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**ON ROUGH MAXIMAL INEQUALITIES: AN EXTENSION OF
FEFFERMAN–STEIN RESULTS**

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We prove some vector-valued inequalities for a rough maximal operator on Lebesgue spaces. These results are an extension of Fefferman–Stein (1971) and Sawano (2006) since the rough maximal operator is a generalization of the Hardy–Littlewood maximal operator and also a fractional maximal operator, respectively. We also establish some vector-valued inequalities for a rough maximal operator on Morrey spaces.

1. Introduction. Let f be a measurable function on \mathbb{R}^n . Let also Ω be a homogeneous function of degree zero on \mathbb{R}^n . For $\alpha \in [0, n)$, the *rough maximal operator* $M_{\Omega, \alpha}$ maps the function f to the rough maximal function $M_{\Omega, \alpha} f$ which is given by

$$M_{\Omega, \alpha} f(x) := \sup_{r > 0} r^{\alpha - n} \int_{B(x, r)} |\Omega(x - y)| |f(y)| dy, \quad x \in \mathbb{R}^n$$

where $B(x, r)$ denotes the open ball centered in x with radius $r > 0$. For $\alpha \in (0, n)$, the operator $M_{1, \alpha}$ is known as a fractional maximal operator. The operator $M_{1, 0}$ is well known as the Hardy–Littlewood maximal operator.

We write \mathbf{f} to denote a sequence of measurable functions $\{f_j\}_{j=1}^\infty$ on \mathbb{R}^n . We also write $M_{\Omega, \alpha} \mathbf{f}$ to denote the rough maximal function sequence $\{M_{\Omega, \alpha} f_j\}_{j=1}^\infty$. We use $a \lesssim b$ to indicate that there exists $c > 0$ such that $a \leq cb$. We express Fefferman–Stein maximal inequalities using this notations as follows.

Proposition 1 ([5], Theorem 1). *Let $u \in (1, \infty)$. Then, for $p \in (1, \infty)$*

$$\|M_{1, 0} \mathbf{f}\|_{L^p(\ell^u)} := \left\| \|M_{1, 0} \mathbf{f}(\cdot)\|_{\ell^u} \right\|_{L^p} = \left(\int_{\mathbb{R}^n} \left(\sum_{j=1}^\infty M_{1, 0} f_j(x) \right)^{\frac{p}{u}} dx \right)^{\frac{1}{p}} \lesssim \|\mathbf{f}\|_{L^p(\ell^u)}, \quad (1)$$

and

$$\|M_{1, 0} \mathbf{f}\|_{WL^1(\ell^u)} := \left\| \|M_{1, 0} \mathbf{f}(\cdot)\|_{\ell^u} \right\|_{WL^1} = \sup_{\tau > 0} \tau \left(\int_{\{x \in \mathbb{R}^n; \|M_{1, 0} \mathbf{f}(x)\|_{\ell^u} > \tau\}} dx \right) \lesssim \|\mathbf{f}\|_{L^1(\ell^u)}. \quad (2)$$

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For any $j \in \mathbb{N}$, we have

$$|f_j(x)| \leq \sup_{j \in \mathbb{N}} |f_j(x)| = \|\mathbf{f}(x)\|_{\ell^\infty}$$

for almost every $x \in \mathbb{R}^n$. Therefore,

$$\|M_{1,0}\mathbf{f}(x)\|_{\ell^\infty} \leq \|M_{1,0}\|\|\mathbf{f}(\cdot)\|_{\ell^\infty(x)}\|_{\ell^\infty} = M_{1,0}(\|\mathbf{f}(\cdot)\|_{\ell^\infty})(x). \tag{3}$$

Hence, inequality (1) and (2) hold for $u = \infty$ due to inequality (3) and the boundedness of $M_{1,0}$ on Lebesgue spaces. However, inequality (1) and (2) do not hold for $u = 1$ (see [13]).

In 2006, Sawano proved the following fractional maximal inequalities.

Proposition 2 ([12], Theorem 2). *Let $u \in [1, \infty]$, $\alpha \in (0, n)$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then for $p \in (1, \frac{n}{\alpha})$,*

$$\|M_{1,\alpha}\mathbf{f}\|_{L^q(\ell^u)} \lesssim \|\mathbf{f}\|_{L^p(\ell^u)}, \quad \|M_{1,\alpha}\mathbf{f}\|_{WL^{\frac{n}{n-\alpha}}(\ell^u)} \lesssim \|\mathbf{f}\|_{L^1(\ell^u)}.$$

Our main aim is to prove the vector-valued inequality for the rough maximal operator $M_{\Omega,\alpha}$ as an extension of Proposition 1 and Proposition 2. We use $a \lesssim_\Omega b$ to express that there exists $c > 0$ which depends on Ω such that $a \leq cb$. In Section 2, we prove the following inequalities.

$$\|M_{\Omega,\alpha}\mathbf{f}\|_{L^q(\ell^u)} \lesssim_\Omega \|\mathbf{f}\|_{L^p(\ell^u)}, \tag{4}$$

and

$$\|M_{\Omega,\alpha}\mathbf{f}\|_{WL^{\frac{n}{n-\alpha}}(\ell^u)} \lesssim_\Omega \|\mathbf{f}\|_{L^1(\ell^u)}. \tag{5}$$

