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ON THE SPECTRUM OF LINEAR OPERATOR PENCILS

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We consider a linear operator pencil $L(\lambda) = A - \lambda B$, $\lambda \in \mathbb{C}$, where A and B are bounded operators on Hilbert space. The purpose of this paper is to study the conditions under which the spectrum of $L(\cdot)$ is the whole complex plane or empty. This leads to some criteria for the spectrum to be bounded.

1. Introduction. Several problems of operator theory and mathematical physics (quantum theory, transport equations, ...) are reduced to the study of certain spectral properties of the linear operator pencil

$$L(\lambda) = A - \lambda B, \quad \lambda \in \mathbb{C}, \quad (1)$$

where A and B are two bounded operators on some Hilbert space \mathcal{H} . See, for instance, [4, 3, 5, 14] and references therein. The spectrum of $L(\cdot)$ is defined by

$$\begin{aligned} \sigma(L(\lambda)) &= \{\lambda \in \mathbb{C} : A - \lambda B \text{ is not invertible in } \mathcal{B}(\mathcal{H})\} = \\ &= \{\lambda \in \mathbb{C} : 0 \in \sigma(L(\lambda))\}. \end{aligned}$$

It well known that the spectrum of bounded operator is never empty nor equal to \mathbb{C} and in the unbounded case we have also, if $\sigma(T) \neq \mathbb{C}$ then T is closed, whereas there exist closed unbounded operators with spectrum may be empty or equal the whole complex plane. These situations occur for the linear operator pencils even for the finite dimensional spaces. According to this fact the following natural question arises:

What conditions on A and B such that the spectrum of $L(\cdot)$ equals the whole complex plane or is empty?

The main objective of this paper is to give some answers of this question. Further, we establish some situations such that the spectrum of $L(\cdot)$ is bounded.

2. Main results. Throughout this paper, \mathcal{H} will denote a separable, complex Hilbert space, and $\mathcal{B}(\mathcal{H})$ is the Banach space of all bounded linear operators on \mathcal{H} . The resolvent set, the spectrum, the kernel and the range of an operator $T \in \mathcal{B}(\mathcal{H})$ are denoted by $\rho(T)$, $\sigma(T)$, $N(T)$ and $R(T)$, respectively.

In what follows we give a well known conditions on A and B such that the spectrum of $L(\cdot)$ is empty.

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Theorem 1. For a pair (A, B) of operators in $\mathcal{B}(\mathcal{H})$, $\sigma(L(\lambda)) = \emptyset$ if and only if A is invertible and $A^{-1}B$ (or BA^{-1}) is quasinilpotent.

Proof. For $\lambda = 0$, we have that A is invertible if and only if $0 \notin \sigma(A - \lambda B) = \sigma(A)$. Now if $\lambda \neq 0$, and A is invertible, we have

$$A - \lambda B = A(I - \lambda A^{-1}B) = (I - \lambda BA^{-1})A.$$

So, $A - \lambda B$ is boundedly invertible if and only if $I - \lambda A^{-1}B$ or $(I - \lambda BA^{-1})$ is invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$, equivalently to $A^{-1}B$ or BA^{-1} is quasinilpotent. \square

Corollary 1. For a pair (A, B) of operators in $\mathcal{B}(\mathcal{H})$, we have

- If A is invertible commutes with B , then $\sigma(L(\lambda)) = \emptyset$ if and only if B is quasinilpotent.
- If A is not invertible, then $A - \lambda B$ is quasinilpotent if and only if B is invertible and AB^{-1} or $B^{-1}A$ is quasinilpotent.
- If A is not invertible commutes with the invertible operator B , then $A - \lambda B$ is quasinilpotent if and only if A is quasinilpotent.

The point spectrum of $L(\cdot)$ is defined by

$$\begin{aligned} \sigma_p(L(\lambda)) &= \{\lambda \in \mathbb{C} : \text{such that } A - \lambda B \text{ is not injective}\} = \\ &= \{\lambda \in \mathbb{C} : \text{such that } 0 \in \sigma_p(L(\lambda))\}. \end{aligned}$$

We know that if $N(A) \cap N(B) \neq \{0\}$ then $\sigma_p(L(\lambda)) = \mathbb{C}$ (and hence $\sigma(L(\lambda)) = \mathbb{C}$). But the converse is not true even if \mathcal{H} is of finite dimension as we can see in the following example.

Example 1. Let A and B defined on \mathbb{C}^2 by

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have

$$A - \lambda B = \begin{pmatrix} 0 & 0 \\ 1 & -\lambda \end{pmatrix} \quad \text{and} \quad \lambda A - B = \begin{pmatrix} 0 & 0 \\ \lambda & -1 \end{pmatrix}.$$

Clearly, $A - \lambda B$ and $\lambda A - B$ are not invertible for all $\lambda \in \mathbb{C}$ but

$$N(A) \cap N(B) = \text{span}\{(0, 1)\} \cap \text{span}\{(1, 0)\} = \{(0, 0)\}.$$

Here $\text{span}(v)$ denote the linear space spanned by a vector v .

