 0$ a.s.;
3) for all $0 < t_1 < t_2 < \ldots < t_m$ the following random variables are independent: $W(t_1, \cdot)$, $W(t_2, \cdot) - W(t_1, \cdot)$, $\ldots$, $W(t_m, \cdot) - W(t_{m-1}, \cdot)$;
4) $W(t, \cdot) - W(s, \cdot) \in N(0, t - s)$ for all $t > s \geq 0$, i.e., its probability density function equals to $q_{s,t}(y) = (2\pi(t - s))^{-1/2}\ \exp(-\frac{|y|^2}{2(t-s)})$, $y \in \mathbb{R}$.

Proposition 3. *If* W *is the Wiener process from Definition 4, then*

(i) (see [11, p. 212] *and Theorem 9.1* [34, p. 234]*)* $W \in L_2(\mathbb{S}; C([0, T]; \mathbb{R}))$*;*
(ii) (see [29, p. 21]*) Since* W *does not depend on the variable* $x \in \Omega$*, we can write, for example, that* $W \in L_2(\Pi_{0,T})$*.*

Now, let us concentrate on the definitions of the stochastic integrals. Take a function $g \in \Psi_0$, where $\Psi_0 := \{g \in C^1[0, T] \ \ | \ \ g(0) = g(T) = 0\}$.

Definition 5. The *Paley-Wiener-Zygmund integral* of the function $g \in \Psi_0$ with respect to the Wiener process W is defined by the rule

$$(\texttt{PWZ}) \int_0^T g(t)\, dW(t, \omega) := -\int_0^T g_t(t) W(t, \omega)\, dt. \qquad (36)$$

In the right hand side of (36) we have the Bochner integral that is well-defined by condition 1 of Definition 4. Further, let $g \in L^2(0,T)$. Then there exists a sequence $\{g_m\}_{m\in\mathbb{N}}$ such that

$$\{g_m\}_{m\in\mathbb{N}} \subset \Psi_0, \qquad g_m \underset{m\to\infty}{\longrightarrow} g \quad \text{in} \quad L^2(0,T).$$

Definition 6. The *Paley-Wiener-Zygmund integral* of the function $g \in L^2(0,T)$ with respect to the Wiener process W is defined by the rule

$$(\mathtt{PWZ})\int_0^T g(t)\,dW(t,\omega) = \underset{m\to\infty}{\text{l.i.m.}}\ (\mathtt{PWZ})\int_0^T g_m(t)\,dW(t,\omega). \tag{37}$$

By [33, p. 59], we get that PWZ-integral (37) is well-defined. For the sake of convenience, we shall write $W(t)$ instead of $W(t,\cdot)$ etc. The properties of PWZ-integrals (36)–(37) are considered in [33] (see also our previous paper [29]). We recall only that if $t_1, t_2 \in [0,T]$ and $t_1 < t_2$, then $(\forall\ A_1, A_2 \in \mathbb{R})$:

$$\int_{t_1}^{t_2}\Big[A_1 g_1(t) + A_2 g_2(t)\Big]\,dW(t,\omega) = A_1\int_{t_1}^{t_2} g_1(t)\,dW(t,\omega) + A_2\int_{t_1}^{t_2} g_2(t)\,dW(t,\omega), \tag{38}$$

and the following Newton-Leibniz formula it holds

$$\int_{t_1}^{t_2} dW(t) = W(t_2) - W(t_1). \tag{39}$$

Now, let us consider the differential properties of the stochastic processes.

Definition 7. A stochastic process $u = u(t,\omega)\colon \Theta_{0,T} \to \mathbb{R}$ is called a *differentiable*, if there exist functions $\alpha \in L^1(0,T;L_1)$ and $\beta \in L^2(0,T)$ such that

$$\forall\ t_1, t_2 \in [0,T], \quad t_1 < t_2\colon \quad u(t_2,\omega) = u(t_1,\omega) + \int_{t_1}^{t_2}\alpha(s,\omega)\,ds + \int_{t_1}^{t_2}\beta(s)\,dW(s,\omega) \tag{40}$$

holds a.s. Then the expression

$$du(t,\omega) = \alpha(t,\omega)\,dt + \beta(t)\,dW(t,\omega), \quad (t,\omega) \in \Theta_{0,T}, \tag{41}$$

is called a *stochastic differential* of the stochastic process u.