Note that if $\mathbf{f} = \{f_j\}_{j=1}^\infty$ with $f_1 = f$, $f_j = 0$ for $j > 2$, then both inequalities (4) and (5) mean the boundedness of $M_{\Omega,\alpha}$ in Lebesgue spaces (see [10, Theorem 2] and [6, Section 5.2.3]). For such \mathbf{f} , the Calderón–Zygmund decomposition and the covering property are main keys to prove inequality (5), which are applied to $\|\mathbf{f}\|_{\ell^u} = |f| \in L^1$. In more general cases of \mathbf{f} , although the covering property applies directly to $\|\mathbf{f}\|_{\ell^u} \in L^1$, the Calderón–Zygmund decomposition does not apply directly to $\|\mathbf{f}\|_{\ell^u}$. To obtain the desired result, we apply Calderón–Zygmund decomposition to each $f_j \in \mathbf{f}$ instead (see Theorem 3).

Let $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$. If $\Omega \in L^\infty(S^{n-1})$, then $M_{\Omega,\alpha}\mathbf{f}(x) \lesssim_\Omega M_{1,\alpha}\mathbf{f}(x)$; which means $M_{\Omega,\alpha}f_j(x) \lesssim_\Omega M_{1,\alpha}f_j(x)$ for any $j \in \mathbb{N}$. Thus, for $\Omega \in L^\infty(S^{n-1})$, we have inequality (4) and (5) immediately from Proposition 1 or Proposition 2. Therefore, we consider a weaker condition than $\Omega \in L^\infty(S^{n-1})$.

Furthermore, we extend inequality (4) and (5) onto Morrey space in Section 3. Let us recall the definition of a Morrey spaces. For $p \in [1, \infty)$ and $\lambda \in [0, n)$, the Morrey norm is defined as

$$\|f\|_{L^{p,\lambda}} := \sup_{B(x,r) \subset \mathbb{R}^n} r^{-\lambda/p} \|f\|_{L^p(B(x,r))}.$$

The collection of any measurable function f such that $\|f\|_{L^{p,\lambda}} < \infty$ is called the Morrey space $L^{p,\lambda}$ [9]. In case of $\lambda = 0$, the Morrey space $L^{p,0}$ is equivalent, by norm value, to the Lebesgue space L^p . The weak Morrey space $WL^{p,\lambda}$ is the set of measurable function f such that

$$\|f\|_{WL^{p,\lambda}} := \sup_{B(x,r)} r^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,r))} < \infty.$$

Let $|\mathbf{f}| = \{|f_j|\}_{j=1}^\infty$, and $\int_{\mathbb{R}^n} \mathbf{f}(x)dx = \{\int_{\mathbb{R}^n} f_j(x)dx\}_{j=1}^\infty$. For $u \in [1, \infty]$, one can observe that for any function $\mathbf{f} \in L^1(\ell^u)$, we have obvious inequality

$$\| |\mathbf{f}| \|_{L^1} \|_{\ell^u} \leq \|\mathbf{f}\|_{L^1(\ell^u)}. \tag{6}$$

2. Extension on Lebesgue spaces. Let us begin this section with the definition of a rough integral operator $T_{\Omega,\alpha}$. Let $\alpha \in [0, n)$, Ω be a homogeneous function of degree zero. The *rough integral operator* $T_{\Omega,\alpha}$ is defined as

$$T_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy.$$

For $\alpha \in (0, n)$, the operator $T_{1,\alpha}$ is known as the fractional integral operator. The operator $T_{\Omega,0}$ is known as the singular integral operator. A boundedness property is known.

Proposition 3 ([10], Theorem 2). *Let $\alpha \in (0, n)$, $p \in (1, \frac{n}{\alpha})$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. The operator $T_{\Omega,\alpha}$ is bounded from L^p to L^q if $\Omega \in L^s(S^{n-1})$ where $s \geq \frac{n}{n-\alpha}$.*

Let $T_{\Omega,\alpha}\mathbf{f}$ be a function sequence $\{T_{\Omega,\alpha}f_j\}_{j=1}^\infty$. It is clear that $M_{\Omega,\alpha}\mathbf{f}(x) \leq T_{|\Omega|,\alpha}(|\mathbf{f}|)(x)$. Under the same argument to obtain inequality (3), we have

$$\|M_{\Omega,\alpha}\mathbf{f}(x)\|_{\ell^\infty} \leq M_{\Omega,\alpha}(\|\mathbf{f}(\cdot)\|_{\ell^\infty})(x). \tag{7}$$

Hence, by inequality (6) and (7), for any $u \in [1, \infty]$

$$\|M_{\Omega,\alpha}\mathbf{f}(x)\|_{\ell^u} \leq T_{|\Omega|,\alpha}(\|\mathbf{f}(\cdot)\|_{\ell^u})(x). \tag{8}$$

With the relation between $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$, we can work on inequality (4) and have the following result.