Recall that, for $A, B \in \mathcal{B}(\mathcal{H})$; A and B commute up to a factor, i.e. $AB = \mu BA$; for some $\mu \in \mathbb{C} \setminus \{0\}$.

Theorem 2. Let \mathcal{H} be a complex Hilbert space with $\dim \mathcal{H} \geq 2$. For a pair (A, B) of operators in $\mathcal{B}(\mathcal{H})$ with B normal (or A normal) and $BA = \mu AB \neq 0$ for some $\mu \in \mathbb{C}$, we have $\sigma_p(L(\lambda)) = \mathbb{C}$ if and only if $N(A) \cap N(B) \neq \{0\}$.

Proof. Suppose that $\sigma_p(L(\lambda)) = \mathbb{C}$. Since $0 \in \sigma_p(L(\lambda))$, A is not injective and $N(A) \neq \{0\}$. Let $(\lambda_n)_n \subset \sigma_p(L(\lambda)) \setminus \{0\}$, then $\frac{1}{\lambda_n} \in \sigma_p(\frac{1}{\lambda_n}A - B)$. Now if $|\lambda_n| \rightarrow \infty$, then $1/\lambda_n \rightarrow 0$, as $n \rightarrow \infty$. So B is not injective and $N(B) \neq \{0\}$. Assume that $N(A) \cap N(B) = \{0\}$. If

$BA = \mu AB$ and B is normal, then $N(B)$ and $\overline{R(B)}$ are reducing subspaces of A and B . Thus A and B have the following forms

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix},$$

with respect to $\mathcal{H} = N(B)^\perp \oplus N(B)$. Therefore, since A_2 is injective, $A - \lambda B$ is not injective for all $\lambda \in \mathbb{C} \setminus \{0\}$ if and only if $A_1 - \lambda B_1$ is not injective for all $\lambda \in \mathbb{C} \setminus \{0\}$, if and only if $B_1(B_1^{-1}A_1 - \lambda I)$ is not injective for all $\lambda \in \mathbb{C} \setminus \{0\}$, equivalently to $\mathbb{C} \setminus \{0\} \subset \sigma_p(B_1^{-1}A_1)$. This is a contradiction with the boundedness of $\sigma_p(B_1^{-1}A_1)$. Now if we assume A is normal, by interchanging the roles of A and B , we obtain the desired result.

The converse follows from the fact that $\{0\} \neq N(A) \cap N(B) \subseteq N(A - \lambda B)$ for all $\lambda \in \mathbb{C}$. □

Remark 1. 1. Yang and Du, [17, Theorem 2.1], proved that if both A and B are normal, then $|\mu| = 1$.

2. Without the commutation of A and B up to a factor, the theorem is not true. For example let the matrices A and B be given by

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have B is symmetric and

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq \mu \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \mu BA$$

for all $\mu \in \mathbb{C}$. On the other hand,

$$A - \lambda B = \begin{pmatrix} 0 & 0 \\ 1 & 1 - \lambda \end{pmatrix} \quad \text{and} \quad \lambda A - B = \begin{pmatrix} 0 & 0 \\ \lambda & \lambda - 1 \end{pmatrix}.$$

Clearly, $A - \lambda B$ and $\lambda A - B$ are not invertible for all $\lambda \in \mathbb{C}$ and

$$N(A) \cap N(B) = \text{span}\{(1, -1)\} \cap \text{span}\{(1, 0)\} = \{(0, 0)\}.$$

By [7, Theorem 2.2] we obtain some conditions equivalent to the condition given in Theorem 2.

Corollary 2. For a pair (A, B) of operators in $\mathcal{B}(\mathcal{H})$ with closed ranges, if B normal and $BA = \mu AB \neq 0$ for some $\mu \in \mathbb{C}$, then the following statements are equivalent,

1. $\sigma_p(L(\lambda)) = \mathbb{C}$,
2. $N(A) \cap N(B) \neq \{0\}$,
3. $R(A^*) + R(B^*)$ is a proper closed subset of \mathcal{H} ,
4. $R((A^*A + B^*B)^{1/2})$ is a proper closed subset of \mathcal{H} .

