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HOMOCLINIC TRAVELING WAVES IN DISCRETE SINE–GORDON EQUATION WITH NONLINEAR INTERACTION ON 2D–LATTICE

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The article deals with the discrete sine–Gordon equation that describes an infinite system of particles with nonlinear interaction and on-site potential on 2D-lattice. The main result concerns the existence of homoclinic traveling waves solutions. By means of critical point theory, we obtain sufficient conditions for the existence of such solutions.

1. Introduction. In the present paper we study the discrete sine–Gordon equation that describes the dynamics of an infinite system of nonlinearly coupled particles on a two dimensional lattice with on-site potential. Let $q_{n,m} = q_{n,m}(t)$ be a generalized coordinate of the (n,m)-th particle at time t. It is assumed that each particle interacts nonlinearly with its four nearest neighbors. The equation of motion of the system considered is of the form

$$\ddot{q}_{n,m} = U'(q_{n+1,m} - q_{n,m}) - U'(q_{n,m} - q_{n-1,m})+$$
$$+U'(q_{n,m+1} - q_{n,m}) - U'(q_{n,m} - q_{n,m-1}) - K\sin(q_{n,m}), (n,m) \in \mathbb{Z}^2, \qquad (1)$$

where U is the potential of interaction, $K > 0$. Equations (1) form an infinite system of ordinary differential equations.

This system can be considered as a 2D version of the Frenkel-Kontorova model (see [15]). Systems of such type are of interest in view of numerous applications in physics [1], [15], [16]. Notice that this system is a representative of a wide class of systems called lattice dynamical systems extensively studied in last decades. Among solutions of such systems, traveling waves deserve special attention. A comprehensive presentation of results about traveling waves for 1D Fermi–Pasta–Ulam lattices is given in [24]. The existence of periodic and solitary traveling waves in Fermi–Pasta–Ulam system on 2D-lattice is studied in [3] and [13]. On the other hand, some results on chains of oscillators are known in the literature. In particular, in [19] certain results of such type are obtained by means of bifurcation theory, while in [9] and [12] the existence of periodic and solitary traveling waves is studied by means of critical point theory. In papers [4], [14], [17], [18] traveling waves for infinite systems of linearly coupled oscillators on 2D-lattice are studied, while [8] and [25] deal with periodic in time solutions for such systems. Paper [23] is devoted to periodic and homoclinic traveling waves for infinite one-dimensional chain of nonlinearly coupled nonlinear particles. In [6] and [7] the existence of subsonic and supersonic periodic traveling waves for the system of nonlinearly coupled

nonlinear oscillators on 2D-lattice are studied, while in [5] it is obtained a result on the existence of solitary traveling waves for such systems.

Paper [20] contains a result on the existence of heteroclinic traveling waves for the discrete sine–Gordon equation with linear interaction, while in [21] periodic, homoclinic and heteroclinic traveling waves for such systems with nonlinear interaction are studied. In paper [10] it is obtained a result on the existence of periodic traveling waves for the discrete sine–Gordon equation with nonlinear interaction on 2D-lattice, while [2] and [11] are devoted to the existence of heteroclinic traveling waves for such equations with linear and nonlinear interaction.

In the present paper we obtain, by means of critical point theory, a result on the existence of homoclinic traveling waves for the dicrete sine–Gordon equation on 2D-lattice. This paper extends some of results obtained in [21].

2. Statement of a problem. A traveling wave solution of equation (1) is a function of the form

$$q_{n,m}(t) = v(n \cos \varphi + m \sin \varphi - ct), \qquad (2)$$

where the profile function $v(s)$ of the wave, or simply profile, satisfies the equation

$$c^2 v''(s) = U'(v(s + \cos \varphi) - v(s)) - U'(v(s) - v(s - \cos \varphi)) +$$
$$+ U'(v(s + \sin \varphi) - v(s)) - U'(v(s) - v(s - \sin \varphi)) - K \sin(v(s)). \qquad (3)$$

The constant $c \neq 0$ is called the speed of the wave. If $c > 0$, then the wave moves to the right, otherwise to the left.

We consider the case of homoclinic traveling waves. The profile function of such a wave satisfies the following boundary condition

$$\lim_{s \to \pm\infty} v(s) = v(\pm\infty) = \pi. \qquad (4)$$

Note that by replacement $v(s) = u(s) + \pi$, we have equation

$$c^2 u''(s) = U'(u(s + \cos \varphi) - u(s)) - U'(u(s) - u(s - \cos \varphi)) +$$
$$+ U'(u(s + \sin \varphi) - u(s)) - U'(u(s) - u(s - \sin \varphi)) + K \sin(u(s)) \qquad (5)$$

with boundary condition

$$\lim_{s \to \pm\infty} u(s) = u(\pm\infty) = 0. \qquad (6)$$

In fact, this waves are the solitary waves. Furthermore, equation (5) is a partial case of the equation studied in [5] with on-site potential $V(r) = K(\cos r - 1)$. But this potential does not satisfy the conditions of the article [5].