Theorem 1. *Let $\alpha \in [0, n)$, $p \in (1, \frac{n}{\alpha})$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, and $\Omega \in L^s(S^{n-1})$ with $s \geq \frac{n}{n-\alpha}$. Then, for $\alpha \in (0, n)$ and $u \in [1, \infty]$, inequality (4) holds. Moreover, for $\alpha = 0$ and $u \in (1, \infty]$, inequality (4) holds.*

Proof. By inequality (8) and Proposition 3, we can verify the case of $\alpha \in (0, \infty)$ as follows.

$$\|M_{\Omega,\alpha}\mathbf{f}\|_{L^q(\ell^u)} \leq \|T_{|\Omega|,\alpha}(\|\mathbf{f}(\cdot)\|_{\ell^u})\|_{L^q} \lesssim_\Omega \|\mathbf{f}\|_{L^p(\ell^u)}.$$

We now need to clarify the case of $\alpha = 0$. For each $j \in \mathbb{N}$, we have

$$M_{\Omega,0}f_j(x) \leq \int_{S^{n-1}} |\Omega(\theta)| M_\theta f_j(x) d\sigma(\theta) \tag{9}$$

where

$$M_\theta f_j(x) = \sup_{r>0} r^{-1} \int_0^r |f_j(x - R\theta)| dR.$$

By the rotation method and Proposition 1 we obtain

$$\|M_\theta\mathbf{f}\|_{L^p(\ell^u)} \lesssim \|\mathbf{f}\|_{L^p(\ell^u)} \tag{10}$$

Using inequalities (9) and (6), we get

$$\|M_{\Omega,0}\mathbf{f}(x)\|_{\ell^u} \leq \left(\int_{S^{n-1}} |\Omega(\theta)| \|M_\theta\mathbf{f}(x)\|_{\ell^u} d\sigma(\theta) \right). \tag{11}$$

By the Minkowski inequality, and inequality (10), we proceed as follows

$$\begin{aligned} \|M_{\Omega,0}\mathbf{f}\|_{L^p(\ell^u)} &\leq \left\| \left(\int_{S^{n-1}} |\Omega(\theta)| \|M_{\theta}\mathbf{f}(\cdot)\|_{\ell^u} d\sigma(\theta) \right) \right\|_{L^p} \\ &\leq \left(\int_{S^{n-1}} |\Omega(\theta)| \|M_{\theta}\mathbf{f}\|_{L^p(\ell^u)} d\sigma(\theta) \right) \lesssim_{\Omega} \|\mathbf{f}\|_{L^p(\ell^u)}. \end{aligned}$$

This completes the proof of Theorem 1. □

We can also work on inequality (5) for $\alpha \in (0, n)$ with the relation between $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ as follows.

Theorem 2. *Let $\alpha \in (0, n)$, and $u \in [1, \infty]$. If $\Omega \in L^s(S^{n-1})$ with $s \geq \frac{n}{n-\alpha}$, then inequality (5) holds.*

Proof. We can rewrite the function $T_{\Omega,\alpha}f$ as $K_{\Omega,\alpha} * f$ where $K_{\Omega,\alpha}(x) = \Omega(x)|x|^{\alpha-n}$. Since $\Omega \in L^{\frac{n}{n-\alpha}}(S^{n-1})$, we have $K_{\Omega,\alpha} \in WL^{\frac{n}{n-\alpha}}$ (see [1, p. 224]). By Young’s inequality, the operator $T_{\Omega,\alpha}$ is bounded from L^1 to $WL^{\frac{n}{n-\alpha}}$ due to

$$\|T_{\Omega,\alpha}f\|_{WL^{\frac{n}{n-\alpha}}} \leq \|K_{\Omega,\alpha}\|_{WL^{\frac{n}{n-\alpha}}} \|f\|_{L^1}.$$

Therefore, by inequality (8),

$$\|M_{\Omega,\alpha}\mathbf{f}\|_{WL^{\frac{n}{n-\alpha}}(\ell^u)} \leq \|T_{|\Omega|,\alpha}(\|\mathbf{f}(\cdot)\|_{\ell^u})\|_{WL^{\frac{n}{n-\alpha}}} \lesssim_{\Omega} \|\mathbf{f}\|_{L^1(\ell^u)}.$$

□

The function Ω is said to satisfy L^1 -Dini condition if $\Omega \in L^1(S^{n-1})$ and

$$\int_0^1 \frac{w(\delta)}{\delta} d\delta < \infty$$

where

$$w(\delta) = \sup_{h \in \mathbb{R}^n, |h| \leq \delta} \int_{S^{n-1}} |\Omega(\theta + h) - \Omega(\theta)| d\sigma(\theta).$$

Proposition 4 ([3], Theorem 1.5). $M_{\Omega,0}$ is bounded from L^1 to WL^1 if Ω satisfies L^1 -Dini condition.

We borrow some notations from [3, proof of Theorem 1.5]. Let ϕ be a nonnegative, radial, radially decreasing C^∞ -function such that $\phi(x) = 1$ for $|x| \leq 1$, and ϕ supported in $\{x \in \mathbb{R}^n; |x| < \frac{3}{2}\}$. Let also $\phi_r(x) = r^{-n}\phi(\frac{x}{r})$. For $\Omega \geq 0$ and $f \geq 0$, we have $M_{\Omega,0}f(x) \leq M_{\Omega,0,\phi}f(x)$ where

$$M_{\Omega,0,\phi}f(x) = \sup_{r>0} \left| \int_{\mathbb{R}^n} \phi_r(x-y)\Omega(x-y)f(y)dy \right|. \tag{12}$$

Suppose that $A^c := \mathbb{R}^n \setminus A$ for any $A \subset \mathbb{R}^n$. From [3, proof of Theorem 1.5], we can conclude the following corollary.