Note carefully that the differential symbols are simply an abbreviation for the integral expressions above: strictly speaking "du", "dt", and "dW" have no meaning alone (see [33, p. 69]). Clearly,

$$\forall\ A_1, A_2 \in \mathbb{R}\colon \quad d\Big(A_1u_1(t,\omega) + A_2u_2(t,\omega)\Big) = A_1\,du_1(t,\omega) + A_2\,du_2(t,\omega). \tag{42}$$

Let us consider few examples. First, let $\alpha \in L^1(0,T;L_1)$ and

$$\eta(t,\omega) := \int_0^t \alpha(s,\omega)\,ds, \quad (t,\omega) \in \Theta_{0,T}. \tag{43}$$

By the properties of the Bochner integral, we get

$$\eta(t_2,\omega) - \eta(t_1,\omega) = \int_0^{t_2}\alpha(s,\omega)\,ds - \int_0^{t_1}\alpha(s,\omega)\,ds = \int_{t_1}^{t_2}\alpha(s,\omega)\,ds.$$

Then Definition 7 implies that

$$d\eta(t,\omega) = \alpha(t,\omega)\,dt + 0\,dW(t,\omega) = \alpha(t,\omega)\,dt. \tag{44}$$

Remark 5. Let $D(0,T)$ is the space of the test functions, X is some Banach space, and $D^*(0,T;X)$ is the space of the vector valued distributions (see, for example, [24, p. 186] for more details). Then for every $u \in L^1_{\mathrm{loc}}(0,T;X)$ we have that $u \in D^*(0,T;X)$ and

$$\langle u, \varphi \rangle_{D(0,T)} := \int_0^T u(t)\varphi(t)\, dt, \quad \varphi \in D(0,T). \tag{45}$$

Since $\alpha \in L^1(0,T;L_1) \subset D^*(0,T;L_1)$, then, for the function η from (43), the distributional derivative $\eta_t \in D^*(0,T;L_1)$ is well-defined and $\eta_t = \alpha$. So, we can rewrite stochastic differential (44) to the form

$$d\eta = \eta_t\, dt + 0\, dW = \eta_t\, dt. \tag{46}$$

Further, let $\beta \in \mathbb{R}$ be a number. Then (42) and Newton-Leibniz formula (39) imply that

$$d\Big(\beta\, W(t,\omega)\Big) = 0\, dt + \beta\, dW(t,\omega) = \beta\, dW(t,\omega). \tag{47}$$

On the other hand, $W \in C([0,T];L_2) \subset D^*(0,T;L_2)$. So, there exists a distributional derivative $W_t \in D^*(0,T;L_2)$ such that

$$\langle W_t, \phi \rangle_{D(0,T)} = -\langle W, \phi_t \rangle_{D(0,T)}, \quad \phi \in D(0,T). \tag{48}$$

Definition 8. The vector valued distributional derivative $W_t \in D^*(0,T;L_2)$ of the Winer process W is called a *white noise*.

Using representation of type (45), we obtain that the right hand side of (48) equals to the Bochner integral $-\int_0^T W(t)\phi_t(t)\, dt$. From (36) we get that it also equals to

$$\int_0^T \phi(t)\, dW(t). \tag{49}$$

So, similarly as in [12] and [13], by (48)–(49), we obtain that the white noise W_t satisfies

$$\langle W_t, \phi \rangle_{D(0,T)} = \int_0^T \phi(t)\, dW(t) = -\int_0^T W(t)\phi_t(t)\, dt, \quad \phi \in D(0,T). \tag{50}$$

Remark 6. Let X be some Banach space such that $L_2 \subset X$ (for example, $X = L_1$). Since $L_2 \subset X$, we have that $D^*(0,T;L_2) \subset D^*(0,T;X)$ (see [24, p. 187]). Then W_t also is a derivative in sence of the space $D^*(0,T;X)$.

