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DISCRETE LEGENDRE MODIFIED PROJECTION-TYPE METHODS FOR HAMMERSTEIN INTEGRAL EQUATIONS

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We investigate discrete modified projection-type methods for the numerical approximation of nonlinear Hammerstein integral equations with sufficiently smooth kernels. Such equations arise in various applications and require efficient numerical techniques for their accurate resolution. The proposed approach is based on Legendre polynomial bases, which provide a suitable framework for constructing approximate solutions in appropriate function spaces. By combining these bases with sufficiently accurate numerical quadrature rules, we derive discrete formulations of modified Galerkin-type and modified collocation-type methods.

These methods are designed to improve the accuracy of classical projection techniques while maintaining computational efficiency. A comprehensive convergence analysis is performed for both approximate and iterated approximate solutions.

Under suitable regularity assumptions on the kernel and the exact solution, we establish superconvergence results, showing that the proposed methods achieve higher-order accuracy compared to standard approaches.

Moreover, we provide a rigorous error analysis that highlights the role of discretization and quadrature in the overall approximation process. The obtained theoretical estimates demonstrate that the use of Legendre-based discretization leads to significant improvements in convergence behavior. These results will contribute to the development of efficient numerical approaches for solving nonlinear Hammerstein-type integral equations.

1. Introduction. Nonlinear integral equations arise from different fields in mathematical physics like potential problems, electromagnetic fluid dynamics and transport problems (see [8]). The *Hammerstein* integral equations have been recognized among the important special equations in terms of nonlinear functional analysis. This equation is as follows

$$x - \mathcal{K}x = f, \tag{1}$$

where $\mathcal{K}$ is the integral operator defined on $\mathbb{X} = \mathcal{C}[-1,1]$ by

$$(\mathcal{K}x)(s) = \int_{-1}^{1} \varkappa(s,t)\psi(t,x(t))dt, \quad s \in [-1,1], \quad x \in \mathbb{X}, \tag{2}$$

where $\varkappa$, f, and ψ are known functions and x is the unknown function to be determined.

Actually, there is a huge literature on numerical methods for approximating the solutions of Hammerstein integral equation (1) with smooth kernels. The superconvergent degenerate kernel and Nyström methods for nonlinear integral equations were presented in [2] and the superconvergence of the iterated Galerkin solutions for Hammerstein equations with smooth

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as well as weakly singular kernels was considered in [18]. Moreover, a new collocation method was presented by Kumar and Sloan ([19]), while its superconvergence properties were studied in [21]. Some other authors proposed discrete methods to solve nonlinear integral equations with orthogonal and interpolatory projection operators (see [6,7,11,12,14,20,23]). There have been many approaches to improve the accuracy of numerical solutions. In this framework, the authors in [9] studied a discrete version of collocation and iterated collocation methods to obtain superconvergence results. Recently, a new method called discrete modified projection (multi-projection method) which aims to improve the convergence of the classical methods, was introduced in [22]. It is widely known that in order to obtain higher accuracy in these methods, the number of partition points must be increased. As a results, we must solve a large number of linear equations, which requires lots of computation.

A discrete version of the polynomially-based modified projection-type method to solve (1) discussed in [3] is considered in this paper. Particularly, Legendre polynomials can be used as basis functions for approximating subspace. These polynomials can easily be generated recursively and require less computational complexity than piecewise polynomials (see [4]). We prove that using a sufficiently accurate numerical quadrature rule the orders of convergence of the proposed method are still valid in its discrete version.

A number of recent papers have studied polynomially based projection methods for nonlinear integral equations. In Das et al. [17], the discrete Legendre-Galerkin and discrete Legendre collocation methods for nonlinear Hammerstein equations were proposed. Moreover, the Legendre-Galerkin solution as well as its iterated version were studied in [15] for Urysohn integral equations. The discrete multi-projection methods using Legendre polynomials was introduced in [16] and the case of weakly singular kernels was treated in [13].

The organization of this paper is as follows. In Section 2, we set up the notations and discuss the discrete Legendre modified projection-type methods for Hammerstein integral equation. In Section 3 we establish the convergence orders of the approximate and the iterated solutions of the proposed methods.

2. Description of the methods.

2.1. Preliminaries and notations. We consider the Banach space $\mathbb{X} = \mathcal{C}[-1,1]$ endowed with the infinity norm defined by $\|x\|_\infty := \max_{s\in[-1,1]} |x(s)|$. Define $\Omega = [-1,1]\times\mathbb{R}$. Throughout the paper, the following assumptions are made on $\varkappa, f$ and ψ:

(i) $f \in \mathbb{X}$ and $\psi \in \mathcal{C}(\Omega)$. (ii) $\varkappa \in \mathcal{C}([-1,1]^2)$, and $M\colon \equiv \max_{(s,t)\in[-1,1]^2} |\varkappa(s,t)|$.

(iii) The functions $\psi(t,u)$ and $\partial\psi(t,u)/\partial u$ are Lipschitz continuous in $u \in \mathbb{R}$, i.e., there exists $\delta_1, \delta_2 > 0$ such that for any $t \in [-1,1], u_1, u_2 \in \mathbb{R}$,

$$|\psi(t,u_1) - \psi(t,u_2)| \le \delta_1 |u_1 - u_2| \quad \text{and} \quad \Big|\frac{\partial\psi}{\partial u}(t,u_1) - \frac{\partial\psi}{\partial u}(t,u_2)\Big| \le \delta_2 |u_1-u_2|.$$

If the condition (iii) holds, the operator $\mathcal{K}$ is Fréchet differentiable and $\mathcal{K}'$ is $M\delta_2$-Lipschitz. The Fréchet derivative at $x \in \mathbb{X}$ is the linear operator $\mathcal{K}'(x)$ given by

$$(\mathcal{K}'(x)g)(s) = \int_{-1}^{1} \varkappa(s,t)\frac{\partial\psi}{\partial u}(t,x(t))g(t)dt, \quad g \in \mathbb{X}.$$

Next we show that equation (1) possess unique solution in $\mathbb{X}$.

Theorem 1. *Assume $f, \varkappa$ and ψ satisfy the conditions (i)–(iii) respectively. Moreover, assume $2\delta_1 M < 1$. Then the integral equation (1) has a unique solution $x_0 \in \mathbb{X}$.*

Proof. Consider the nonlinear operator $\mathcal{T}\colon \mathbb{X} \to \mathbb{X}$ defined by $\mathcal{T}(x) = \mathcal{K}x + f, \quad x \in \mathbb{X}$. Then equation (1) becomes $\mathcal{T}(x) = x$.

For $x, y \in \mathbb{X}$, using the Lipschitz's continuity of ψ in (iii), we obtain

$$\left\|\mathcal{T}(x) - \mathcal{T}(y)\right\|_\infty = \sup_{s\in[-1,1]} \left| \int_{-1}^{1} \varkappa(s,t)[\psi(t,x(t)) - \psi(t,y(t))]dt \right| \leq$$

$$\leq \sup_{s,t\in[-1,1]} |\varkappa(s,t)| \int_{-1}^{1} |\psi(t,x(t)) - \psi(t,y(t))|\, dt \leq \delta_1 M \int_{-1}^{1} |x(t) - y(t)|\, dt \leq 2\delta_1 M \|x-y\|_\infty.$$

By assumption $2\delta_1 M < 1$, hence $\mathcal{T}$ is a contraction mapping on $\mathbb{X}$. So, $\mathcal{T}$ has a unique fixed point in $\mathbb{X}$, by Banach contraction theorem. Hence the proof follows. □

Remark 1. The condition $2\delta_1 M < 1$ is only a sufficient condition for the existence and uniqueness of a solution to equation (1), it is not necessary. In the subsequent analysis, this condition is not required. Instead, we assume that equation (1) has a unique solution, denoted by x_0.