Proof. If A and B have closed ranges then also $R(A^*)$ and $R(B^*)$ are closed. By [10, Theorem 4.8, Chapter IV], we have $\mathcal{H} \neq (N(A) \cap N(B))^\perp = N(A)^\perp + N(B)^\perp = R(A^*) + R(B^*)$. This proves that items (1) and (2) are equivalent. Now by [7, Theorem 2.2], we have $R(A^*) + R(B^*) = R((A^*A + B^*B)^{1/2})$, so by this and the preceding theorem we get the desired result. □

Theorem 3. For a pair (A, B) of positive self-adjoint operators in $\mathcal{B}(\mathcal{H})$ we have if $N(A) \cap N(B) = \{0\}$ then $\sigma_p(L(\lambda)) \subseteq \mathbb{R}$.

Proof. Assume that there exists λ_0 an eigenvalue of $A - \lambda B$ with $\text{Im}(\lambda_0) \neq 0$. Let x_0 be an eigenvector corresponding to λ_0 . Then

$$\langle (A - \lambda_0 B)x_0, x_0 \rangle = 0,$$

Since $\langle Ax_0, x_0 \rangle$ and $\langle Bx_0, x_0 \rangle$ are real, we obtain

$$\langle Ax_0, x_0 \rangle - \text{Re}(\lambda_0)\langle Bx_0, x_0 \rangle = 0$$

and

$$\text{Im}(\lambda_0)\langle Bx_0, x_0 \rangle = 0.$$

By the second equation, we get $\langle Bx_0, x_0 \rangle = 0$, so $x_0 \in R(B)^\perp = N(B^*) = N(B)$. Now, the first equation yields $Ax_0 = 0$, which contradicts our assumption. \square

Brook et al, [2, Theorem 1.1], proved that if both A and B are self-adjoint and $BA = \mu AB \neq 0$ for some $\mu \in \mathbb{C}$, then $\mu = 1$ or $\mu = -1$. Furthermore, if one of them positive then $\mu = 1$. So by Theorems 2 and 3, we have

Corollary 3. For a pair (A, B) of positive self-adjoint operators in $\mathcal{B}(\mathcal{H})$ such that $AB = BA$, we have $N(A) \cap N(B) = \{0\}$ if and only if $\sigma_p(L(\lambda)) \subseteq \mathbb{R}$.

For the finite dimensional case, we have

Corollary 4. For a pair (A, B) of positive hermitian matrices in $\mathcal{B}(\mathbb{C}^n)$ such that $AB = BA$, we have $N(A) \cap N(B) = \{0\}$ if and only if $A - \lambda B$ is invertible for some $\lambda \in \mathbb{C}$. In this case $\sigma_p(L(\lambda))$ contains no more than n real points.

Remark 2. Without the sign-definite; this corollary is not true, even if A and B are real. For example we take the following symmetric matrices

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We have $\sigma_p(L(\lambda)) = \{i, -i\}$. Here $AB = -BA$ and $N(A) \cap N(B) = \{0\}$.

Theorem 4. For a pair (A, B) of operators in $\mathcal{B}(\mathcal{H})$ with B has closed range, if $\sigma_p(L(\lambda)) = \mathbb{C}$ then $N(A) \cap N(B) \neq \{0\}$ or $A(N(B))$ is not closed.

Proof. Suppose that $\sigma_p(L(\lambda)) = \mathbb{C}$, as before we have $N(A) \neq \{0\}$ and $N(B) \neq \{0\}$. Since $R(B)$ has closed range, A and B as operators from $R(B^*) \oplus N(B)$ into $R(B) \oplus N(B^*)$ have the following forms

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

with B_1 is an invertible operator from $R(B^*)$ to $R(B)$. Since $\sigma_p(L(\lambda)) = \mathbb{C}$, take $(\lambda_n)_n \subset \sigma_p(L(\lambda))$ and $|\lambda_n| \rightarrow \infty$. There exists a sequence of unit vectors $x_n = y_n + z_n$, where $y_n \in R(B^*)$ and $z_n \in N(B)$ such that $(A - \lambda_n B)x_n = 0$ for all n .

$$\begin{cases} (B_1^{-1}A_1 - \lambda_n I)y_n + B_1^{-1}A_2z_n = 0, \\ A_3y_n + A_4z_n = 0, \end{cases} \quad (2)$$

This implies that $\|y_n\| \rightarrow 0$ and $\|z_n\| \rightarrow 1$, as $n \rightarrow \infty$, we have $B_1^{-1}A_2z_n \rightarrow 0$ and $A_4z_n \rightarrow 0$, so the operator column

$$\begin{pmatrix} B_1^{-1}A_2 \\ A_4 \end{pmatrix} : N(B) \rightarrow R(B^*) \times N(B^*)$$

is not bounded from below. Since B_1 is invertible,

$$\mathcal{A} = \begin{pmatrix} A_2 \\ A_4 \end{pmatrix} : N(B) \rightarrow R(B) \times N(B^*)$$

is not bounded from below. Thus \mathcal{A} is not injective or $R(\mathcal{A})$ is not closed. Clearly, if \mathcal{A} is not injective then $N(\mathcal{A}) \cap N(B) \neq \{0\}$. Now, assume that \mathcal{A} is injective with nonclosed range. In this case we have $R(\mathcal{A}) = A(N(B))$ is not closed in $R(B) \times N(B^*)$. This implies that $A(N(B))$ is also not closed in \mathcal{H} , since both $R(B)$ and $N(B^*)$ are closed in \mathcal{H} . \square