Finally, let $\alpha \in L^1(0,T;L_1)$, $\beta \in \mathbb{R}$, and

$$u(t,\omega) = u(0,\omega) + \int_0^t \alpha(s,\omega)\, ds + \beta\, W(t,\omega), \quad (t,\omega) \in \Theta_{0,T}. \tag{51}$$

Then for $t_1, t_2 \in [0,T]$, $t_1 < t_2$, using (39), we get

$$u(t_2) - u(t_1) = u(0) + \int_0^{t_2} \alpha(s)\, ds + \beta\, W(t_2) - u(0) - \int_0^{t_1} \alpha(s)\, ds - \beta\, W(t_1) =$$
$$= \int_{t_1}^0 \alpha(s)\, ds + \int_0^{t_2} \alpha(s)\, ds + \beta(W(t_2) - W(t_1)) = \int_{t_1}^{t_2} \alpha(s)\, ds + \int_{t_1}^{t_2} \beta\, dW(s).$$

So, using (41), we have the following stochastic differential of random process (51):

$$du(t,\omega) = \alpha(t,\omega)\,dt + \beta\,dW(t,\omega). \tag{52}$$

On the other hand, stochastic process (51) belongs to $C([0,T];L_1)$ and has in $D^*(0,T;L_1)$ the distributional derivative $u_t = (u(0))_t + (\int_0^t \alpha(s)\,ds)_t + (\beta\,W)_t$, i.e., (see Remark 6),

$$u_t = \alpha + \beta\,W_t. \tag{53}$$

Definition 9. The distribution u_t from (53) is called a *generalized stochastic derivative* of the stochastic process u from (51).

Finally, let us consider the functions of more than two variables. For example, take a function of three variables.

Remark 7. Suppose that $\alpha \in L^1(0,T;L_1(\Omega\times\mathbb{S}))$, $\beta\in\mathbb{R}$, and

$$u(x,t,\omega) = u(x,0,\omega) + \int_0^t \alpha(x,s,\omega)\,ds + \beta\,W(t,\omega), \quad (x,t,\omega)\in\Pi_{0,T}. \tag{54}$$

Since we can consider the Winer process as a function from the space $L_2(\Pi_{0,T})$ (see Proposition 3), then, similarly to (53) and Definition 9, we obtain that $u\in D^*(0,T;L_1(\Omega\times\mathbb{S}))$ from (54) has a generalized stochastic derivative of type (53).

3. Main results. Let us again $(\mathbb{S},\mathcal{F},\mathbb{P})$ be a complete probability space, $T>0$ be some number, $\Omega_i := (0,\ell_i)$ be the edges of our graph,

$$m_i\in\mathbb{N} \quad \text{such that} \quad \frac{1}{m_i} < \frac{\ell_i}{2}, \tag{55}$$

$$\Pi^i_{0,T} = \Omega_i\times(0,T)\times\mathbb{S}, \quad Q^i_{0,T} = \Omega_i\times(0,T) \quad (i=\overline{1,n}), \quad \Theta_{0,T} = (0,T)\times\mathbb{S}. \tag{56}$$

Now, we seek a function $\boldsymbol{u} = (u^1,\dots,u^n)$ such that $u^i\colon \Pi^i_{0,T}\to\mathbb{R}$,

$$u^i_t - a\,u^i_{xx} = f^i(x,t,\omega) + b^i_t(x,t,\omega), \quad (x,t,\omega)\in\Pi^i_{0,T}, \quad i=\overline{1,n}, \tag{57}$$

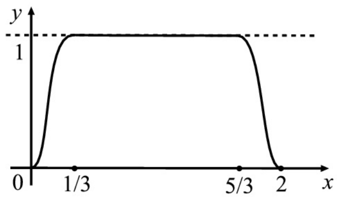
$$\begin{cases} u^k(\ell_k,t,\omega) = u^d(\ell_d,t,\omega) = u^r(0,t,\omega) = u^s(0,t,\omega), & k,d\in J^-_j, \quad r,s\in J^+_j, \\ \sum\limits_{k\in J^-_j} u^k_x(\ell_k,t,\omega) - \sum\limits_{r\in J^+_j} u^r_x(0,t,\omega) = 0, & j=\overline{1,M}, \quad (t,\omega)\in\Theta_{0,T}, \end{cases} \tag{58}$$

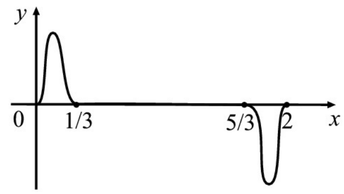
$$u^i(x,0,\omega) = u^i_0(x,\omega), \quad x\in\Omega_i, \quad \omega\in\mathbb{S}, \quad i=\overline{1,n}, \tag{59}$$

where $a>0$ is a number, f^i, b^i, u^i_0 are some given functions, in particular, b^i_t is the function of the white noise type (see Section 2 above and condition **(W)** below for more details).