Let $\mathbb{X}_n$ denote the space of all polynomials of degree $\leq n$ defined on $[-1,1]$. Then the dimension of $\mathbb{X}_n$ is $n+1$, and the Legendre polynomials $\{L_0, L_1, L_2, \ldots, L_n\}$ defined by

$$L_0(s) = 1, \quad L_1(s) = s, \quad s \in [-1,1],$$
$$(i+1)L_{i+1}(s) = (2i+1)sL_i(s) - iL_{i-1}(s), \quad i \in \{1, 2, \ldots, n-1\},$$

form an orthogonal basis for $\mathbb{X}_n$ with respect to the inner product

$$\langle f, g\rangle := \int_{-1}^{1} f(t)g(t)\, dt, \quad f, g \in \mathbb{X}.$$

Since $\langle L_i, L_j\rangle = \begin{cases} 2/(2i+1), & i = j \\ 0, & i \neq j, \end{cases}$ then, an orthonormal basis for $\mathbb{X}_n$ is given by

$$\left\{\varphi_i(s) = \sqrt{i+1/2}\, L_i(s) \colon i \in \{0, 1, \ldots, n\}\right\}.$$

The integral in the definition of $\mathcal{K}$ involved in equation (1) is not computed exactly in practice. A discrete projection-type method is formed by replacing these integrals with numerical quadrature, as described below. We begin by choosing a numerical integration scheme

$$\int_{-1}^{1} f(t)dt \simeq \sum_{i=1}^{m} \omega_i f(t_i), \tag{3}$$

where the weights satisfy $\omega_i > 0$ for $i \in \{1, 2, \ldots, m\}$. The number of nodes is denoted simply by m (with $m \gg n$), and the dependence on n is understood implicitly. We assume that this formula has degree of precision $d \geq 2n$, that is

$$\int_{-1}^{1} P(t)dt = \sum_{i=1}^{m} \omega_i P(t_i),$$

for all polynomials P of degree $\leq d$. Throughout this paper, the parameters d and m are not chosen independently of n, but rather in such a way that $d \geq 2n$ and $m \to \infty$ as $n \to \infty$. This guarantees the validity of the approximation properties used in the following analysis.

Let $f, g \in \mathbb{X}$. According to Golberg [12] and Sloan [23], the discrete inner product is defined as follows

$$\langle f, g\rangle_m := \sum_{i=1}^{m} \omega_i f(t_i)g(t_i) \tag{4}$$

and the associated norm is

$$\|f\|_{\mathscr{L}^2} := \left(\sum_{i=1}^{m} \omega_i f^2(t_i)\right)^{\frac{1}{2}}.$$

Throughout this paper, we assume that $r \geq 2$ and $0 \leq p \leq r$.

If $\varkappa \in \mathcal{C}^p([-1,1]^2)$ and $\psi \in \mathcal{C}^p(\Omega)$ then $\mathcal{R}(\mathcal{K}) \subset \mathcal{C}^p[-1,1]$. Thus, if $f \in \mathcal{C}^p[-1,1]$ then $x \in \mathcal{C}^p[-1,1]$. We set

$$\|\varkappa\|_{p,\infty} := \max\left\{\left|\frac{\partial^{i+j}\varkappa}{\partial s^i \partial t^j}(s,t)\right| : 0 \leq i+j \leq p \ \text{ and } \ s,t \in [-1,1]\right\},$$

$$\|x\|_{p,\infty} := \max\left\{\|x^{(i)}\|_\infty : i \in \{0,1,\ldots,p\}\right\} \text{ and } \Psi_p := \max_{t\in[-1,1]}\left|\frac{\partial^p \psi}{\partial t^p}(t, x_0(t))\right|.$$

In terms of the numerical integration method (3), the Nyström operator can be defined as

$$(\mathcal{K}_m^N x)(s) = \sum_{i=1}^{m} \omega_i \varkappa(s,t_i)\psi(t_i, x(t_i)), \quad s \in [-1,1].$$

The Fréchet derivative of $\mathcal{K}_m^N$ at x_0 is given by

$$(\mathcal{K}_m^{N'}(x_0)g)(s) = \sum_{i=1}^{m} \omega_i \varkappa(s,t_i)\frac{\partial \psi}{\partial u}(t_i, x_0(t_i))g(t_i).$$

Since $\int_{-1}^{1} dt = \sum_{i=1}^{m} \omega_i = 2$ and $w_j > 0$, for $j \in \{0,1,2,\ldots,r\}$ we get

$$\|[\mathcal{K}_m^{N'}(x_0)g]^{(j)}\|_\infty = \max_{s\in[-1,1]}\Big|\sum_{i=1}^{m} \omega_i \frac{\partial^j \varkappa}{\partial s^j}(s,t_i)\frac{\partial \psi}{\partial u}(t_i, x_0(t_i))g(t_i)\Big| \leq$$

$$\leq \max_{s\in[-1,1]}\sum_{i=1}^{m} \omega_i \Big|\frac{\partial^j \varkappa}{\partial s^j}(s,t_i)\Big|\Big|\frac{\partial \psi}{\partial u}(t_i, x_0(t_i))\Big||g(t_i)| \leq 2\|\varkappa\|_{j,\infty}\zeta_1\|g\|_\infty,$$

where $\zeta_1 := \max_{t\in[-1,1]}\left|\frac{\partial \psi}{\partial u}(t, x_0(t))\right|$. Hence, we deduce that $\|\mathcal{K}_m^{N'}(x_0)g\|_\infty \leq 2\zeta_1\|\varkappa\|_{0,\infty}\|g\|_\infty$. This implies,

$$\|\mathcal{K}_m^{N'}(x_0)\|_\infty \leq 2\zeta_1\|\varkappa\|_{0,\infty}. \tag{5}$$

The operator $\mathcal{K}_m^{N'}(x_0)$ is a compact operator on $\mathbb{X}$.

As a next step, we demonstrate an important lemma that will be useful in this paper.

Lemma 1. *Let $x, y, g \in \mathbb{X}$. Then for $j \in \{0,1,2,\ldots,r\}$, the following estimate hold*

$$\|[(\mathcal{K}_m^{N'}(x) - \mathcal{K}_m^{N'}(y))g]^{(j)}\|_\infty \leq 2\delta_2\|\varkappa\|_{j,\infty}\|x-y\|_\infty\|g\|_\infty. \tag{6}$$

Proof. Using the Lipschitz continuity of $\frac{\partial \psi}{\partial u}(t,u)$, we get for $j \in \{0,1,\ldots,r\}$

$$\|(\mathcal{K}_m^{N'}(x) - \mathcal{K}_m^{N'}(y))g]^{(j)}\|_\infty =$$

$$= \max_{s\in[-1,1]}\Big|\sum_{i=1}^{m} \omega_i \frac{\partial^j \varkappa}{\partial s^j}(s,t_i)\Big[\frac{\partial \psi}{\partial u}(t_i, x(t_i)) - \frac{\partial \psi}{\partial u}(t_i, y(t_i))\Big]g(t_i)\Big| \leq$$

$$\leq \|\varkappa\|_{j,\infty}\max_{s\in[-1,1]}\sum_{i=1}^{m} \omega_i \Big|\frac{\partial \psi}{\partial u}(t_i, x(t_i)) - \frac{\partial \psi}{\partial u}(t_i, y(t_i))\Big|\Big|g(t_i)\Big| \leq 2\delta_2\|\varkappa\|_{j,\infty}\|x-y\|_\infty\|g\|_\infty.$$

□

We consider two types of projections from $\mathbb{X}$ to $\mathbb{X}_n$.

Discrete orthogonal projection. Let $\mathscr{Q}_n^G : \mathbb{X} \to \mathbb{X}_n$ be the hyperinterpolation operator introduced by Sloan ([23]) as

$$(\mathscr{Q}_n^G x)(s) = \sum_{i=0}^{n} \langle x, \varphi_i \rangle_m \varphi_i(s), \tag{7}$$

and satisfying $\langle \mathscr{Q}_n^G x, \varphi_i \rangle_m = \langle x, \varphi_i \rangle_m$, $i \in \{0,1,\ldots,n\}$. For any $x \in \mathcal{C}^r[-1,1]$, we have also

$$\|\mathscr{Q}_n^G x\|_{\mathscr{L}^2} \leq \sqrt{2}\|x\|_\infty, \quad \text{and} \quad \langle x - \mathscr{Q}_n^G x, x - \mathscr{Q}_n^G x \rangle_m^{\frac{1}{2}} \leq c_1\sqrt{2}n^{-r}\|x^{(r)}\|_\infty, \tag{8}$$

where c_1 is a constant independent of n and $n \geq r$.