For $T \in \mathcal{B}(\mathcal{H})$, an element $S \in \mathcal{B}(\mathcal{H})$ is the Drazin inverse of T , if the following hold:

$$STS = S; ST = TS; \text{ and } T^{m+1}S = T^m;$$

for some nonnegative integer m . The smallest such m is called the Drazin index of T . We know that if $T \in \mathcal{B}(\mathcal{H})$ is a Drazin invertible then there exists a nonnegative integer m , such that $\mathcal{H} = R(T^m) \oplus N(T^m)$, the restriction of T on $R(T^m)$ is invertible, and the restriction of T on $N(T^m)$ is a nilpotent operator of order m (i.e. $T^{m-1} \neq 0$ and $T^m = 0$).

Theorem 5. For a pair (A, B) of operators in $\mathcal{B}(\mathcal{H})$ with B is a Drazin invertible with index $m \geq 1$, we have if $\sigma_p(L(\lambda)) = \mathbb{C}$ then $N(A) \cap N(B^m) \neq \{0\}$ or $A(N(B^m))$ is not closed.

Proof. Suppose that $\sigma_p(L(\lambda)) = \mathbb{C}$, as before we have $N(A) \neq \{0\}$ and $N(B) \neq \{0\}$, hence $N(A^m) \neq \{0\}$ and $N(B^m) \neq \{0\}$. So A and B as operators from $R(B^m) \oplus N(B^m)$ into $R(B^m) \oplus N(B^m)$ have the following form

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

with B_1 is an invertible operator on $R(B^m)$ and B_2 is a nilpotent operator of order m . Since $\sigma_p(L(\lambda)) = \mathbb{C}$, take $(\lambda_n)_n \subset \sigma_p(L(\lambda))$ and $|\lambda_n| \rightarrow \infty$. There exists a sequence of unit vectors $x_n = y_n + z_n$, where $y_n \in R(B^m)$ and $z_n \in N(B^m)$ such that $(A - \lambda_n B)x_n = 0$ for all n .

$$\begin{cases} (B_1^{-1}A_1 - \lambda_n I)y_n + B_1^{-1}A_2z_n & = 0, \\ A_3y_n + (A_4 - \lambda_n B_2)z_n & = 0. \end{cases} \tag{3}$$

This implies that $\|y_n\| \rightarrow 0$ and $\|z_n\| \rightarrow 1$, as $n \rightarrow \infty$, we have $B_1^{-1}A_2z_n \rightarrow 0$ and $A_4z_n \rightarrow 0$, so the operator column

$$\begin{pmatrix} B_1^{-1}A_2 \\ A_4 \end{pmatrix} : N(B^m) \rightarrow R(B^m) \times N(B^m)$$

is not bounded from below. Since B_1 is invertible,

$$\mathcal{A} = \begin{pmatrix} A_2 \\ A_4 \end{pmatrix} : N(B^m) \rightarrow R(B^m) \times N(B^m)$$

is not bounded from below. Thus \mathcal{A} is not injective or $R(\mathcal{A})$ is not closed. Clearly, if \mathcal{A} is not injective then $N(A) \cap N(B^m) \neq \{0\}$. Now, assume that \mathcal{A} is injective with nonclosed range. In this case, we have $R(\mathcal{A}) = A(N(B^m))$ is not closed in $R(B^m) \times N(B^m)$. This implies that $A(N(B^m))$ is also not closed in \mathcal{H} , since both $R(B^m)$ and $N(B^m)$ are closed in \mathcal{H} . \square

The approximate spectrum of $L(\cdot)$ is defined by

$$\begin{aligned} \sigma_{ap}(L(\lambda)) &= \{\lambda \in \mathbb{C} : A - \lambda B \text{ is not injective or its range is not closed}\} = \\ &= \{\lambda \in \mathbb{C} : 0 \in \sigma_{ap}(L(\lambda))\}. \end{aligned}$$

Theorem 6. *Let \mathcal{H} be a complex Hilbert space. For a pair (A, B) of operators in $\mathcal{B}(\mathcal{H})$ with B has closed range, the following statements are equivalent*

1. $\sigma_{ap}(L(\lambda)) = \mathbb{C}$,
2. $\sigma_{ap}(L(\lambda))$ is not bounded,
3. B is not injective and A restricted to $N(B)$ is not bounded from below,
4. $N(A) \cap N(B) \neq \{0\}$ or $A(N(B))$ is not closed.