To formulate the main results, let us introduce an additional notation. For every edge Ω_i and number m_i from (55), we put

$$B^{m_i}(x) = \begin{cases} 3(m_i x)^2 - 2(m_i x)^3, & 0\le x\le \frac{1}{m_i}, \\ 1, & \frac{1}{m_i}\le t\le \ell_i - \frac{1}{m_i}, \\ 3(m_i(\ell_i - x))^2 - 2(m_i(\ell_i-x))^3, & \ell_i - \frac{1}{m_i}\le t\le \ell_i, \end{cases} \quad i=\overline{1,n}. \tag{60}$$





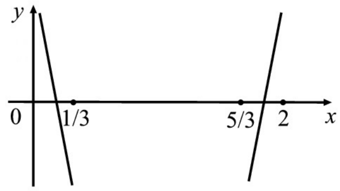


Fig. 3: Function B^3. Fig. 4: Function B^3_x.

Fig. 5: Function B^3_{xx}.

Plotting of function (60) and its derivatives (for $m_i = 3$ and $\ell_i = 2$) is shown on Fig. 3–5. Clearly, since (55) holds, then $B^{m_i} \in C^1(\overline{\Omega_i})$, $B^{m_i}_{xx} \in L^\infty(\Omega_i)$, and

$$B^{m_i}(0) = B^{m_i}_x(0) = 0, \quad B^{m_i}(\ell_i) = B^{m_i}_x(\ell_i) = 0, \quad i = \overline{1,n}. \tag{61}$$

Suppose that condition **(A)** holds and the following conditions are satisfied.

(F): $\boldsymbol{f} := (f^1, \dots, f^n) \in L_2(\mathbb{S}; L^2(0,T;H))$, where H is taken from (7);

(U): $\boldsymbol{u}_0 := (u^1_0, \dots, u^n_0) \in L_2(\mathbb{S}; H)$;

(W): $\boldsymbol{b} = (b^1, \dots, b^n)$, where $b^i(x,t,\omega) = B^{m_i}(x)W(t,\omega)$, $(x,t,\omega) \in \Pi^i_{0,T}$, W is taken from Definition 4, m_i is taking from (55), and B^{m_i} is taken from (60), $i = \overline{1,n}$.

Note that, if the function B^{m_i} is taking from (60), then there exists a constant $C_5 > 0$ such that for every $x \in \overline{\Omega^i}$ it holds: $|B^{m_i}(x)| \le 1$, $|B^{m_i}_x(x)| \le C_5$, $i = \overline{1,n}$. Then the function $\boldsymbol{b}$ from condition **(W)** satisfies the following estimates

$$|b^i(x,t,\omega)| \le |W(t,\omega)|, \quad |b^i_x(x,t,\omega)| \le C_5|W(t,\omega)|, \quad i = \overline{1,n}. \tag{62}$$

Estimates (62) and smoothness of the Winer process (see Definition 4) imply that

$$\sum_{i=1}^{n} \int_{\Pi^i_{0,T}} \Big[|b^i_x(x,t,\omega)|^2 + |b^i(x,t,\omega)|^2\Big]\, dxdt\, \mathbb{P}(d\omega) \le C_6, \tag{63}$$

where $C_6 > 0$ is a constant. So, Lemma 3.4 [30, p. 19] implies that $\boldsymbol{b} \in L_2(\mathbb{S}; L^2(0,T;Z_{01}))$, where Z_{01} is taken from (34).