Interpolatory projection. Let $\mathscr{Q}_n^C\colon \mathbb{X} \to \mathbb{X}_n$ be the interpolatory operator defined by Sloan ([23]) as

$$(\mathscr{Q}_n^C u)(\tau_i) = u(\tau_i), \quad i \in \{0, 1, \ldots, n\},$$

where $\{\tau_i\colon\ i \in \{0, 1, ..., n\}\}$ are the zeros of the Legendre polynomial L_{n+1}. In the Lagrange form, $\mathscr{Q}_n^C u$ is

$$(\mathscr{Q}_n^C u)(s) = \sum_{i=0}^{n} u(\tau_i)\ell_i(s), \tag{9}$$

where ℓ_i is the unique polynomial of degree n that satisfies $\ell_i(\tau_j) = \delta_{ij}$. Here, δ_{ij} denotes the Kronecker delta function.

For notational convenience from now on we write $\mathscr{Q}_n \equiv \mathscr{Q}_n^G$ or $\mathscr{Q}_n^C$. According to the analysis of (Golberg [11] and Sloan [23]), $\mathscr{Q}_n$ satisfies the following lemma.

Lemma 2. *Let $\mathscr{Q}_n\colon \mathbb{X} \to \mathbb{X}_n$ be the the hyperinterpolation or interpolatory projection operator defined by (7) and (9). There exists a constant $p > 0$ independent of n such that for $x \in \mathbb{X}$*

$$\|\mathscr{Q}_n x\|_{\mathscr{L}^2} \leq p\|x\|_\infty, \qquad \|x - \mathscr{Q}_n x\|_{\mathscr{L}^2} \leq (1+p) \inf_{\phi\in\mathbb{X}_n} \|x - \phi\|_\infty. \tag{10}$$

In particular, for any $x \in \mathcal{C}^r[-1,1]$, there exists a constant c_1 independent of n such that

$$\|x - \mathscr{Q}_n x\|_{\mathscr{L}^2} \leq c_1 n^{-r}\|x^{(r)}\|_{\mathscr{L}^2}, \tag{11}$$

$$\|x - \mathscr{Q}_n x\|_\infty \leq c_1 n^{\gamma-r}\|x^{(r)}\|_\infty, \tag{12}$$

where $\gamma = 1$ for the hyperinterpolation operator and $\gamma = \frac{1}{2}$ for the interpolatory projection.

Remark 2 (Sloan [23])**.** If $m = n + 1$ and the quadrature points used in the discrete inner product (4) and (9) are the same, the hyperinterpolation operator $\mathscr{Q}_n^G$ reduces to the $\mathscr{Q}_n^C$. Then we can write $\mathscr{Q}_n^G \equiv \mathscr{Q}_n^C$.

For a fixed $s \in [-1, 1]$, define $\varkappa_s(t) := \varkappa(s, t)$ for $t \in [-1, 1]$ to be the s section of $\varkappa$. In the Kumar and Sloan [19] technique for finding the approximate solution of (1), the function

$$z(t) = \psi(t, x(t)), \qquad t \in [-1, 1]$$

is approximated by the polynomial $z_n = \mathscr{Q}_n z$ of degree $\leq n$.

Recall that the modified projection-type method introduced in [3, 10] consist of approximating $\mathcal{K}$ by

$$\mathscr{Q}_n\mathcal{K} + \mathcal{K}_n - \mathscr{Q}_n\mathcal{K}_n,$$

where $\mathcal{K}_n$ is the nonlinear operator given by

$$(\mathcal{K}_n x)(s) = \int_{-1}^{1} \varkappa(s,t)\mathscr{Q}_n z(t)dt, \qquad s \in [-1, 1].$$

In this framework, we propose to approximate $\mathcal{K}$ by the following discrete finite rank operator

$$\mathcal{K}_n^M = \mathscr{Q}_n\mathcal{K}_m^N + \mathcal{K}_n^D - \mathscr{Q}_n\mathcal{K}_n^D, \tag{13}$$

where $\mathcal{K}_n^D$ is the discrete nonlinear operator given by

$$(\mathcal{K}_n^D x)(s) = \langle \varkappa_s, \mathscr{Q}_n z\rangle_m = \sum_{i=1}^{m} \omega_i \varkappa_s(t_i)(\mathscr{Q}_n z)(t_i), \quad s \in [-1, 1].$$

Then the discrete Legendre modified projection-type method for equation (1) is seeking an approximate solution x_n to x_0 such that

$$x_n - \mathcal{K}_n^M x_n = f, \tag{14}$$

and the discrete iterated solution is given by

$$\widetilde{x}_n = \mathcal{K}_m^N x_n + f. \tag{15}$$

Throughout the paper, this method will be called a modified Galerkin-type or modified collocation-type method when the discrete orthogonal projection or the interpolatory projection is used, respectively.

In the next section we consider the reduction of (14) to a system of nonlinear equations, and we give some details on the numerical implementation.

2.2. Implementation note. Let $\mathscr{Q}_n^G$ be the hyperinterpolation operator defined by (7) and $\varkappa_j(s) := \langle \varkappa(s,.), \varphi_j \rangle$, in order to give more details about the implementation of x_n, it is easy to show from (13) and (14), that x_n has the following form

$$x_n = f + \sum_{k=0}^{n} a_k \varphi_k + \sum_{k=0}^{n} b_k \varkappa_k, \tag{16}$$

where the coefficients $\{a_i, b_i\colon\ i \in \{0,1,...,n\}\}$ are obtained by substituting x_n from equation (16) into equation (14) then, we successively have

$$(\mathscr{Q}_n^G \mathcal{K}_m^N) x_n = \sum_{i=0}^{n} \langle \mathcal{K}_m^N x_n, \varphi_i \rangle_m \varphi_i = \sum_{i=0}^{n} \Big\{ \sum_{j=1}^{m} \Big[\sum_{k=1}^{m} \omega_k \varkappa(t_j, t_k) \overline{z}(t_k) \Big] \omega_j \varphi_i(t_j) \Big\} \varphi_i,$$

$$\mathcal{K}_n^D x_n = \sum_{i=0}^{n} \langle \overline{z}, \varphi_i \rangle_m \varkappa_i = \sum_{i=0}^{n} \Big[\sum_{j=1}^{m} \omega_j \overline{z}(t_j) \varphi_i(t_j) \Big] \varkappa_i,$$

$$(\mathscr{Q}_n^G \mathcal{K}_n^D) x_n = \sum_{i=0}^{n} \langle \mathcal{K}_n^D x_n, \varphi_i \rangle_m \varphi_i = \sum_{i=0}^{n} \Big\{ \sum_{j=1}^{m} \Big[\sum_{k=0}^{n} \Big(\sum_{l=1}^{m} \omega_l \overline{z}(t_l) \varphi_k(t_l) \Big) \varkappa_k(t_j) \Big] \omega_j \varphi_i(t_j) \Big\} \varphi_i,$$

where $\overline{z}(t_j) = \psi\Big(t_j, f(t_j) + \sum\limits_{k=0}^{n} a_k \varphi_k(t_j) + \sum\limits_{k=0}^{n} b_k \varkappa_k(t_j)\Big)$. Therefor we can identify the coefficients of φ_i and $\varkappa_j$ respectively, and we obtain the nonlinear system

$$\begin{cases} a_i = \sum\limits_{j=1}^{m} \Big[\sum\limits_{k=1}^{m} \omega_k \varkappa(t_j, t_k) \overline{z}(t_k) - \sum\limits_{k=0}^{n} b_k \varkappa_k(t_j) \Big] \omega_j \varphi_i(t_j), \\ b_i = \sum\limits_{j=1}^{m} \omega_j \overline{z}(t_j) \varphi_i(t_j). \end{cases}$$

For the interpolatory projection given by (9), we apply $\mathscr{Q}_n^C$ and $(\mathcal{I} - \mathscr{Q}_n^C)$ to equation (14), to obtain

$$\mathscr{Q}_n^C x_n - \mathscr{Q}_n^C \mathcal{K}_m^N = \mathscr{Q}_n^C f, \tag{17}$$

$$(\mathcal{I} - \mathscr{Q}_n^C) x_n - (\mathcal{I} - \mathscr{Q}_n^C) \mathcal{K}_n^D x_n = (\mathcal{I} - \mathscr{Q}_n^C) f. \tag{18}$$