Proof. (1) \implies (2). It is clear.

(2) \implies (3). Assume that $\sigma_{ap}(L(\lambda))$ is not bounded and B is injective. So B is bounded from below and hence by [11, Theorem 1] there exists a positive number c such that $A - \lambda B$ or $\frac{1}{\lambda}A - B$ is bounded from below for all $|\lambda| > \frac{\|A\|}{c}$, this contradicts the fact $\sigma_{ap}(L(\lambda))$ is not bounded. So B is not injective. Now, since $R(B)$ has closed range, A and B as operators from $R(B^*) \oplus N(B)$ into $R(B) \oplus N(B^*)$ have the following form

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

with B_1 is an invertible operator from $R(B^*)$ to $R(B)$. Since $\sigma_{ap}(L(\lambda))$ is not bounded, there exists a sequence $(\lambda_n)_n \subset \sigma_{ap}(L(\lambda))$ such that $|\lambda_n| \rightarrow \infty$. For a fixed n , there exists a sequence $(x_n^k)_k$ of unit vectors such that

$$\|(A - \lambda_n B)x_n^k\| < \frac{1}{k}$$

for all n . x_n^k has the decomposition $x_n^k = y_n^k + z_n^k$, where $y_n^k \in R(B^*)$ and $z_n^k \in N(B)$. Thus we have

$$\delta|\lambda_n|\|y_n^k\| \leq \|\lambda_n B y_n^k\| = \|\lambda_n B x_n^k\| \leq \|(A - \lambda_n B)x_n^k\| + \|A x_n^k\| < \frac{1}{k} + \|A\|,$$

hence

$$\|y_n^k\| < \frac{1}{\delta|\lambda_n|} \left(\frac{1}{k} + \|A\| \right).$$

This implies that $\|y_n^k\| \rightarrow 0$ and $\|z_n^k\| \rightarrow 1$, as $n \rightarrow \infty$. On the other hand, we have

$$\|(A - \lambda_n B)z_n^k\| = \|A z_n^k\| < \frac{1}{k}.$$

For n sufficiently large, we can construct a sequence $(z_n^k)_k$ of unit vectors of $N(B)$ such that $\|A z_n^k\| < \frac{1}{k}$. This implies that A restricted to $N(B)$ is not bounded from below.

(3) \implies (4). B is not injective, implies that $N(B) \neq \{0\}$ and A restricted to $N(B)$ is not bounded from below it means that $N(A) \cap N(B) \neq \{0\}$ or $A(N(B))$ is not closed.

(4) \implies (1). If $N(A) \cap N(B) \neq \{0\}$, the statement follows from the fact that $N(A) \cap N(B) \subseteq N(A - \lambda B)$ for all $\lambda \in \mathbb{C}$. Now, assume that $A(N(B))$ is not closed. Therefore, A restricted to $N(B)$ is not bounded from below, hence there exists a sequence $(x_n)_n$ of unit vectors of $N(B)$ such that $\|Ax_n\| < \frac{1}{n}$. But $\|(L(\lambda))x_n\| = \|Ax_n\| < \frac{1}{n}$. Thus $\lambda \in \sigma_{ap}(A - \lambda B)$, for all $\lambda \in \mathbb{C}$, that is $\sigma_{ap}(A - \lambda B) = \mathbb{C}$. \square

We recall that, if the range $R(T)$ of T is closed and $\alpha(T) := \dim N(T) < \infty$ (resp. $\beta(T) := \dim N(T^*) < \infty$), then T is called an upper (resp. a lower) semi-Fredholm operator. If T is either upper or lower semi-Fredholm, then T is called a semi-Fredholm operator, and the index of T is defined by $\text{ind}(T) := \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a Fredholm operator. An operator $R \in \mathcal{B}(\mathcal{H})$ is said to be a Riesz operator if $\lambda I - R$ is a Fredholm operator for all $\lambda \neq 0$.

Theorem 7. *For a pair (A, B) of operators in $\mathcal{B}(\mathcal{H})$, we have $\sigma(L(\lambda)) = \mathbb{C}$, proved that one of the following conditions hold.*

1. *There exists a natural number $m \geq 1$ such that $A(N(B^m))$ or $B(N(A^m))$ is not closed.*
2. *$R(A) \subset R(B)$ and B is not invertible.*
3. *$R(B) \subset R(A)$ and A is not invertible.*
4. *A is not an invertible semi-Fredholm operator and B is a compact operator.*
5. *A is not an invertible semi-Fredholm operator and B is a Riesz operator commuting with B .*
6. *$(A^*A + B^*B)^{1/2}$ is not a Fredholm operator.*
7. *$(AA^* + BB^*)^{1/2}$ is not a Fredholm operator.*

Proof. (1) If $A(N(B^m))$ (resp. $B(N(A^m))$) is not closed, then B (resp. A) must be not injective. Now observe that $(A - \lambda B)(N(B^m)) = A(N(B^m))$ (resp. $(A - \lambda B)(N(A^m)) = -\lambda B(N(A^m))$) for all $\lambda \in \mathbb{C}$. So if $\sigma(A - \lambda B) \neq \mathbb{C}$, there exists a λ such that the operator $(A - \lambda B)$ has a continuous inverse, so it maps the closed subspace $N(B^m)$ (resp. $N(A^m)$) to a closed subspace. This is a contradiction.