In what follows, a solution of equation (5) is understood as a function $u(s)$ from the space $C^2(\mathbb{R})$ satisfying equation (5) for all $s \in \mathbb{R}$.

3. Variational setting. To equation (5) we associate the functional

$$J(u) := \int_{-\infty}^{+\infty} \left[\frac{c^2}{2}(u'(s))^2 - U(Au(s)) - U(Bu(s)) + K(1 - \cos(u(s))) \right] ds, \qquad (7)$$

where

$$Au(s) := u(s + \cos\varphi) - u(s) = \int_{s}^{s+\cos\varphi} u'(\tau)d\tau,$$

$$Bu(s) := u(s + \sin\varphi) - u(s) = \int_{s}^{s+\sin\varphi} u'(\tau)d\tau.$$

The functional J is defined on the Hilbert space $E = H^1(\mathbb{R})$ with the scalar product

$$(u, v) = \int_{-\infty}^{+\infty} [u(s)v(s) + u'(s)v'(s)]ds$$

and corresponding norm $\|u\| = (u, u)^{\frac{1}{2}}$. Recall that by the embedding theorem, $E \subset C_b(\mathbb{R})$, where $C_b(\mathbb{R})$ is the space of bounded continuous functions.

Remark 1. According to Lemma 3.1 in [14],

$$\|Au(s)\|_{L^2(\mathbb{R})} \le |\cos\varphi| \cdot \|u'(s)\|_{L^2(\mathbb{R})}, u \in E,$$
$$\|Bu(s)\|_{L^2(\mathbb{R})} \le |\sin\varphi| \cdot \|u'(s)\|_{L^2(\mathbb{R})}, u \in E.$$

We assume that

(h) $U(r) = \frac{c_0^2}{2}r^2 + V(r)$, where $c_0 \ge 0, V \in C^1(\mathbb{R})$, $V(0) = V'(0) = 0$, $V'(r) = o(|r|)$ as $r \to 0$, and there exists $\mu > 2$ such that

$$0 \le \mu V(r) \le rV'(r), r \ne 0.$$

It is easy to verify that under assumption (h) there exists $d > 0$ such that $V(r) \ge d(|r|^\mu - |r|^2)$.

Next, using the approach implemented in [21], we define the modified functional by

$$\tilde{J}(u) := \int_{-\infty}^{+\infty} \left[\frac{c^2}{2}(u'(s))^2 - U(Au(s)) - U(Bu(s)) + K(1 - \cos(u(s))) + f(u(s))\right] ds,$$

where

$$f(r) = \frac{1}{2}\left(\max\{0; |r| - \frac{\pi}{2}\}\right)^2.$$

It is obvious that $J(u) \le \tilde{J}(u)$ and $J(u) = \tilde{J}(u)$ for all u with $\|u\|_{L^\infty(\mathbb{R})} \le \frac{\pi}{2}$. Moreover, $f \in C^1(\mathbb{R})$, $1 - \cos r + f(r) \le \frac{r^2}{2}$ and there is a constant $\omega = \omega(\mu) > 0$ such that for all $r \in \mathbb{R}$

$$\omega r^2 \le 1 - \cos r + f(r) - \frac{1}{\mu}r\sin r - \frac{1}{\mu}rf'(r). \tag{8}$$

An important role is played by the quantity α defined by the equation

$$\alpha := \min\left\{\frac{\mu - 2}{2\mu}(c^2 - c_0^2), \omega K\right\}.$$

The following two lemmas can be obtained by a straightforward calculation.

Lemma 1. *Under assumption (h) the functionals $J, \tilde{J} \in C^1$ on E and*

$$\langle J'(u), h \rangle = \int_{-\infty}^{+\infty} [c^2 u'(s) h'(s) - U'(Au(s)) Ah(s) - U'(Bu(s)) Bh(s) + K \sin(u(s)) h(s)] ds,$$

$$\langle \tilde{J}'(u), h \rangle = \int_{-\infty}^{+\infty} [c^2 u'(s) h'(s) - U'(Au(s)) Ah(s) - U'(Bu(s)) Bh(s) + K \sin(u(s)) h(s) +$$

$$+ f'(u(s)) h(s)] ds$$

for $u, h \in E$.

Lemma 2. *Under assumption (h) any critical point of the functional J is a C^2–solution of equation (5) satisfying (6).*

4. Main result. Thus, to construct the solutions of equation(5), by Lemma 2, it suffices to find nontrivial critical points of the functional J.