By writing $\mathcal{K}_m^N x_n = \mathcal{K}_m^N (I - \mathscr{Q}_n^C) x_n + \mathcal{K}_m^N \mathscr{Q}_n^C x_n$, and replacing $(\mathcal{I} - \mathscr{Q}_n^C) x_n$ by its expression from equation (18), $\mathcal{K}_m^N x_n$ becomes $\mathcal{K}_m^N x_n = \mathcal{K}_m^N \big((\mathcal{I} - \mathscr{Q}_n^C) \mathcal{K}_n^D x_n + \mathscr{Q}_n^C x_n + (\mathcal{I} - \mathscr{Q}_n^C) f \big)$. Now, by replacing $\mathcal{K}_m^N x_n$ in equation (17), we obtain

$$\mathscr{Q}_n^C x_n - \mathscr{Q}_n^C \mathcal{K}_m^N \big((\mathcal{I} - \mathscr{Q}_n^C) \mathcal{K}_n^D x_n + \mathscr{Q}_n^C x_n + (\mathcal{I} - \mathscr{Q}_n^C) f \big) = \mathscr{Q}_n^C f,$$

and then for $i \in \{0,1,...,n\}$, we have

$$x_n(\tau_i) - \mathcal{K}_m^N \big((\mathcal{I} - \mathscr{Q}_n^C) \mathcal{K}_n^D x_n + \mathscr{Q}_n^C x_n + (\mathcal{I} - \mathscr{Q}_n^C) f \big)(\tau_i) = f(\tau_i).$$

From (18), the approximate solution is given by

$$x_n = \mathscr{Q}_n^C x_n + (\mathcal{I} - \mathscr{Q}_n^C) \mathcal{K}_n^D x_n + (\mathcal{I} - \mathscr{Q}_n^C) f = f + \sum_{i=0}^{n} (a_i - f_i) \ell_i +$$

$$+ \sum_{j=1}^{m} \Big[\sum_{i=0}^{n} \psi(\tau_i, a_i) \ell_i(t_j) \Big] \omega_j \varkappa(., t_j) - \sum_{k=0}^{n} \Big\{ \sum_{j=1}^{m} \Big[\sum_{i=0}^{n} \psi(\tau_i, a_i) \ell_i(t_j) \Big] \omega_j \varkappa(\tau_k, t_j) \Big\} \ell_k,$$

where $f_i := f(\tau_i)$ and $a_i := x_n(\tau_i)$. Now, applying $\mathscr{Q}_n^C$ to both sides of equations (14) and (15), we obtain $\mathscr{Q}^C x_n = \mathscr{Q}^C \mathcal{K}_m^N x_n + \mathscr{Q}^C f = \mathscr{Q}^C \widetilde{x}_n$, and this yields

$$x_n(\tau_j) = \widetilde{x}_n(\tau_j), \qquad j \in \{0, 1, ..., n\}. \tag{19}$$

The above formula proves that at the collocation node points, the convergence of x_n to x_0 is as rapid as that of $\widetilde{x}_n$ to x_0.

3. Convergence rates. The purpose of this section is to study the existence and uniqueness of approximate solutions of (1) and to discuss the superconvergence results. In the next theorem, we give first the error estimation for the integral operator $\mathcal{K}$ and the Nyström operator $\mathcal{K}_m^N$. Throughout this paper, we assume that $r \geq 2$ and introduce the following notation:

$$\psi_r(t) := \frac{\partial^r \psi}{\partial u^r}(t, x_0(t)), \quad q(s,t) := \frac{\partial^r \varkappa}{\partial s^r}(s,t), \quad q_s(t) := q(s,t), \quad s, t \in [-1, 1].$$

Theorem 2. *Suppose that* $\varkappa \in \mathcal{C}^d([-1,1]^2)$ *and* $\psi_0, \psi_1, g \in \mathcal{C}^d[-1,1]$, *with* $d \geq 2n > n \geq r \geq 2$. *Then there exists a positive constant* c_2 *independent of* n *such that*

$$\|\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)\|_\infty \leq c_2 n^{-d} \Psi_d \|\varkappa\|_{d,\infty}, \tag{20}$$

$$\|\mathcal{K}'(x_0)g - \mathcal{K}_m^{N'}(x_0)g\|_\infty \leq c_2 n^{-d} \Psi_d^* \|\varkappa\|_{d,\infty} \|g\|_{d,\infty}, \tag{21}$$

where $\Psi_d^* := \max\limits_{t \in [-1,1]} |\frac{\partial^d \psi_1}{\partial t^d}(t)|$.

Proof. It follows that for any $\mathcal{P} \in \mathbb{X}_d$

$$\left| (\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0))(s) \right| = \left| \int_{-1}^1 [\varkappa(s,t)\psi_0(t) - \mathcal{P}(t)]\, dt - \sum_{i=1}^m \omega_i [\varkappa(s,t_i)\psi_0(t_i) - \mathcal{P}(t_i)] \right| \leq$$

$$\leq \|\varkappa_s \psi_0 - \mathcal{P}\|_\infty \Big[\int_{-1}^1 dt + \sum_{i=1}^m \omega_i \Big] \leq 4 \|\varkappa_s \psi_0 - \mathcal{P}\|_\infty.$$

For $x \in \mathcal{C}^r[-1,1]$, we have from Jackson's theorem (see [8])

$$\inf_{\phi \in \mathbb{X}_n} \|x - \phi\|_\infty \leq c n^{-r} \|x^{(r)}\|_\infty. \tag{22}$$

Thus

$$\left| (\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0))(s) \right| \leq 4 \inf_{\mathcal{P} \in \mathbb{X}_d} \|\varkappa_s \psi_0 - \mathcal{P}\|_\infty \leq 4 c d^{-d} \|[\varkappa_s \psi_0]^{(d)}\|_\infty \leq 4 c d^{-d} \Psi_d \|\varkappa\|_{d,\infty}.$$

Since $d \geq 2n$, we have

$$\left| (\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0))(s) \right| \leq 4c(2n)^{-d} \Psi_d \|\varkappa\|_{d,\infty}. \tag{23}$$

Thus,

$$\|\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)\|_\infty \leq c_2 n^{-d} \Psi_d \|\varkappa\|_{d,\infty}. \tag{24}$$

Similarly, it can be demonstrated that $\|(\mathcal{K}'(x_0) - \mathcal{K}_m^{N'}(x_0))g\|_\infty \leq c_2 n^{-d} \Psi_d^* \|\varkappa\|_{d,\infty} \|g\|_{d,\infty}$, and the proof is completed. □

Lemma 3. *Assume that* $q \in \mathcal{C}^d([-1,1]^2)$ *and* $\psi_0, \psi_1, g \in \mathcal{C}^d[-1,1]$, *then*

$$\|[\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)]^{(r)}\|_\infty \le c_2 n^{-d} \Psi_d \|q\|_{d,\infty}, \tag{25}$$

$$\|[\mathcal{K}'(x_0)g - \mathcal{K}_m^{N'}(x_0)g]^{(r)}\|_\infty \le c_2 n^{-d} \Psi_d^* \|q\|_{d,\infty} \|g\|_{d,\infty}, \tag{26}$$

where c_2 *is a constant independent of* n.

Proof of the above lemma can be easily done using similar technique given in Theorem 2. The following lemma is crucial.