(2) If $R(A) \subset R(B)$, then by [6, Theorem 1] there exists a bounded operator C on \mathcal{H} such that $A = BC$. Hence $A - \lambda B = B(C - \lambda I)$, since B is not invertible, $A - \lambda B$ for all $\lambda \in \mathbb{C}$.

(3) By interchanging the roles of A and B in (2), we obtain the desired result.

(4) It follows from the fact that A is not invertible, $K = -\lambda B$ and A is a semi-Fredholm operator if and only if $A + K$ is semi-Fredholm for all compact operator K .

(5) As consequence of the commuting Riesz perturbations of semi-Fredholm operators, we have $A - \lambda B$ is not invertible for all $\lambda \neq 0$. Since A is not invertible we have also $A - \lambda B$ is not invertible for $\lambda = 0$.

(6) Let $T = (A^*A + B^*B)^{1/2}$. Since T is not Fredholm, by [8, Proposition 2] there exists a quasinilpotent operator Q on \mathcal{H} such that $T^2 = QQ^*$. Since $A^*A \leq T^2 = QQ^*$ and $B^*B \leq T^2 = QQ^*$, it follows again from [6, Theorem 1] that there exist operators C_1 and C_2 such that $A = C_1Q$ and $B = C_2Q$. Hence $A - \lambda B = (C_1 - \lambda C_2)Q$, since Q is not invertible, $A - \lambda B$ for all $\lambda \in \mathbb{C}$.

(7) By duality in (6). \square

The items (4) and (5) of the preceding theorem may be extended to polynomial operator pencils

$$P(\lambda) = \lambda^n A_0 + \lambda^{n-1} A_1 + \dots + A_n$$

of degree n , with coefficients A_i belong to the space $\mathcal{B}(\mathcal{H})$, as follows,

Theorem 8. *If $(\sum_{i=0}^n A_i^* A_i)^{1/2}$ (or $(\sum_{i=0}^n A_i A_i^*)^{1/2}$ is not Fredholm operator, then $\sigma(P(\lambda)) = \mathbb{C}$.*

Proof. Let $T = (\sum_{i=0}^n A_i^* A_i)^{1/2}$. Since T is not Fredholm, by [8, Proposition 2] there exists a quasinilpotent operator Q on \mathcal{H} such that $T^2 = QQ^*$,

$$A_i^* A_i \leq T^2 = QQ^* \quad \text{for all } i.$$

It follows by [6, Theorem 1] that there exist operators C_i such that $A_i = C_i Q$. Hence

$$P(\lambda) = (\lambda^n C_0 + \lambda^{n-1} C_1 + \dots + C_n) Q,$$

and then because Q is not invertible, $P(\lambda)$ is not invertible for all $\lambda \in \mathbb{C}$. □

By Theorem 7 and Theorem 5, we have

Corollary 5. *For a pair (A, B) of operators in $\mathcal{B}(\mathcal{H})$, if B is a Drazin invertible with index $m \geq 1$ and $N(A) \cap N(B^m) = \{0\}$, then $\sigma(L(\lambda)) = \mathbb{C}$ if and only if $A(N(B^m))$ is not closed.*

In the sequel we derive a criteria for the invertibility of the operator pencil $L(\cdot)$.

Let (A, B) a pair of matrix in $\mathcal{B}(\mathbb{C}^n)$. Assume that there exists $\lambda_0 \in \mathbb{C}$ such that $A - \lambda_0 B$ is invertible. Then

$$A - \lambda B = (A - \lambda_0 B)(I + (\lambda_0 - \lambda)(A - \lambda_0 B)^{-1} B). \tag{4}$$

The next step is to transform $B_{\lambda_0} = (A - \lambda_0 B)^{-1} B$ to Jordan canonical form. Then there exists a nonsingular matrix P such that $B_{\lambda_0} = P \text{diag}(J, N) P^{-1}$, where J is nonsingular (i.e., the part belonging to the nonzero eigenvalues) and N is a nilpotent, strictly upper triangular matrix. We obtain

$$A - \lambda B = (A - \lambda_0 B) P \begin{pmatrix} I + (\lambda_0 - \lambda) J & 0 \\ 0 & I + (\lambda_0 - \lambda) N \end{pmatrix} P^{-1}.$$