If we formally take in (57)–(59) $\boldsymbol{u} = \widetilde{\boldsymbol{u}} + \boldsymbol{b}$, i.e.,

$$u^i(x,t,\omega) = \widetilde{u}^i(x,t,\omega) + b^i(x,t,\omega), \quad (x,t,\omega) \in \Pi^i_{0,T}, \quad i = \overline{1,n}, \tag{64}$$

where $\boldsymbol{u} = (u^1, \dots, u^n)$, $\widetilde{\boldsymbol{u}} = (\widetilde{u}^1, \dots, \widetilde{u}^n)$, and $\boldsymbol{b}$ is taken from condition **(W)**, we obtain $\boldsymbol{u}_t = \widetilde{\boldsymbol{u}}_t + \boldsymbol{b}_t$ and the following problem (with the Winer process but without the white noise)

$$\widetilde{u}^{\,i}_t - a\, \widetilde{u}^{\,i}_{xx} = \widetilde{f}^{\,i}(x,t,\omega), \quad (x,t,\omega) \in \Pi^i_{0,T}, \quad i = \overline{1,n}, \tag{65}$$

$$\begin{cases} \widetilde{u}^{\,k}(\ell_k,t,\omega) = \widetilde{u}^{\,d}(\ell_d,t,\omega) = \widetilde{u}^{\,r}(0,t,\omega) = \widetilde{u}^{\,s}(0,t,\omega), & k,d \in J^-_j, \quad r,s \in J^+_j, \\ \sum\limits_{k \in J^-_j} \widetilde{u}^{\,k}_x(\ell_k,t,\omega) - \sum\limits_{r \in J^+_j} \widetilde{u}^{\,r}_x(0,t,\omega) = 0, & j = \overline{1,M}, \quad (t,\omega) \in \Theta_{0,T}, \end{cases} \tag{66}$$

$$\widetilde{u}^{\,i}(x,0,\omega) = u^i_0(x,\omega), \quad x \in \Omega_i, \quad i = \overline{1,n}, \quad \omega \in \mathbb{S}, \tag{67}$$

where $(\widetilde{f}^{\,1}, \dots, \widetilde{f}^{\,n}) =: \widetilde{\boldsymbol{f}} = \boldsymbol{f} + a\,\widetilde{\boldsymbol{b}}_{xx}$, i.e.,

$$\widetilde{f}^{\,i}(x,t,\omega) = f^i(x,t,\omega) + a\, b^i_{xx}(x,t,\omega), \quad (x,t,\omega) \in \Pi^i_{0,T}, \quad i = \overline{1,n}. \tag{68}$$

Clearly, for fixed $\omega \in \mathbb{S}$, problem (65)–(67) coincides with problem (3)–(5), where (33) holds.

Let V is taken from (6) and H is taken from (7).

Definition 10. A function $\boldsymbol{u}=(u^1,\dots,u^n)$ is called a *weak solution to problem (57)–(59)* if

$$\boldsymbol{u} \in L_2\big(\mathbb{S}; C([0,T];H) \cap L^2(0,T;V)\big) \tag{69}$$

and equality (64) holds, where the function $\widetilde{\boldsymbol{u}} = (\widetilde{u}^{\,1},\dots,\widetilde{u}^{\,n})$ for $\mathbb{P}$-a.e. $\omega \in \mathbb{S}$ is a weak solution of problem (65)–(67) as a function of the variables (x,t).

Theorem 4. *Suppose that conditions **(A)**, **(F)**, **(U)**, and **(W)** hold. Then problem (57)–(59) has an unique weak solution $\boldsymbol{u}$. Moreover,*

$$\Big\|\boldsymbol{u}; L_2\Big(\mathbb{S}; C([0,T];H) \cap L^2(0,T;V)\Big)\Big\| \le C_7\Big\{\Big\|\boldsymbol{u}_0; L_2(\mathbb{S};H)\Big\| + \Big\|\boldsymbol{f}; L_2(\mathbb{S}; L^2(0,T;H))\Big\| +$$

$$+ \Big\|\boldsymbol{b}; L_2\Big(\mathbb{S}; C([0,T];H) \cap L^2(0,T;V)\Big)\Big\|\Big\}, \tag{70}$$

where the constant $C_7 > 0$ is independent of $\boldsymbol{u}, \boldsymbol{u}_0, \boldsymbol{f}, \boldsymbol{b}$.