Lemma 4. *Let* $x_0 \in \mathbb{X}$ *be the unique solution of (1). Then for* n *large enough,* $(\mathcal{I} - \mathcal{K}_n^{M'}(x_0))^{-1}$ *exists and is uniformly bounded, i.e., there exists a constant* $A > 0$ *such that*

$$\|(\mathcal{I} - \mathcal{K}_n^{M'}(x_0))^{-1}\|_\infty \le A. \tag{27}$$

Proof. Let

$$q_j(s,t) := \frac{\partial^j \varkappa}{\partial s^j}(s,t) \frac{\partial \psi}{\partial u}(t, x_0(t)), \quad j \in \{0, 1, 2, ..., r\}.$$

For each $g \in \mathcal{C}^d[-1,1]$, it follows from the estimates (12), (21) and (26) that

$$\begin{aligned}
\|(\mathcal{K}'(x_0)g - \mathcal{K}_n^{M'}(x_0)g\|_\infty &= \|\mathscr{Q}_n[\mathcal{K}'(x_0) - \mathcal{K}_m^{N'}(x_0)]g\|_\infty + \\
&+ \|(\mathcal{I} - \mathscr{Q}_n)[\mathcal{K}'(x_0) - \mathcal{K}_n^{D'}(x_0)]g\|_\infty = \\
= \|(\mathcal{I} - \mathscr{Q}_n)[\mathcal{K}'(x_0) - \mathcal{K}_m^{N'}(x_0)]g\|_\infty &+ \|[\mathcal{K}'(x_0) - \mathcal{K}_m^{N'}(x_0)]g\|_\infty + \|(\mathcal{I} - \mathscr{Q}_n)\mathcal{K}'(x_0)g\|_\infty + \\
+ \|(\mathcal{I} - \mathscr{Q}_n)\mathcal{K}_n^{D'}(x_0)g\|_\infty &\le c_1 n^{\gamma - r} \|[[\mathcal{K}'(x_0) - \mathcal{K}_m^{N'}(x_0)]g]^r\|_\infty + c_2 n^{-d} \Psi_d^* \|\varkappa\|_{d,\infty} \|g\|_{d,\infty} + \\
+ c_1 n^{\gamma - r} \|[\mathcal{K}'(x_0)g]^r\|_\infty + c_1 n^{\gamma - r} \|[\mathcal{K}_n^{D'}(x_0)g]^r\|_\infty &\le c_1 c_2 n^{\gamma - r - d} \Psi_d^* \|q\|_{d,\infty} \|g\|_{d,\infty} + \\
+ c_2 n^{-d} \Psi_d^* \|\varkappa\|_{d,\infty} \|g\|_{d,\infty} + 2c_1 \zeta_2 n^{\gamma - r} \|g\|_\infty &+ c_1 n^{\gamma - r} \|[\mathcal{K}_n^{D'}(x_0)g]^r\|_\infty,
\end{aligned} \tag{28}$$

where $\zeta_2 := \max\limits_{s,t \in [-1,1]} |q_r(s,t)|$. Using Cauchy-Bunyakovsky-Schwarz inequality and estimate (10), we obtain

$$\begin{aligned}
\|[\mathcal{K}_n^{D'}(x_0)g]^{(r)}\|_\infty &= \max_{s \in [-1,1]} \Big| \sum_{i=1}^m \omega_i \frac{\partial^r \varkappa}{\partial s^r}(s, t_i) \mathscr{Q}_n \psi_1(t_i) g(t_i) \Big| \le \\
\le \|\varkappa\|_{r,\infty} \Big(\sum_{i=1}^m \omega_i \left(\mathscr{Q}_n \psi_1(t_i) \right)^2 \Big)^{\frac{1}{2}} \Big(\sum_{i=1}^m \omega_i \left(g(t_i) \right)^2 \Big)^{\frac{1}{2}} &\le \|\varkappa\|_{r,\infty} \langle \mathscr{Q}_n \psi_1, \mathscr{Q}_n \psi_1 \rangle_m^{\frac{1}{2}} \langle g, g \rangle_m^{\frac{1}{2}} \le \\
\le \sqrt{2} p \|\varkappa\|_{r,\infty} \|\psi_1\|_\infty \|g\|_\infty &= \sqrt{2} p \zeta_1 \|\varkappa\|_{r,\infty} \|g\|_\infty.
\end{aligned} \tag{29}$$

Since $0 < \gamma \le 1 < r$, $d \ge 2r$, the estimates (29) and (28) imply that

$$\begin{aligned}
\|(\mathcal{K}'(x_0) - \mathcal{K}_n^{M'}(x_0)\|_\infty &\le c_2 n^{-d} \Psi_d^* \big[c_1 n^{\gamma - r} \|q\|_{d,\infty} + \|\varkappa\|_{d,\infty} \big] + \\
&+ c_1 n^{\gamma - r} \big[2\zeta_2 + \sqrt{2} p \zeta_1 \|\varkappa\|_{r,\infty} \big] \to 0
\end{aligned}$$

as $n \to +\infty$. It then follows from the results of [1] that, $(\mathcal{I} - \mathcal{K}_n^{M'}(x_0))^{-1}$ exists and are uniformly bounded. □

Lemma 5. *let* $x_0 \in \mathbb{X}$ *be the unique solution of (1). Assume that* $q \in \mathcal{C}^d([-1,1]^2)$. *Then*

$$\|[\mathcal{K}_n(x_0) - \mathcal{K}_n^D(x_0)]^{(r)}\|_\infty = \mathcal{O}(n^{-d+1}). \tag{30}$$

Proof. Let $\mathscr{Q}_n^C$ be the interpolatory operator defined by (9). By the formula

$$[\mathcal{K}_n(x_0) - \mathcal{K}_n^D(x_0)](s) = \sum_{i=1}^{n} \psi_0(t_i)\Big[\int_0^1 \varkappa_s(t)\ell_i(t)dt - \langle \varkappa_s, \ell_i\rangle_m\Big],$$

it holds that for any $\mathcal{P} \in \mathbb{X}_d$

$$[\mathcal{K}_n(x_0) - \mathcal{K}_n^D(x_0)]^{(r)}(s) = \sum_{i=1}^{n} \psi_0(t_i)\Big[\int_0^1 q_s(t)\ell_i(t)dt - \sum_{j=1}^{m} w_j q_s(t_j)\ell_i(t_j)\Big] =$$

$$= \sum_{i=1}^{n} \psi_0(t_i)\Big[\int_{-1}^1 [q_s(t)\ell_i(t)dt - \mathcal{P}(t)]\,dt - \sum_{j=1}^{m} w_j\,[q(s,t_j)\ell_i(t_j) - \mathcal{P}(t_j)]\Big] \leq$$

$$\leq \sum_{i=1}^{n} \psi_0(t_i)\Big\{\|q_s\ell_i - \mathcal{P}\|_\infty\Big[\int_{-1}^1 dt + \sum_{j=1}^{m} w_j\Big]\Big\} \leq 4\sum_{i=1}^{n} \psi_0(t_i)\|q_s\ell_i - \mathcal{P}\|_\infty.$$

Now following the steps of (22) to (24), we can show that

$$\big|\,[\mathcal{K}_n(x_0) - \mathcal{K}_n^D(x_0)]^{(r)}(s)\,\big| \leq n\|\psi_0\|_\infty c_2 n^{-d}\|(q_s\ell_i)^{(d)}\|_\infty,$$

which means that

$$\|[\mathcal{K}_n(x_0) - \mathcal{K}_n^D(x_0)]^{(r)}\|_\infty = \mathcal{O}(n^{-d+1}). \tag{31}$$

A similar estimate can be obtained in the case of the hyperinterpolation projection. □

Given $x_0 \in \mathbb{X}$ and $\delta_0 > 0$, the set

$$\mathcal{B}(x_0,\delta_0) := \{x \in \mathbb{X}\colon \|x - x_0\|_\infty < \delta_0\}$$

is called the open ball centered at x_0 with radius δ_0. Using Theorem 2 given in [24], the following theorem can be proved.

Theorem 3. *Let $x_0 \in \mathbb{X}$ be an isolated solution of (1). Assume that 1 is not an eigenvalue of $\mathcal{K}'(x_0)$. Then there exists a real number $\delta_0 > 0$ such that the approximate equation (14) has a unique solution x_n in $\mathcal{B}(x_0,\delta_0)$ for a sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n such that*

$$\frac{\alpha_n}{1+q} \leq \|x_0 - x_n\|_\infty \leq \frac{\alpha_n}{1-q}, \tag{32}$$

where $\alpha_n = \big\|(\mathcal{I} - \mathcal{K}_n^{M\prime}(x_0))^{-1}(\mathcal{K}(x_0) - \mathcal{K}_n^M(x_0))\big\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$.

The next theorem establish the rate of convergence of the discrete Legendre Galerkin solutions of equation (14) to the exact solution x_0.