This implies that

$$\det(A - \lambda B) = \det(A - \lambda_0 B) \det(I + (\lambda_0 - \lambda) J) = c(1 + (\lambda_0 - \lambda)\mu_1)^{\nu_1} \dots (1 + (\lambda_0 - \lambda)\mu_k)^{\nu_k},$$

where $c \neq 0$ and μ_i are the nontrivial entries of J and ν_i is corresponding algebraic multiplicity, $i = 1 \dots k$, with $k \leq d$ and d is the size of J . So, if $A - \lambda B$ is not invertible, then the zero is an eigenvalue of algebraic multiplicity $\leq n - d$ and the nonzero eigenvalues are given by

$$\lambda_i = \lambda_0 + \frac{1}{\mu_i}, \quad i = 1 \dots k; \tag{5}$$

with algebraic multiplicity ν_i . Here μ_i are the nonzero eigenvalues of the matrix B_{λ_0} .

According to (5), by [16, Theorem 1], we have the following result.

Proposition 1. *Let (A, B) be a pair of matrices in $\mathcal{B}(\mathbb{C}^n)$. Assume that there exists $\lambda_0 \in \mathbb{C}$ such that $A - \lambda_0 B$ is invertible. If μ is an eigenvalue of $B_{\lambda_0} = (A - \lambda_0 B)^{-1}B$, then there exist $x, y \in \mathbb{C}^n \setminus \{0\}$ with $\|x\| = \|y\| = 1$ such that*

$$\mu = \frac{\langle Bx, x \rangle}{\langle (A - \lambda_0 B)y, y \rangle}. \quad (6)$$

Furthermore, if $A - \lambda_0 B$ is strictly positive and B is positive (resp. hermitian) then the eigenvalues of $A - \lambda B$ are positive (resp. real).

$\lambda_j(A)$ will represent an eigenvalue of the matrix A . If A is Hermitian, then we require $\lambda_1(A) \leq \lambda_2(A) \leq \dots \leq 0 \leq \dots \leq \lambda_n(A)$ and if A is positive definite, then $0 < \lambda_1(A) \leq \lambda_2(A) \leq \dots \leq \lambda_n(A)$. If we assume that $A - \lambda_0 B$ is positive definite and B is hermitian, then by (5) and (6) we can give a region of the nonzero real eigenvalues of $A - \lambda B$ by

$$|\lambda_i - \lambda_0| \leq \frac{\lambda_{\max}(B_{\lambda_0})}{\min\{|\lambda_{\min}(B)|, |\lambda_{\max}(B)|\}}.$$

We see that the relation (4) is also true if (A, B) is a pair of operators in $\mathcal{B}(\mathcal{H})$, so by the use of (4), [16, Theorem 1] and the corollary given after [16, Theorem 2], we deduce the following results.

Proposition 2. *Let (A, B) be a pair of operators in $\mathcal{B}(\mathcal{H})$. Assume that there exists $\lambda_0 \in \mathbb{C}$ such that $A - \lambda_0 B$ is invertible. We have*

1. $\lambda \in \sigma(L(\lambda))$ if and only if $\frac{1}{\lambda_0 - \lambda} \in \sigma(B_{\lambda_0})$ with $B_{\lambda_0} = (A - \lambda_0 B)^{-1}B$,
2. if $\mu \in \sigma(B_{\lambda_0})$, there exist two sequences $(x_n)_n$ and $(y_n)_n$ in $\mathcal{H} \setminus \{0\}$, with $\|x_n\| = \|y_n\| = 1$, such that

$$\frac{\langle Bx_n, x_n \rangle}{\langle (A - \lambda_0 B)y_n, y_n \rangle} \longrightarrow \mu,$$

3. if $A - \lambda_0 B$ is strictly positive and B is positive (resp. hermitian) then the spectrum of $A - \lambda B$ is positive (resp. real).

Now, we derive necessary and sufficient conditions for $A - \lambda B$ to be invertible for a pair (A, B) of bounded operators and Drazin invertible.

Theorem 9. *Let \mathcal{H} be a complex Hilbert space. For a pair (A, B) of operators in $\mathcal{B}(\mathcal{H})$, we suppose that A and B are Drazin invertible operators with indices k and m , respectively. If $N(A^k) \cap N(B^m) = \{0\}$, then the following conditions are equivalent*

1. $A - \lambda B$ is invertible for $\lambda \in \mathbb{C} \setminus \{0\}$.
2. $(A - [(\frac{1}{\lambda}I - B)|_{R(B^m)}]^{-1}) P_{N(A^k)}$ is invertible for $\frac{1}{\lambda} \in \rho(B) \setminus \{0\}$.
3. $([(\lambda I - A)|_{R(A^k)}]^{-1} - B) P_{N(B^m)}$ is invertible for $\lambda \in \rho(A) \setminus \{0\}$.