Proof. Existence. First, let us take Z_{01} from (34) and let us define the vector-valued function $\Bbbk \colon \mathbb{S} \to H \times L^2(0,T;H) \times L^2(0,T;Z_{01})$ of the random variable $\omega \in \mathbb{S}$ by the rule

$$\Bbbk(\omega) := \Big(\boldsymbol{u}_0(\cdot,\omega),\ \ \boldsymbol{f}(\cdot,\cdot,\omega),\ \ \boldsymbol{b}(\cdot,\cdot,\omega)\Big), \quad \omega \in \mathbb{S}. \tag{71}$$

Since the functions $\boldsymbol{u}_0$, $\boldsymbol{f}$, and $\boldsymbol{b}$ are measurable, we have that $\Bbbk$ is a measurable function (see Corollary 2 [26, p. 80]).

Further, let us solve problem (65)–(67). Take an operator

$$\mathfrak{R} \colon H \times L^2(0,T;H) \times L^2(0,T;Z_{01}) \to C([0,T];H) \cap L^2(0,T;V)$$

such that

$$\mathfrak{R}(\boldsymbol{u}_0,\ \boldsymbol{f},\ \boldsymbol{b}) = \widetilde{\boldsymbol{u}}, \tag{72}$$

where $\widetilde{\boldsymbol{u}}$ is an unique weak solution of problem (3)–(5) and (33) holds. Remark 3 and Theorem 3 imply that $\mathfrak{R}$ is well-defined and it is continuous.

Using (71) and (72), we define the function $\mathfrak{P} \colon \mathbb{S} \to C([0,T];H) \cap L^2(0,T;V)$ such that

$$\mathfrak{P}(\omega) := (\mathfrak{R} \circ \Bbbk)(\omega) = \mathfrak{R}(\Bbbk(\omega)), \quad \omega \in \mathbb{S}. \tag{73}$$

Thus, for every $\omega \in \mathbb{S}$ the value $\mathfrak{P}(\omega)$ is $\widetilde{\boldsymbol{u}}(\cdot,\cdot,\omega)$, where $\widetilde{\boldsymbol{u}}$ is a solution of deterministic problem (65)–(67) with the random parameter $\omega \in \mathbb{S}$, where (68) holds.

Since the function $\Bbbk$ from (71) is measurable and the function $\mathfrak{R}$ from (72) is continuous (so, $\mathfrak{R}$ is a measurable function), we have that the function $\mathfrak{P}$ from (73) is measurable (see Theorem 1 [26, p. 82]). Then the function $\widetilde{\boldsymbol{u}}$ is a $\big(C([0,T];H) \cap L^2(0,T;V)\big)$-valued random variable. Moreover, $\widetilde{\boldsymbol{u}}$ satisfies the following estimates of type (35)

$$\max_{0\le t\le T} |\widetilde{\boldsymbol{u}}(\cdot,t,\omega)|_H^2 + \int_0^T \|\widetilde{\boldsymbol{u}}(\cdot,t,\omega)\|^2\,dt \le$$

$$\le C_8\Big\{|\boldsymbol{u}_0(\cdot,\omega)|_H^2 + \int_0^T \Big[|\boldsymbol{f}(\cdot,t,\omega)|_H^2 + |\boldsymbol{b}_x(\cdot,t,\omega)|_H^2\Big]\,dt\Big\}.$$

Clearly, $\max\limits_{0\le t\le T}|\widetilde{\boldsymbol{u}}(\cdot,t,\omega)|_H^2=\big(\max\limits_{0\le t\le T}|\widetilde{\boldsymbol{u}}(\cdot,t,\omega)|_H\big)^2$. Then

$$\|\widetilde{\boldsymbol{u}}(\cdot,\cdot,\omega);C([0,T];H)\|^2+\|\widetilde{\boldsymbol{u}}(\cdot,\cdot,\omega);L^2(0,T;V)\|^2\le C_9\mathbf{F}(\omega), \tag{74}$$

where the constant $C_9>0$ depends only on the constants T and a from (57),

$$\mathbf{F}(\omega)=|\boldsymbol{u}_0(\cdot,\omega)|_H^2+\|\boldsymbol{f}(\cdot,\cdot,\omega);L^2(0,T;H)\|^2+\|\boldsymbol{b}_x(\cdot,\cdot,\omega);L^2(0,T;H)\|^2,\quad \omega\in\mathbb{S}.$$