Theorem 4. *Let x_0 be an isolated solution of the equation (1) and x_n^G be the discrete Legendre modified-Galerkin approximation given by (14). We assume that $\varkappa, q \in C^d([-1,1]^2)$ and $\psi_0, \psi_1 \in \mathcal{C}^d[-1,1]$, with $d \geq 2n > n \geq r \geq 2$. Then*

$$\|x_0 - x_n^G\|_\infty = \mathcal{O}\left(\exp\{-\min\{3r-1, d\}\ln n\}\right). \tag{33}$$

Proof. We observe from Theorem 3 that to estimate $\|x_0 - x_n\|_\infty$ we need to estimate $\|\mathcal{K}(x_0) - \mathcal{K}_n^M(x_0)\|_\infty$. By using (13), we have

$$\|\mathcal{K}(x_0) - \mathcal{K}_n^M(x_0)\|_\infty \leq \|\mathscr{Q}_n^G(\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0))\|_\infty + \|(\mathcal{I} - \mathscr{Q}_n^G)(\mathcal{K}(x_0) - \mathcal{K}_n^D(x_0))\|_\infty \leq$$

$$\leq \|(\mathcal{I} - \mathscr{Q}_n^G)(\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0))\|_\infty + \|(\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0))\|_\infty +$$

$$+\|(\mathcal{I}-\mathscr{Q}_n^G)(\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0))\|_\infty.$$

Using estimates (12), (20), (25) and Lemma 4, we get

$$\begin{aligned}\|x_0-x_n^G\|_\infty \le A\big\{c_1 n^{-r+1}\|[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)]^r\|_\infty+\|(\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0))\|_\infty+\\ +c_1 n^{-r+1}\|[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty\big\}\le Ac_1c_2n^{-r-d+1}\Psi_d\|q\|_{d,\infty}+Ac_2n^{-d}\Psi_d\|\varkappa\|_{d,\infty}+\\ +Ac_1n^{-r+1}\|[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty.\end{aligned} \tag{34}$$

We quote the following estimate from the Theorem 3.2 in [3]

$$\|[\mathcal{K}(x_0)-\mathcal{K}_n(x_0)]^{(r)}\|_\infty=\mathcal{O}(n^{-2r}). \tag{35}$$

Now from the above estimate and (30), we obtain

$$\begin{aligned}\|[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty=\|[\mathcal{K}(x_0)-\mathcal{K}_n(x_0)+\mathcal{K}_n(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty\le\\ \le\|[\mathcal{K}(x_0)-\mathcal{K}_n(x_0)]^{(r)}\|_\infty+\|[\mathcal{K}_n(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty=\\ =\mathcal{O}\left(\exp\{-\min\{2r,d-1\}\ln n\}\right).\end{aligned} \tag{36}$$

This result, together with (34), proves (33) in view of $d\ge 2r$ and $r\ge 2$. □

Theorem 5. *Let x_0 be an isolated solution of the equation (1) and x_n^C be the discrete Legendre modified-collocation approximation given by (14). If $\varkappa, q\in\mathcal{C}^d([-1,1]^2)$ and $\psi_0,\psi_1\in\mathcal{C}^d[-1,1]$, with $d\ge 2n>n\ge r\ge 2$, then*

$$\|x_0-x_n^C\|_\infty=\mathcal{O}\big(n^{-2r+\frac{1}{2}}\big). \tag{37}$$

Proof. Using estimates (12), (13), (20), (25), (27) and Theorem 3, we have

$$\begin{aligned}\|x_0-x_n^C\|_\infty\le A\big\{\|\mathscr{Q}_n^C[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)]\|_\infty+\|(\mathcal{I}-\mathscr{Q}_n^C)(\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0))\|_\infty\big\}\le\\ \le A\big\{\|(\mathcal{I}-\mathscr{Q}_n^C)(\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0))\|_\infty+\|(\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0))\|_\infty+\\ +\|(\mathcal{I}-\mathscr{Q}_n^C)(\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0))\|_\infty\big\}\le\\ \le A\big\{c_1n^{-r+\frac{1}{2}}\|[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)]^r\|_\infty+\|(\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0))\|_\infty+\\ +c_1n^{-r+\frac{1}{2}}\|[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty\big\}\le\\ \le Ac_1c_2n^{-r-d+\frac{1}{2}}\Psi_d\|q\|_{d,\infty}+Ac_2n^{-d}\Psi_d\|\varkappa\|_{d,\infty}+Ac_1n^{-r+\frac{1}{2}}\|[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty.\end{aligned} \tag{38}$$

From the proof of Theorem 3.3 in [3] we have

$$\|[\mathcal{K}(x_0)-\mathcal{K}_n(x_0)]^{(r)}\|_\infty=\mathcal{O}(n^{-r}). \tag{39}$$

Similarly to (36), combining this estimate with (30), we obtain

$$\begin{aligned}\|[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty\le\|[\mathcal{K}(x_0)-\mathcal{K}_n(x_0)]^{(r)}\|_\infty+\|[\mathcal{K}_n(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty=\\ =\mathcal{O}\left(\exp\{-\min\{r,d-1\}\ln n\}\right).\end{aligned} \tag{40}$$

Thus, (37) follows from (38) and (40). □

The following results are needed to obtain the order of convergence of $\widetilde{x}_n$ to x_0.

Lemma 6. *Assume that $\varkappa\in\mathcal{C}^r([-1,1]^2)$ and $\frac{\partial\psi}{\partial u}\in\mathcal{C}^r(\Omega)$. Then, the operator $\mathcal{K}_n^{M'}$ is Lipschitz continuous in a neighborhood $\mathcal{B}(x_0,\delta_0)$ of x_0, that is, there exists a constant $\delta_3>0$ independent of n such that*

$$\|\mathcal{K}_n^{M'}(x_0)-\mathcal{K}_n^{M'}(x)\|_\infty\le\delta_3\|x_0-x\|_\infty,\quad x\in\mathcal{B}(x_0,\delta_0). \tag{41}$$

Proof. From equation (13), we have

$$\mathcal{K}_n^{M'}(y) = \mathscr{Q}_n\mathcal{K}_m^{N'}(y) + (\mathcal{I} - \mathscr{Q}_n)\mathcal{K}_n^{D'}(y), \qquad y \in \mathbb{X}. \tag{42}$$

Hence, for any $g \in \mathbb{X}$,

$$\|[\mathcal{K}_n^{M'}(x_0) - \mathcal{K}_n^{M'}(x)]g\|_\infty = \|\mathscr{Q}_n(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g\|_\infty + \|(\mathcal{I} - \mathscr{Q}_n)(\mathcal{K}_n^{D'}(x_0) - \mathcal{K}_n^{D'}(x))g\|_\infty.$$

Now using the Lipschitz's continuity of $\frac{\partial\psi}{\partial u}$ and estimates (6) and (12), we get

$$\begin{gathered}\|\mathscr{Q}_n(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g\|_\infty \le \|(\mathscr{Q}_n - \mathcal{I})(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g\|_\infty + \\ +\|(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g\|_\infty \le c_1 n^{\gamma-r}\|[(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g]^{(r)}\|_\infty + \\ +\|(\mathcal{K}_m^{N'}(x_0) - \mathcal{K}_m^{N'}(x))g\|_\infty \le 2c_1 n^{\gamma-r}\delta_2\|\varkappa\|_{r,\infty}\|x_0 - x\|_\infty\|g\|_\infty + 2\delta_2\|\varkappa\|_{0,\infty}\|x_0 - x\|_\infty\|g\|_\infty.\end{gathered} \tag{43}$$

On the other hand, using the estimate (10) and the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

$$\begin{aligned}&\|(\mathcal{I} - \mathscr{Q}_n)(\mathcal{K}_n^{D'}(x_0) - \mathcal{K}_n^{D'}(x))g\|_\infty \le c_1 n^{\gamma-r}\|[(\mathcal{K}_n^{D'}(x_0) - \mathcal{K}_n^{D'}(x))g]^{(r)}\|_\infty = \\ &= c_1 n^{\gamma-r}\max_{s\in[-1,1]}\Big|\sum_{i=1}^m \omega_i\frac{\partial^r\varkappa}{\partial s^r}(s,t_i)\mathscr{Q}_n\big[\frac{\partial\psi}{\partial u}(t_i, x_0(t_i)) - \frac{\partial\psi}{\partial u}(t_i, x(t_i))\big]g(t_i)\Big| \le \\ &\le c_1 n^{\gamma-r}\|\varkappa\|_{r,\infty}\Big\|\mathscr{Q}_n\big[\frac{\partial\psi}{\partial u}(., x_0(.)) - \frac{\partial\psi}{\partial u}(., x(.))\big]\Big\|_{\mathscr{L}^2}\|g\|_{\mathscr{L}^2} \le \\ &\le \sqrt{2}c_1 p\delta_2 n^{\gamma-r}\|\varkappa\|_{r,\infty}\|x_0 - x\|_\infty\|g\|_\infty.\end{aligned} \tag{44}$$

Since $0 < \gamma \le 1 < r$, the desired result follows now from (43) and (44) with

$$\delta_3 = 2\delta_2\|\varkappa\|_{0,\infty} + (2 + \sqrt{2}p)\delta_2 c_1 n^{\gamma-r}\|\varkappa\|_{r,\infty}.$$

□

In the next theorem we give the approximation error of the discrete iterated Legendre modified projection-type method.