Corollary 1. *Suppose f is supported in a cube Q with center y_0 such that $\int f = 0$. Then,*

$$\begin{aligned} M_{\Omega,0,\phi}f(x) &\lesssim \int_{|x-y|\leq 2|y-y_0|} \left(\frac{|\Omega(x-y)|}{|x-y|^n} + \frac{|\Omega(x-y_0)|}{|x-y_0|^n} \right) |f(y)|dy + \\ &\quad + \int_{|x-y|>2|y-y_0|} \frac{|\Omega(x-y) - \Omega(x-y_0)|}{|x-y|^n} |f(y)|dy + \\ &\quad + \int_{|x-y|>2|y-y_0|} \frac{|y-y_0|}{|x-y_0|^{n+1}} |\Omega(x-y_0)| |f(y)|dy = F_1(f)(x) + F_2(f)(x) + F_3(f)(x). \end{aligned}$$

Let \tilde{Q} with the same center as cube Q , such that the sides of \tilde{Q} are parallel to the sides of Q , and the side-length $\ell(\tilde{Q}) = 2\ell(Q)$. If Ω satisfies L^1 -Dini condition, then for each $i \in \{1, 2, 3\}$

$$\int_{\tilde{Q}^c} |F_i(f)(x)|dx \lesssim_{\Omega} \|f\|_{L^1}.$$

We now can prove inequality (5) for $\alpha = 0$.

Theorem 3. *For $\alpha = 0$, and $u \in (1, \infty]$, if Ω satisfies L^1 -Dini condition, then inequality (5) holds.*

Proof. The case of $u = \infty$ is clear by Proposition 4 and inequality (7).

$$\|M_{\Omega,0}\mathbf{f}\|_{WL^1(\ell^\infty)} \leq \|M_{\Omega,0}(\|\mathbf{f}(\cdot)\|_{\ell^\infty})\|_{WL^1} \lesssim_{\Omega} \|\mathbf{f}\|_{L^1(\ell^\infty)}.$$

What is left is the case of $u \in (1, \infty)$.

Without loss of generality, we may suppose $\Omega \geq 0$ and $\mathbf{f} \geq \mathbf{0}$ where $\mathbf{0}$ is the zero function sequence. Fix $\lambda > 0$. Since $\|\mathbf{f}(\cdot)\|_{\ell^u} \in L^1$, by applying the covering property (see [4, Theorem 2.11]), we have a collection of dyadic cubes $\{Q_k\}$ (may be empty) such that

(i) $\|\mathbf{f}(x)\|_{\ell^u} \leq \lambda$ for almost every $x \notin \Psi = \bigcup_k Q_k$.

(ii) $|\Psi| < \frac{1}{\lambda} \|\mathbf{f}\|_{L^1(\ell^u)}$.

(iii) $\lambda < \frac{1}{|Q_k|} \int_{Q_k} \|\mathbf{f}(y)\|_{\ell^u} dy \leq 2^n \lambda$.

We decompose $\mathbf{f} = \{f_j\} = \{g_j\} + \{b_j\} = \mathbf{g} + \mathbf{b}$ where

$$g_j = f_j \chi_{\Psi^c} + \sum_k \frac{\chi_{Q_k}}{|Q_k|} \int_{Q_k} |f_j(y)| dy$$

and $b_j = \sum_k b_{j,k}$ with

$$b_{j,k} = f_j \chi_{Q_k} - \frac{\chi_{Q_k}}{|Q_k|} \int_{Q_k} |f_j(y)| dy.$$

We write $\mathbf{b} = \{b_j\} = \sum_k \{b_{j,k}\} = \sum_k \mathbf{b}_k$. Let us take a look on behavior of \mathbf{g} and \mathbf{b} . By inequality (6), (i) and (iii)

$$\|\mathbf{g}\|_{L^u(\ell^u)}^u = \int_{\Psi^c} \|\mathbf{g}(x)\|_{\ell^u}^u dx + \sum_k \int_{Q_k} \|\mathbf{g}(x)\|_{\ell^u}^u dx \leq$$

$$\leq \int_{\tilde{\Psi}^c} \|\mathbf{f}(x)\|_{\ell^u}^u dx + \sum_k \int_{Q_k} \frac{1}{|Q_k|^u} \left(\int_{Q_k} \|\mathbf{f}(y)\|_{\ell^u} dy \right)^u dx \lesssim \lambda^{u-1} \|\mathbf{f}\|_{L^1(\ell^u)}. \quad (13)$$

For any k , we have $\int b_{j,k} = 0$. Moreover, by inequality (6)

$$\sum_k \|\mathbf{b}_k\|_{L^1(\ell^u)} \lesssim \|\mathbf{f}\|_{L^1(\ell^u)}. \quad (14)$$