Since $\mathbf{F}\in L_1(\mathbb{S})$, integrating (74) in $\omega\in\mathbb{S}$, we get

$$\widetilde{\boldsymbol{u}}\in L_2\big(\mathbb{S};C([0,T];H)\cap L^2(0,T;V)\big). \tag{75}$$

Thus, equality (64), condition **(W)**, Proposition 3, and (75) yield (69). From (64) and (74), we easily obtain (70).
(Uniqueness). Using (64), similarly as in the proof of Theorem 3, we get the following estimate of type (35) with $\boldsymbol{u}_0^1=\boldsymbol{u}_0^2$, $\boldsymbol{f}^1=\boldsymbol{f}^2$, $\boldsymbol{b}^1=\boldsymbol{b}^2$, and with the parameter $\omega\in\mathbb{S}$:

$$\max_{0\le t\le T}|\boldsymbol{u}^1(\cdot,t,\omega)-\boldsymbol{u}^2(\cdot,t,\omega)|_H^2+\int_0^T\|\boldsymbol{u}^1(\cdot,t,\omega)-\boldsymbol{u}^2(\cdot,t,\omega)\|^2\,dt\le 0.$$

From this it follows, that $\boldsymbol{u}^1=\boldsymbol{u}^2$ and Theorem 4 is proved. □

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References

1. B. Zugec, *Regularity of a parabolic differential equation on graphs*, Mathematics, **11** (2023), no.21, Article ID 4453, 1–10. doi:10.3390/math11214453
2. M. Stoll, M. Winkler, *Optimal Dirichlet control of partial differential equations on networks*, Electron. Trans. Numer. Anal., **54** (2021), 392–419. doi:10.1553/etna_vol54s392
3. C. Chen, Y. Xiong, Z. Li, Y. Chen, *Flexibility of biodegradable polymer stents with different strut geometries*, Materials., **13** (2020), no.15, article ID 3332, 1–12. doi:10.3390/ma13153332
4. A. Colombo, E. Karvouni, *Biodegradable stents: fulfilling the mission and stepping away*, Circulation (J. Am. Heart Assoc.), **102** (2000), no.4, 371–73. doi:10.1161/01.CIR.102.4.371
5. S. Čanić, J. Tambaca, *Cardiovascular stents as PDE nets: 1D vs. 3D*, IMA J. Appl. Math., **77** (2012), no.6, 748–770. doi:10.1093/imamat/hxs047
6. M.K. Fijavz, D. Mugnolo, E. Sikolya, *Variational and semigroup methods for waves and diffusion in networks*, Appl. Math. Optim., **55** (2007), no.2, 219–240. doi:10.1007/s00245-006-0887-9
7. Yu. Golovaty, V. Flyud, *Singularly perturbed hyperbolic problems on metric graphs: asymptotics of solutions*, Open Math., **15** (2017), no.1, 404–419. doi:10.1515/math-2017-0030
8. Yu. Golovaty, *Quantum graphs: Coulomb-type potentials and exactly solvable models*, Annales Henri Poincaré, **24** (2023), 2557–2585. doi:10.1007/s00023-023-01270-9
9. Kh. Buhrii, Yu. Golovaty, *SIR models on complex networks and impact of vaccination*, 2023 IEEE 13th Inter. Conf. on Electronics and Information Technologies, ELIT 2023, Proceedings, 2023, 37–42.
10. A. Weller, *Numerical methods for parabolic partial differential equations on metric graphs*, PhD Thesis, Universität zu Köln, 2024.
https://kups.ub.uni-koeln.de/73182/1/weller_NumPDEMeGra_dissertation_24.pdf
11. A. Bensoussan, R. Temam, *Equations stochastiques du type Navier-Stokes*, J. Funct. Anal., **13** (1973), no.2, 195–222. doi:10.1016/0022-1236(73)90045-1