Theorem 6. *We suppose that $\varkappa \in \mathcal{C}^r([-1,1]^2)$ and $\frac{\partial\psi}{\partial u} \in \mathcal{C}(\Omega)$. Let $x_0 \in \mathbb{X}$ be the unique solution of the integral equation (1). Then, for n sufficiently large, the iterated solution $\widetilde{x}_n$ given by (15), satisfies*

$$\|x_0 - \widetilde{x}_n\|_\infty \le c_3\|x_0 - x_n\|_\infty^2 + A\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0) - \mathcal{K}_n^M(x_0)]\|_\infty + \|\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)\|_\infty, \tag{45}$$

where c_3 is a constant independent of n.

Proof. Note that from (1) and (15) we have

$$x_0 - \widetilde{x}_n = \mathcal{K}x_0 - \mathcal{K}_m^N x_n = \mathcal{K}_m^N x_0 - \mathcal{K}_m^N x_n - \mathcal{K}_m^N x_0 + \mathcal{K}x_0.$$

Therefore, for some $0 < \theta < 1$, we get $\mathcal{K}_m^N x_0 - \mathcal{K}_m^N x_n = \mathcal{K}_m^{N'}(x_0 + \theta(x_0 - x_n))(x_0 - x_n) =$ $= [\mathcal{K}_m^{N'}(x_0 + \theta(x_0 - x_n)) - \mathcal{K}_m^{N'}(x_0) + \mathcal{K}_m^{N'}(x_0)](x_0 - x_n)$. Taking the norm on both sides of the above equation and applying the Lipschitz's continuity of $\mathcal{K}_m^{N'}$, we can show by using (6) that

$$\|x_0 - \widetilde{x}_n\|_\infty \le 2\delta_2\theta\|\varkappa\|_{0,\infty}\|x_0 - x_n\|_\infty^2 + \|\mathcal{K}_m^{N'}(x_0)(x_0 - x_n)\|_\infty + \|\mathcal{K}(x_0) - \mathcal{K}_m^N(x_0)\|_\infty. \tag{46}$$

Let

$$(\mathcal{I} - \mathcal{K}_n^{M'}(x_0))(x_0 - x_n) = \mathcal{K}(x_0) - \mathcal{K}_n^M(x_0) - \mathcal{K}_n^{M'}(x_0)(x_0 - x_n) + \mathcal{K}_n^M(x_0) - \mathcal{K}_n^M(x_n).$$

Applying $\mathcal{K}_m^{N'}(x_0)$ to both sides of the above equation and using the mean value theorem, we deduce that

$$\mathcal{K}_m^{N'}(x_0)(x_0 - x_n) = \mathcal{K}_m^{N'}(x_0)(\mathcal{I} - \mathcal{K}_n^{M'}(x_0))^{-1}[\mathcal{K}(x_0) - \mathcal{K}_n^M(x_0) - \mathcal{K}_n^{M'}(x_0)(x_0 - x_n) +$$

$$+\mathcal{K}_n^M(x_0)-\mathcal{K}_n^M(x_n)] = \mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathcal{K}_n^{M'}(x_0))^{-1}[\mathcal{K}(x_0)-\mathcal{K}_n^M(x_0)]+$$
$$+\mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathcal{K}_n^{M'}(x_0))^{-1}[\mathcal{K}_n^{M'}(x_0+\theta(x_0-x_n))-\mathcal{K}_n^{M'}(x_0)](x_0-x_n),$$

where $0<\theta<1$. Now from estimates (5), (27) and Lemma 6 one has

$$\|\mathcal{K}_m^{N'}(x_0)(x_0-x_n)\|_\infty \leq A\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0)-\mathcal{K}_n^M(x_0)]\|_\infty + 2A\theta\zeta_1\delta_3\|\varkappa\|_{0,\infty}\|x_0-x_n\|_\infty^2.$$

By combining this inequality with estimate (46), we get

$$\|x_0-\widetilde{x}_n\|_\infty \leq c_3\|x_0-x_n\|_\infty^2 + A\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0)-\mathcal{K}_n^M(x_0)]\|_\infty + \|\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)\|_\infty,$$

with $c_3 = 2\theta\|\varkappa\|_{0,\infty}(\delta_2+A\zeta_1\delta_3)$. □

Remark 3. Estimate (45) shows that the convergence of the iterated solution $\widetilde{x}_n$ depends on the convergence of three terms. We now justify that each of these tends to zero as $n\to\infty$:

- From Theorem 3, we know that $\|x_0-x_n\|_\infty \to 0$, since $\alpha_n\to 0$ and $0<q<1$.
- From Theorem 2, we have $\|\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)\|_\infty = \mathcal{O}\left(n^{-d}\right)$.
- For the remaining term, under the assumptions $d\geq 2r$ and $r\geq 2$, we distinguish the two types of projection:
  – If the method is based on orthogonal projection, then as we will prove later (see equation (50)), $\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0)-\mathcal{K}_n^M(x_0)]\|_\infty = \mathcal{O}\left(\exp\{-\min\{4r,d+2r-1,d\}\ln n\}\right)$.
  – If the method is based on interpolatory projection, then the following estimate will be established later (see equation (56)):
  $$\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0)-\mathcal{K}_n^M(x_0)]\|_\infty = \mathcal{O}\left(\exp\{-\min\{2r,r+d-1,d\}\ln n\}\right).$$

Therefore, in both cases, all terms on the right-hand side of inequality (45) converge to zero, which implies $\|x_0-\widetilde{x}_n\|_\infty \to 0 \quad \text{as } n\to\infty$.

The theorem below state that the iterated discrete Legendre modified Galerkin-type solution defined by (15) converges to x_0 faster than x_n^G.

Theorem 7. *Let x_0 be an isolated solution of the equation (1). We assume that $\varkappa, q\in C^d([-1,1]^2)$ and $\psi_0,\psi_1\in\mathcal{C}^d[-1,1]$, with $d\geq 2n>n\geq r\geq 2$. Then, for n sufficiently large, the iterated discrete Legendre modified-Galerkin solution $\widetilde{x}_n^G$ given by (15), satisfies*

$$\|x_0-\widetilde{x}_n^G\|_\infty = \mathcal{O}\left(\exp\{-\min\{4r,d\}\}\ln n\right). \tag{47}$$

Proof. For the second term of the estimate (45), we can write

$$\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0)-\mathcal{K}_n^M(x_0)]\|_\infty = \|\mathcal{K}_m^{N'}(x_0)\mathscr{Q}_n^G[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)]\|_\infty +$$
$$+\|\mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathscr{Q}_n^G)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]\|_\infty. \tag{48}$$

First, by the Cauchy-Bunyakovsky-Schwarz inequality and estimate (8), we get

$$\left|\mathcal{K}_m^{N'}(x_0)\mathscr{Q}_n^G[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)](s)\right| = \Big|\sum_{i=1}^m \omega_i q_0(s,t_i)\mathscr{Q}_n^G[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)](t_i)\Big| \leq$$
$$\leq \Big(\sum_{i=1}^m \omega_i[q_0(s,t_i)]^2\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^m \omega_i\Big\{\mathscr{Q}_n^G\big[\mathcal{K}(x_0)(t_i)-\mathcal{K}_m^N(x_0)(t_i)\big]\Big\}^2\Big)^{\frac{1}{2}} \leq$$
$$\leq \sqrt{2}\|q_0(s,.)\|_\infty\Big\langle \mathscr{Q}_n^G\big[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)\big], \mathscr{Q}_n^G\big[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)\big]\Big\rangle_m^{\frac{1}{2}} \leq$$
$$\leq 2\zeta_3\|\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)\|_\infty,$$

where $\zeta_3 := \max_{s,t\in[-1,1]}\left|q_0(s,t)\right|$. Thus, by using (20), we have

$$\|\mathcal{K}_m^{N'}(x_0)\mathscr{Q}_n^G[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)]\|_\infty \leq \Big\{2c_2\zeta_3\Psi_d\|\varkappa\|_{d,\infty}\Big\}n^{-d}. \tag{49}$$