Since $M_{\Omega,0}f(x) \leq M_{\Omega,0,\phi}f(x)$ (see (12)), we now only need to show

$$\left| \left\{ x \in \mathbb{R}^n; \|M_{\Omega,0,\phi}\mathbf{g}(x)\|_{\ell^u} > \frac{\lambda}{2} \right\} \right| \lesssim_{\Omega} \frac{1}{\lambda} \|\mathbf{f}\|_{L^1(\ell^u)} \quad (15)$$

and

$$\left| \left\{ x \in \mathbb{R}^n; \|M_{\Omega,0,\phi}\mathbf{b}(x)\|_{\ell^u} > \frac{\lambda}{2} \right\} \right| \lesssim_{\Omega} \frac{1}{\lambda} \|\mathbf{f}\|_{L^1(\ell^u)}. \quad (16)$$

First, let us prove inequality (15). By Chebychev's inequality, inequality (6), Minkowski's inequality, inequalities (10) and (13), we have

$$\begin{aligned} & \left| \left\{ x \in \mathbb{R}^n; \|M_{\Omega,0,\phi}\mathbf{g}(x)\|_{\ell^u} > \frac{\lambda}{2} \right\} \right| \lesssim \frac{1}{\lambda^u} \|M_{\Omega,0,\phi}\mathbf{g}\|_{L^u(\ell^u)}^u \lesssim \\ & \lesssim \frac{1}{\lambda^u} \left(\int_{S^{n-1}} |\Omega(\theta)| \|M_{\theta}\mathbf{g}\|_{L^u(\ell^u)} d\sigma(\theta) \right)^u \lesssim_{\Omega} \frac{1}{\lambda^u} \|\mathbf{g}\|_{L^u(\ell^u)}^u \lesssim \frac{1}{\lambda} \|\mathbf{f}\|_{L^1(\ell^u)}. \end{aligned}$$

Recall \tilde{Q} from Corollary 1. Suppose that $\tilde{\Psi} = \bigcup_k \tilde{Q}_k$. From (ii), we have

$$|\tilde{\Psi}| \lesssim |\Psi| \leq \frac{1}{\lambda} \|\mathbf{f}\|_{L^1(\ell^u)}. \quad (17)$$

Since we have inequality (17), we can get inequality (16) once we show the following.

$$\left| \left\{ x \in \tilde{\Psi}^c; \|M_{\Omega,0,\phi}\mathbf{b}(x)\|_{\ell^u} > \frac{\lambda}{2} \right\} \right| \lesssim_{\Omega} \frac{1}{\lambda} \|\mathbf{f}\|_{L^1(\ell^u)}.$$

By Chebychev's inequality and Minkowski's inequality

$$\begin{aligned} & \left| \left\{ x \in \tilde{\Psi}^c; \|M_{\Omega,0,\phi}\mathbf{b}(x)\|_{\ell^u} > \frac{\lambda}{2} \right\} \right| \lesssim \frac{1}{\lambda} \int_{\tilde{\Psi}^c} \|M_{\Omega,0,\phi}\mathbf{b}(x)\|_{\ell^u} dx \leq \\ & \leq \frac{1}{\lambda} \sum_k \int_{\tilde{\Psi}^c} \|M_{\Omega,0,\phi}\mathbf{b}_k(x)\|_{\ell^u} dx. \end{aligned} \quad (18)$$

Let us consider a fixed $b_{j,k}$ which is supported in Q_k . Suppose that y_0 is the center of Q_k . Since the integral of $b_{j,k}$ is zero, by Corollary 1

$$M_{\Omega,0,\phi}b_{j,k}(x) \lesssim F_1(b_{j,k})(x) + F_2(b_{j,k})(x) + F_3(b_{j,k})(x).$$

By inequality (6), and the boundedness of F_i for each $i \in \{1, 2, 3\}$ from Corollary 1, we can conclude that

$$\int_{\tilde{\Psi}^c} \left(\sum_{j=1}^{\infty} |F_i(b_{j,k})(x)|^u \right)^{\frac{1}{u}} dx \leq \int_{\tilde{Q}_k^c} F_i(\|\mathbf{b}_k(\cdot)\|_{\ell^u})(x) dx \lesssim_{\Omega} \|\mathbf{b}_k\|_{L^1(\ell^u)}.$$

Hence, from inequality (18), and (14),

$$\left| \left\{ x \in \tilde{\Psi}^c; \|M_{\Omega,0,\phi} \mathbf{b}(x)\|_{\ell^u} > \frac{\lambda}{2} \right\} \right| \lesssim_{\Omega} \frac{1}{\lambda} \sum_k \|\mathbf{b}_k\|_{L^1(\ell^u)} \lesssim \frac{1}{\lambda} \|\mathbf{f}\|_{L^1(\ell^u)}.$$

Thus, inequality (16) is verified. This completes the proof of Theorem 3. \square

3. Extension on Morrey spaces. Let us start this section with some properties of the rough integral operator $T_{\Omega,\alpha}$. Let p' be the exponent conjugate to p where $p' = \frac{p}{p-1}$.