Proof. Let $\lambda \in \mathbb{C} \setminus \{0\}$. Then $A - \lambda B$ is an invertible operator if and only if $\begin{pmatrix} A & B \\ \lambda I_{\mathcal{H}} & I_{\mathcal{H}} \end{pmatrix}$ is an invertible operator on $\mathcal{H} \oplus \mathcal{H}$. This directly follows from

$$\begin{pmatrix} A - \lambda B & 0 \\ 0 & I_{\mathcal{H}} \end{pmatrix} = \begin{pmatrix} I_{\mathcal{H}} & -B \\ 0 & I_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} A & B \\ \lambda I_{\mathcal{H}} & I_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} I_{\mathcal{H}} & 0 \\ -\lambda I_{\mathcal{H}} & I_{\mathcal{H}} \end{pmatrix}.$$

A as an operator on $R(A^k) \oplus N(A^k)$ has the following form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

with A_1 is an invertible operator on $R(A^k)$ and A_2 is a nilpotent operator of order k . Similarly, B as an operator on $R(B^m) \oplus N(B^m)$ take the following form

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

with B_1 is an invertible operator on $R(B^m)$ and B_2 is a nilpotent operator of order m .

Now, the operator

$$\begin{pmatrix} A & B \\ \lambda I_{\mathcal{H}} & I_{\mathcal{H}} \end{pmatrix}$$

has the following matrix representation

$$T = \begin{pmatrix} A_1 & 0 & \tilde{B}_1 & \lambda \tilde{B}_2 \\ 0 & A_2 & \tilde{B}_3 & 0 \\ \lambda C_1 & \lambda C_2 & I_{R(B^m)} & 0 \\ \lambda C_3 & 0 & 0 & I_{N(B^m)} \end{pmatrix}$$

on $R(A^k) \oplus N(A^k) \oplus R(B^m) \oplus N(B^m)$ and taking in account $N(A^k) \cap N(B^m) = \{0\}$. Since A_1 is invertible, by using the Schur complement, we can assert that T is invertible if and only if

$$T_1 = \begin{pmatrix} A_2 & \tilde{B}_3 & 0 \\ \lambda C_2 & I_{R(B^m)} - \lambda C_1 A_1^{-1} \tilde{B}_1 & -\lambda C_1 A_1^{-1} \tilde{B}_2 \\ 0 & -\lambda C_3 A_1^{-1} \tilde{B}_1 & I_{N(B^m)} - \lambda C_3 A_1^{-1} \tilde{B}_2 \end{pmatrix}$$

is invertible on $N(A^k) \oplus R(B^m) \oplus N(B^m)$. The fact that B is Drazin invertible and $R(B^m) \cap N(B^m) = \{0\}$; we obtain that $B_1 = C_1 A_1^{-1} \tilde{B}_1$, $B_2 = C_3 A_1^{-1} \tilde{B}_2$ and $C_1 A_1^{-1} \tilde{B}_2 = C_3 A_1^{-1} \tilde{B}_1 = 0$. Hence T_1 is invertible operator if an only if

$$\tilde{T} = \begin{pmatrix} A_2 & \tilde{B}_3 \\ \lambda C_2 & I_{R(B^m)} - \lambda B_1 \end{pmatrix}$$

is invertible on $N(A^k) \oplus R(B^m)$.

If we assume that $\frac{1}{\lambda} \in \rho(B_1) = \rho(B) \setminus \{0\}$, then \tilde{T} is an invertible operator if and only if $A_2 - \lambda \tilde{B}_3 (I_{R(B^m)} - \lambda B_1)^{-1} C_2$ is invertible in $N(A^k)$. By the definitions of A_2 , \tilde{B}_3 , B_1 and C_2 , we write

$$A_2 - \lambda \tilde{B}_3 (I_{R(B^m)} - \lambda B_1)^{-1} C_2 = \left(A - \left[\left(\frac{1}{\lambda} I - B \right) |_{R(B^m)} \right]^{-1} \right) P_{N(A^k)}.$$

This shows the equivalence between (1) and (2).

In a similar way one can show that the equivalence between (1) and (3), by considering now the operator pencil $B - \frac{1}{\lambda} A$ which is invertible if and only if the if $\begin{pmatrix} B & A \\ \frac{1}{\lambda} I_{\mathcal{H}} & I_{\mathcal{H}} \end{pmatrix}$ is an invertible operator on $\mathcal{H} \oplus \mathcal{H} = R(B^m) \oplus N(B^m) \oplus R(A^k) \oplus N(A^k)$. \square

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