Taking use of the Cauchy-Bunyakovsky-Schwarz inequality and estimate (8), we obtain

$$\begin{aligned}\Big|\mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathscr{Q}_n^G)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)](s)\Big| &= \Big|\sum_{i=1}^m \omega_i q_0(s,t_i)(\mathcal{I}-\mathscr{Q}_n^G)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)](t_i)\Big| = \\ &= \Big|\big\langle q_0(s,.),(\mathcal{I}-\mathscr{Q}_n^G)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]\big\rangle_m\Big| = \\ &= \Big|\big\langle(\mathcal{I}-\mathscr{Q}_n^G)q_0(s,.),(\mathcal{I}-\mathscr{Q}_n^G)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]\big\rangle_m\Big| \leq \\ &\leq \Big(\sum_{i=1}^m \omega_i\big[(\mathcal{I}-\mathscr{Q}_n^G)q_0(s,(t_i))\big]^2\Big)^{\frac12}\Big(\sum_{i=1}^m \omega_i\big[(\mathcal{I}-\mathscr{Q}_n^G)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)](t_i)\big]^2\Big)^{\frac12} = \\ &= \big\langle(\mathcal{I}-\mathscr{Q}_n^G)q_0(s,.),(\mathcal{I}-\mathscr{Q}_n^G)q_0(s,.)\big\rangle_m^{\frac12}\times \\ &\times\big\langle(\mathcal{I}-\mathscr{Q}_n^G)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)],(\mathcal{I}-\mathscr{Q}_n^G)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]\big\rangle_m^{\frac12} \leq \\ &\leq 2c_1^2n^{-2r}\|[q_0(s,.)]^{(r)}\|_\infty\|[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty = 2c_1^2\zeta_4 n^{-2r}\|[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty,\end{aligned}$$

where $\zeta_4 := \max_{s,t\in[-1,1]}|\frac{\partial^r q_0}{\partial t^r}(s,t)|$. Hence, from (36),

$$\|\mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathscr{Q}_n^G)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]\|_\infty = \mathcal{O}\big(\exp\{-\min\{4r,d+2r-1\}\ln n\}\big).$$

Then by combining (48), (49) and the above estimate, we obtain

$$\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0)-\mathcal{K}_n^M(x_0)]\|_\infty = \mathcal{O}\big(\exp\{-\min\{4r,d+2r-1,d\}\ln n\}\big). \tag{50}$$

This together with (20), (33) and Theorem 6 gives (47). □

The following theorem give the superconvergence of the iterated discrete Legendre modified collocation-type solution $\widetilde{x}_n^C$ to x_0.

Theorem 8. *Let x_0 be an isolated solution of the equation (1). We assume that $\varkappa, q \in C^d([-1,1]^2)$ and $\psi_0,\psi_1 \in \mathcal{C}^d[-1,1]$, with $d \geq 2n > n \geq r \geq 2$. Then, for n sufficiently large, the iterated discrete Legendre modified-collocation solution $\widetilde{x}_n^C$ given by (15), satisfies*

$$\|x_0-\widetilde{x}_n^C\|_\infty = \mathcal{O}\big(n^{-2r}\big). \tag{51}$$

Moreover, we have the following superconvergence result for x_n^C at the collocation points

$$\max_{0\leq i\leq n}|x_0(\tau_i)-x_n^C(\tau_i)| = \mathcal{O}\big(n^{-2r}\big). \tag{52}$$

Proof. Consider

$$\begin{aligned}\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0)-\mathcal{K}_n^M(x_0)]\|_\infty &= \|\mathcal{K}_m^{N'}(x_0)\mathscr{Q}_n^C[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)]\|_\infty + \\ &+\|\mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathscr{Q}_n^C)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]\|_\infty.\end{aligned} \tag{53}$$

By using the estimates (10) and (20), we obtain

$$\begin{aligned}&\|\mathcal{K}_m^{N'}(x_0)\mathscr{Q}_n^C[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)]\|_\infty \leq \\ &\leq \Big(\sum_{i=1}^m \omega_i[q_0(s,t_i)]^2\Big)^{\frac12}\Big(\sum_{i=1}^m \omega_i\Big\{\mathscr{Q}_n^C\big[\mathcal{K}(x_0)(t_i)-\mathcal{K}_m^N(x_0)(t_i)\big]\Big\}^2\Big)^{\frac12} = \\ &= \|q_0(s,.)\|_{\mathscr{L}^2}\big\|\mathscr{Q}_n^C\big[\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)\big]\big\|_{\mathscr{L}^2} \leq \\ &\leq \sqrt{2}p\|q_0(s,.)\|_\infty\|\mathcal{K}(x_0)-\mathcal{K}_m^N(x_0)\|_\infty \leq \Big\{\sqrt{2}pc_2\zeta_3\Psi_d\|\varkappa\|_{d,\infty}\Big\}n^{-d}.\end{aligned} \tag{54}$$

For the second term in (53), by applying (11) and the Cauchy-Bunyakovsky-Schwarz inequality, we can write

$$\left|\mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathscr{Q}_n^C)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)](s)\right| = \left|\big\langle q_0(s,.),(\mathcal{I}-\mathscr{Q}_n^C)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]\big\rangle_m\right| \leq$$
$$\leq \|q_0(s,.)\|_{\mathscr{L}^2}\|(\mathcal{I}-\mathscr{Q}_n^C)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]\|_{\mathscr{L}^2} \leq$$
$$\leq \sqrt{2}c_1 n^{-r}\|q_0(s,.)\|_\infty \|[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_{\mathscr{L}^2} \leq 2c_1\zeta_3 n^{-r}\|[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]^{(r)}\|_\infty. \quad (55)$$

Therefore, using (40) we deduce that

$$\|\mathcal{K}_m^{N'}(x_0)(\mathcal{I}-\mathscr{Q}_n^C)[\mathcal{K}(x_0)-\mathcal{K}_n^D(x_0)]\|_\infty = \mathcal{O}\big(\exp\{-\min\{2r, r+d-1\}\ln n\}\big).$$

Now using the above estimate and (54) in the estimate (53), we obtain

$$\|\mathcal{K}_m^{N'}(x_0)[\mathcal{K}(x_0)-\mathcal{K}_n^M(x_0)]\|_\infty = \mathcal{O}\big(\exp\{-\min\{2r, r+d-1, d\}\ln n\}\big). \quad (56)$$

Combining (20), (37), (45) and (56), the estimate (51) is proved. We recall from (19), that the solutions x_n^C and $\widetilde{x}_n^C$ agree at the collocation node points, and therefore (52) comes from (51) and this concludes the proof. □

Remark 4. If the assumptions of Remark 2 holds, the hyperinterpolation operator $\mathscr{Q}_n^G$ reduces to the interpolatory projection operator $\mathscr{Q}_n^C$ and consequently the iterated discrete Legendre modified collocation-type approximation exhibits the following superconvergence rate

$$\|x_0-\widetilde{x}_n^C\|_\infty = \mathcal{O}\big(\exp\{-\min\{4r, d\}\ln n\}\big), \quad \text{with } d \geq 2n > n \geq r \geq 2.$$

Remark 5. The parameter r in the error estimates plays the role of a regularity index that reflects the smoothness of the exact solution. Although the optimal convergence rate is achieved when $r = n$, keeping r as a free parameter allows the analysis to cover a broader class of solutions with different smoothness levels. Similarly, while the choice $d = 2n$ yields optimal accuracy, we prefer to adopt the more general condition $d \geq 2n$, which is standard in the literature on discrete projection methods (see, e.g., [5,16,22]). This formulation provides greater flexibility without altering the validity of the theoretical results.

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