Proposition 5 ([11], Lemma 3.2). *Let $\alpha \in [0, n)$, $\lambda, \mu \in [0, n)$, $p \in (1, \frac{n-\lambda}{\alpha})$, and $\frac{n-\mu}{q} = \frac{n-\lambda}{p} - \alpha$. Let also $\Omega \in L^s(S^{n-1})$ with $s \geq p'$. Then for any $z \in \mathbb{R}^n$ and $r > 0$*

$$r^{-\frac{\mu}{q}} \|T_{\Omega,\alpha}(f\chi_{B^c(z,2r)})\|_{L^q(B(z,r))} \lesssim_{\Omega} \|f\|_{L^{p,\lambda}}.$$

Lemma 1. *Let $\alpha \in [0, n)$, $s \in (1, \infty)$, $p \in [1, \infty)$, $\lambda \in [0, n - \frac{np}{s} - \alpha p)$, $\mu \in [0, n)$, and $\frac{n-\mu}{q} = \frac{n-\lambda}{p} - \alpha$. Let also $\Omega \in L^s(S^{n-1})$. Then for any $z \in \mathbb{R}^n$ and $r > 0$*

$$r^{-\frac{\mu}{q}} \|T_{\Omega,\alpha}(f\chi_{B^c(z,2r)})\|_{L^q(B(z,r))} \lesssim_{\Omega} \|f\|_{L^{p,\lambda}}.$$

Proof. For any $x \in B(z, r)$ we have $B(x, t) \subset B(z, 2t)$ for $t > r$. By Fubini's theorem,

$$\begin{aligned} |T_{\Omega,\alpha}(f\chi_{B^c(z,2r)})(x)| &\leq \int_{B^c(z,2r)} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy \lesssim \\ &\lesssim \int_{B^c(z,2r)} |\Omega(x-y)| |f(y)| \int_{|x-y|}^{\infty} t^{\alpha-n-1} dt dy \leq \int_r^{\infty} t^{\alpha-n-1} \int_{B(x,t)} |\Omega(x-y)| |f(y)| dy dt \leq \\ &\leq \int_r^{\infty} t^{\alpha-n-1} \int_{B(z,2t)} |\Omega(x-y)| |f(y)| dy dt. \end{aligned} \quad (19)$$

Since $y \in B(z, 2t)$, $B(z, r) \subset B(y, 4t)$ for $t > r$. Since $\lambda < n - \frac{np}{s} - \alpha p$ and $\frac{n-\mu}{q} = \frac{n-\lambda}{p} - \alpha$, we have $s > q$. Applying inequality (19), Minkowski and Hölder inequalities, we have

$$\begin{aligned} &r^{-\frac{\mu}{q}} \|T_{\Omega,\alpha}(f\chi_{B^c(z,2r)})\|_{L^q(B(z,r))} \\ &\leq r^{-\frac{\mu}{q}} \int_r^{\infty} t^{\alpha-n-1} \int_{B(z,2t)} |f(y)| \left(\int_{B(z,r)} |\Omega(x-y)|^q dx \right)^{\frac{1}{q}} dy dt \lesssim \end{aligned}$$

$$\begin{aligned}
 &\lesssim r^{\frac{n-\mu}{q}-\frac{n}{s}} \int_r^\infty t^{\alpha-n-1} \int_{B(z,2t)} |f(y)| \left(\int_{B(z,r)} |\Omega(x-y)|^s dx \right)^{\frac{1}{s}} dy dt \leq \tag{20} \\
 &\leq r^{\frac{n-\mu}{q}-\frac{n}{s}} \int_r^\infty t^{\alpha-n-1} \int_{B(z,2t)} |f(y)| \left(\int_{B(y,4t)} |\Omega(x-y)|^s dx \right)^{\frac{1}{s}} dy dt \lesssim_\Omega \\
 &\lesssim_\Omega r^{\frac{n-\mu}{q}-\frac{n}{s}} \int_r^\infty t^{\alpha-n+\frac{n}{s}-1} \int_{B(z,2t)} |f(y)| dy dt \lesssim_\Omega \\
 &\lesssim r^{\frac{n-\mu}{q}-\frac{n}{s}} \int_r^\infty t^{\alpha-\frac{n}{p}+\frac{n}{s}-1} \|f\|_{L^p(B(z,2t))} dt \leq \|f\|_{L^{p,\lambda}} r^{\frac{n-\mu}{q}-\frac{n}{s}} \int_r^\infty t^{\alpha-\frac{n-\lambda}{p}+\frac{n}{s}-1} dt.
 \end{aligned}$$

Since $\lambda < n - \frac{np}{s} - \alpha p$, we get the desired result. □

We note that the proof of Lemma 1 is similar to Guliyev’s result in [7, p. 217]. Guliyev used $\|\Omega(\cdot - y)\|_{L^s(B(z,r))} \sim r^{\frac{n}{s}} \|\Omega\|_{L^s(S^{n-1})}$. However, we can not simply do the same way since $\|\Omega(\cdot - y)\|_{L^s(B(z,r))}$ depends on y . Therefore, we instead treat inequality (20) by the fact that $B(z, r) \subset B(y, 4t)$, and

$$\|\Omega(\cdot - y)\|_{L^s(B(y,4t))} = \|\Omega\|_{L^s(B(0,4t))} \sim t^{\frac{n}{s}} \|\Omega\|_{L^s(S^{n-1})}.$$

We have two following boundedness properties of $T_{\Omega,\alpha}$.