12. G. Vage, Stochastic differential equations and Kondratiev spaces, Dr. Ing. Thesis. Trondheim, 1995. https://www.nb.no/items/5ae9361e3c1ce3db1c4fe812813d881c?page=0
13. H. Liang, L. Hou, J. Ming, *The velocity tracking problem for Wick-stochastic Navier–Stokes flows using Weiner chaos expansion*, J. Computat. Appl. Math., **307** (2016), 25–36. doi:10.1016/j.cam.2016.04.030
14. D. Breit, E. Feireisl, M. Hofmanova, Stochastically forced compressible fluid flows, Berlin, Boston: Walter de Gruyter, 2018.
15. N. Buhrii, O. Buhrii, O. Domanska, *Semilinear stochastic parabolic equation with variable exponent of nonlinearity*, Visn. Lviv Univ., Ser. Mech.-Math., **93** (2022), 108–121. doi:10.1016/0022-1236(73)90045-1
16. O. Buhrii, M. Khoma, I.-M. Vovk, *Stochastic differentiations in equations with variable exponents of nonlinearity*, Visn. Lviv Univ., Ser. Mech.-Math., **96** (2024), 83–104. doi:10.30970/vmm.2024.96.071-092
17. M.I. Freidlin, A.D. Wentzell, *Diffusion processes on graphs and the averaging principle*, Ann. Probab., **21** (1993), no.4, 2215–2245. doi:10.1214/aop/1176989018
18. M. Freidlin, S.-J. Sheu, *Diffusion processes on graphs: stochastic differential equations, large deviation principle*, Probab. Theory Relat. Fields, **116** (2000), no.2, 181–220. doi:10.1007/PL00008726
19. M. Thorpe, T. Nguyen, H. Xia, T. Strohmer, A. Bertozzi, S. Osher, B. Wang *GRAND++: Graph neural diffusion with a source term*, Conference paper at ICLR-2022, 1–21.
20. H. Hajri, O. Raimond *Stochastic flows on metric graphs*, Electron. J. Probab., **19** (2014), no.12, 1–20. doi:10.1214/EJP.v19-2773
21. H. Hajri, O. Raimond, *Stochastic flows and an interface SDE on metric graphs*, Stochastic processes and their applications, **126** (2016), no.1, 33–65. doi:10.1016/j.spa.2015.07.014
22. R.A. Adams, Sobolev spaces, New York, San Francisco, London: Academic Press, 1975.
23. G. Leoni, A first course in Sobolev spaces, AMS, Providence, RI, 2010.
24. M. Chipot, Elements of nonlinear analysis, Basel, Boston, Berlin: Birkhauser, 2012.
25. O. Buhrii, N. Buhrii, *Integro-differential systems with variable exponents of nonlinearity*, Open Math., **15** (2017), no.1, 859–883. doi:10.1515/math-2017-0069
26. V. Kadets, A course in functional analysis and measure theory, Springer International Publ., 2018.
27. S.J. Dilworth, *Some probabilistic inequalities with applications to functional analysis*, Contemporary Mathematics, **144** (1993), AMS, Providence, RI, 53–67.
28. V. Lupulescu, C. Lungan, *Random integral equations on time scales*, Opuscula Math., **33** (2013), no.2, 323–335. doi:10.7494/OpMath.2013.33.2.323
29. O. Buhrii, N. Buhrii, V. Vlasov, *On stochastic space-time Paley-Wiener-Zygmund integral*, Mathematics, Informatics, Physics: Science and Education, **1** (2024), no.1, 13–26. doi:10.31652/3041-1955-2024-01-02
30. M. Siebenmorgen, Quadrature methods for elliptic PDEs with random diffusion, PhD Thesis. Bonn, 2016.
31. Q. Lü, J. Yong, X. Zhang, *Representation of Itô integrals by Lebesgue-Bochner integrals*, J. European Math. Soc., **14** (2012), no.6, 1795–1823. doi:10.4171/JEMS/347
32. Y. Wang, X. Zhu, P. Kloeden, *Compactness in Lebesgue–Bochner spaces of random variables and the existence of mean-square random attractors*, Stochastics and Dynamics, **19** (2019), no.4, 1950032. doi:10.1142/S0219493719500321
33. L.C. Evans, An introduction to stochastic differential equations, V.82, AMS, 2012.
34. Y. Mishura, G. Shevchenko, Theory and statistical applications of stochastic processes, London, Hoboken: ISTE Ltd and John Wiley and Sons, 2017.

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