Proposition 6 ([11], Theorem 3.3). *Let $\alpha \in (0, n)$, $0 \leq \lambda \leq \mu < n$, $p \in (1, \frac{n-\lambda}{\alpha})$, and $\frac{n-\mu}{q} = \frac{n-\lambda}{p} - \alpha$. Let $\Omega \in L^s(S^{n-1})$ where $s \geq p'$. Then $T_{\Omega,\alpha}$ is bounded from $L^{p,\lambda}$ to $L^{q,\mu}$.*

Theorem 4. *Let $\alpha \in (0, n)$, $p, s \in (1, \infty)$, $\lambda \in [0, n - \frac{np}{s} - \alpha p]$, $\mu \in [\frac{\lambda q}{p}, n)$, and $\frac{n-\mu}{q} = \frac{n-\lambda}{p} - \alpha$. Let also $\Omega \in L^s(S^{n-1})$. Then $T_{\Omega,\alpha}$ is bounded from $L^{p,\lambda}$ to $L^{q,\mu}$.*

Proof. Since $\mu \geq \frac{\lambda q}{p}$, we have $q_1 \geq q$ with $\frac{1}{q_1} = \frac{1}{p} - \frac{\alpha}{n}$. Fix $z \in \mathbb{R}^n$ and $r > 0$. We define $f_1 = f \chi_{B(z,2r)}$ and $f_2 = f - f_1$. By the Hölder inequality and Proposition 3, we have

$$\begin{aligned}
 r^{-\frac{\mu}{q}} \|T_{\Omega,\alpha} f_1\|_{L^q(B(z,r))} &\lesssim r^{\frac{n-\mu}{q}-\frac{n}{q_1}} \|T_{\Omega,\alpha} f_1\|_{L^{q_1}(B(z,r))} \lesssim_\Omega \\
 &\lesssim_\Omega r^{\frac{n-\mu}{q}-\frac{n}{v}} \|f_1\|_{L^p(B(z,2r))} \lesssim \|f\|_{L^{p,\lambda}}. \tag{21}
 \end{aligned}$$

Due to inequality (21) and Lemma 1, we have

$$r^{-\frac{\mu}{q}} \|T_{\Omega,\alpha} f\|_{L^q(B(z,r))} \leq r^{-\frac{\mu}{q}} (\|T_{\Omega,\alpha} f_1\|_{L^q(B(z,r))} + \|T_{\Omega,\alpha} f_2\|_{L^q(B(z,r))}) \lesssim_\Omega \|f\|_{L^{p,\lambda}}.$$

□

By the properties of $T_{\Omega,\alpha}$, we can extend inequality (4) onto Morrey spaces as follows.

Theorem 5. *Let $\alpha \in [0, n)$, $0 \leq \lambda \leq \mu < n$, $p \in (1, \frac{n-\lambda}{\alpha})$, $\frac{n-\mu}{q} = \frac{n-\lambda}{p} - \alpha$, and $\Omega \in L^s(S^{n-1})$ with $s \geq p'$. Then, for $\alpha \in (0, n)$ and $u \in [1, \infty]$,*

$$\|M_{\Omega,\alpha} \mathbf{f}\|_{L^{q,\mu}(\ell^u)} := \| \|M_{\Omega,\alpha} \mathbf{f}(\cdot)\|_{\ell^u} \|_{L^{q,\mu}} \lesssim_\Omega \| \mathbf{f} \|_{L^{p,\lambda}(\ell^u)}. \tag{22}$$

Moreover, for $\alpha = 0$ and $u \in (1, \infty]$, inequality (22) holds.

Theorem 6. Let $\alpha \in [0, n)$, $p \in (1, \infty)$, $s \in (1, \infty)$, $\lambda \in [0, n - \frac{np}{s} - \alpha p)$, $\mu \in [\frac{\lambda q}{p}, n)$, $\frac{n-\mu}{q} = \frac{n-\lambda}{p} - \alpha$, and $\Omega \in L^s(S^{n-1})$. Then, for $\alpha \in (0, n)$ and $u \in [1, \infty]$, inequality (22) holds. For $\alpha = 0$ and $u \in (1, \infty]$, inequality (22) holds.

Proof of Theorem 5 and Theorem 6. The case of $\alpha \in (0, n)$ is verified by inequality (8), Proposition 6 (for Theorem 5), and Theorem 4 (for Theorem 6).

$$\|M_{\Omega, \alpha} \mathbf{f}\|_{L^{q, \mu}(\ell^u)} \leq \|T_{|\Omega|, \alpha}(\|\mathbf{f}(\cdot)\|_{\ell^u})\|_{L^{q, \mu}} \lesssim_{\Omega} \|\mathbf{f}\|_{L^{p, \lambda}(\ell^u)}.$$

Let us now treat the case of $\alpha = 0$. Fix $B(z, r) \subset \mathbb{R}^n$. Suppose that $\mathbf{f} = \{f_j\} = \{f_{j, (0)} + f_{j, (1)}\} = \mathbf{f}_{(0)} + \mathbf{f}_{(1)}$, where $f_{j, (0)} = f_j \chi_{B(z, 2r)}$ and $f_{j, (1)} = f_j - f_{j, (0)}$. Since $M_{\Omega, 0}$ is a sublinear operator, we have

$$\|M_{\Omega, 0} \mathbf{f}(x)\|_{\ell^u} \leq \|M_{\Omega, 0} \mathbf{f}_{(0)}(x)\|_{\ell^u} + \|M_{\Omega, 0} \mathbf{f}_{(1)}(x)\|_{\ell^u}. \quad (23)$$

For the first term on the right-hand side of inequality (23), we use the Hölder inequality with order p/q and Theorem 1 to obtain

$$r^{-\frac{\mu}{q}} \left\| \|M_{\Omega, 0} \mathbf{f}_{(0)}(\cdot)\|_{\ell^u} \right\|_{L^q(B(z, r))} \lesssim r^{-\frac{n}{p} + \frac{n-\mu}{q}} \left\| \|M_{\Omega, 0} \mathbf{f}_{(0)}(\cdot)\|_{\ell^u} \right\|_{L^p(B(z, r))} \lesssim_{\Omega} \|\mathbf{f}\|_{L^{p, \lambda}(\ell^u)}.$$

Now, we treat the second term. By virtue of inequality (8), Proposition 5 (for Theorem 5), and Lemma 1 (for Theorem 6), we have

$$r^{-\frac{\mu}{q}} \left\| \|M_{\Omega, 0} \mathbf{f}_{(1)}(\cdot)\|_{\ell^u} \right\|_{L^q(B(z, r))} \leq r^{-\frac{\mu}{q}} \|T_{|\Omega|, \alpha}(\|\mathbf{f}_{(1)}(\cdot)\|_{\ell^u})\|_{L^q(B(z, r))} \lesssim_{\Omega} \|\mathbf{f}\|_{L^{p, \lambda}(\ell^u)}.$$

This completes the proof of both Theorem 5 and Theorem 6. \square

We also can extend inequality (5) onto weak Morrey spaces as follows.

Theorem 7. Let $\alpha \in [0, n)$, $q = \frac{n-\mu}{n-\lambda-\alpha} < s$, $\lambda \in [0, n - \frac{n}{s} - \alpha)$, and $\mu \in [\lambda q, n)$. For $\alpha \in (0, n)$ and $u \in [1, \infty]$, if $\Omega \in L^s(S^{n-1})$, then

$$\|M_{\Omega, \alpha} \mathbf{f}\|_{WL^{q, \mu}(\ell^u)} := \left\| \|M_{\Omega, \alpha} \mathbf{f}(\cdot)\|_{\ell^u} \right\|_{WL^{q, \mu}} \lesssim_{\Omega} \|\mathbf{f}\|_{L^{1, \lambda}(\ell^u)}. \quad (24)$$

For $\alpha = 0$ and $u \in (1, \infty]$, if $\Omega \in L^s(S^{n-1})$ satisfies L^1 -Dini condition, then inequality (24) holds.

Proof. Since $\mu > \lambda q$, we have $q_1 = \frac{n}{n-\alpha} > q$. We use the same decomposition as in inequality (25). For the first term of right-hand side in inequality (23), we use the Hölder inequality with the exponent q_1/q , Theorem 2 (for $\alpha \in (0, n)$) and Theorem 3 (for $\alpha = 0$) to obtain

$$r^{-\frac{\mu}{q}} \left\| \|M_{\Omega, \alpha} \mathbf{f}_{(0)}(\cdot)\|_{\ell^u} \right\|_{WL^q(B(z, r))} \lesssim r^{\frac{n-\mu}{q} - \frac{n}{q_1}} \left\| \|M_{\Omega, \alpha} \mathbf{f}_{(0)}(\cdot)\|_{\ell^u} \right\|_{WL^{q_1}(B(z, r))} \lesssim_{\Omega} \|\mathbf{f}\|_{L^{1, \lambda}(\ell^u)}.$$

Let us now treat the second term. For any $\gamma > 0$, by Chebychev's inequality, inequality (8), and Lemma 1, we have

$$\begin{aligned} & r^{-\frac{\mu}{q}} \left| \left\{ x \in B(z, r); \|M_{\Omega, \alpha} \mathbf{f}_{(1)}(x)\|_{\ell^u} > \gamma \right\} \right|^{\frac{1}{q}} \leq \\ & \leq \frac{r^{-\frac{\mu}{q}}}{\gamma} \|T_{|\Omega|, \alpha}(\|\mathbf{f}_{(1)}(\cdot)\|_{\ell^u})\|_{L^q(B(z, r))} \lesssim_{\Omega} \frac{1}{\gamma} \|\mathbf{f}\|_{L^{1, \lambda}(\ell^u)}. \end{aligned}$$

This completes the proof of Theorem 7. \square

4. Remarks. To the best of our knowledge, there is still no proof for $\Omega \in L^1(S^{n-1})$ to be sufficient condition for boundedness of $M_{\Omega,0}$ from L^1 to WL^1 . Moreover, there is still no counterexample such that $M_{\Omega,0}: L^1 \rightarrow WL^1$ is not bounded under $\Omega \in L^1(S^{n-1})$ either. Therefore, possibility of removing Dini-condition from Theorem 3 and Theorem 6 is still an open problem. Furthermore, the necessary condition (in term of Ω) for inequality (4), inequality (5), inequality (22), or inequality (24) to hold is still open .

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