To get the main results we need the following lemmas.

Lemma 3 (**Mountain pass geometry**). *Assume (h) and $c^2 > c_0^2$. Then there exist $e \in E$ and $r > 0$ such that $\|e\| > r$ and $\beta := \inf_{\|u\|=r} \tilde{J}(u) > 0 = \tilde{J}(0) \geq \tilde{J}(e)$.*

Proof. Let $\varepsilon > 0$ such that $c^2 - c_0^2 - 2\varepsilon > 0$ and choose $r \in (0, \frac{\pi}{2})$ small enough such that $|V(x)| \leq \varepsilon x^2$ for all $x \leq r$. Then, by Remark 1 and the inequality $1 - \cos u \geq \frac{u^2}{4}$ for $|u| < \frac{\pi}{2}$, we have for all u with $\|u\|_{L^\infty(\mathbb{R})} \leq \|u\| \leq r < \frac{\pi}{2}$

$$\tilde{J}(u) \geq \int_{-\infty}^{+\infty} \left[\frac{c^2}{2} |u'(s)|^2 - \frac{c_0^2}{2} (|Au(s)|^2 + |Bu(s)|^2) - \varepsilon (|Au(s)|^2 + |Bu(s)|^2) + \right.$$

$$\left. + K(1 - \cos u(s)) \right] ds \geq \frac{1}{2} (c^2 - c_0^2 - 2\varepsilon) \|u'\|_{L^2(\mathbb{R})}^2 + \frac{K}{4} \|u\|_{L^2(\mathbb{R})}^2 \geq \alpha_0 \|u\|^2,$$

where $\alpha_0 = \min\{\frac{1}{2}(c^2 - c_0^2 - 2\varepsilon), \frac{K}{4}\} > 0$. Thus, for $\|u\| = r$

$$\tilde{J}(u) \geq \alpha_0 r^2 > 0.$$

On the other hand, to find $e \in E$ such that $\|e\| > r$ and $\tilde{J}(e) \leq \tilde{J}(0)$, we fix $u_0 \in E$. Then for every $\lambda \geq 0$, by the inequality $1 - \cos u + f(u) \leq \frac{u^2}{2}$, we have

$$\tilde{J}(\lambda u_0) \leq \frac{c^2}{2} \lambda^2 \|u_0'\|_{L^2(\mathbb{R})}^2 - d\lambda^\mu \|Au_0\|_{L^\mu(\mathbb{R})}^\mu - d\lambda^\mu \|Bu_0\|_{L^\mu(\mathbb{R})}^\mu +$$

$$+ d\lambda^2 \|Au_0\|_{L^2(\mathbb{R})}^2 + d\lambda^2 \|Bu_0\|_{L^2(\mathbb{R})}^2 + \frac{K}{2} \lambda^2 \|u_0\|_{L^2(\mathbb{R})}^2.$$

Since $\mu > 2$ and $d > 0$, $\lim_{\lambda \to +\infty} \tilde{J}(\lambda u_0) = -\infty$. The proof is complete. $\square$

Remark 2. Since $J(u) \leq \tilde{J}(u)$ and $J(u) = \tilde{J}(u)$ for all u with $\|u\|_{L^\infty(\mathbb{R})} \leq \frac{\pi}{2}$, the statement of Lemma 3 holds for J with the same β.

We fix $u_0 \in E$ and set

$$M := \sup_{\tau \geq 0} \tilde{J}(\tau u_0).$$

Now we assume that the parameters satisfy

$$\frac{1}{\alpha\mu} + \sqrt{\frac{M+1}{\alpha}} < \frac{\pi}{2}. \tag{9}$$

From Lemma 3 and the Mountain Pass Theorem (see, [26], Theorem 1.15) we obtain following.

Corollary 1. *Under the assumptions of Lemma 3, there exists a Palais–Smale sequence of the functional $\tilde{J}$, i.e., a sequence $\{u_n\} \subset E$ such that for $n \to \infty$*

$$\tilde{J}(u_n) \to b \ \ \text{and} \ \ \tilde{J}'(u_n) \to 0$$

for some $b \in [\beta, M]$.

Lemma 4. *Assume (h), (9) and $c^2 > c_0^2$. Then the Palais–Smale sequence $\{u_n\} \subset E$ of $\tilde{J}$, obtained in Corollary 1, is bounded, and*

$$\|u_n\|_{L^\infty(\mathbb{R})} < \frac{\pi}{2}$$

for n large enough.

Proof. For n large enough, due to Remark 1 and inequality (8), we have

$$M + 1 + \frac{1}{\mu}\|u_n\| \geq J(u_n) - \frac{1}{\mu}\langle J'(u_n), u_n \rangle =$$

$$= \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{-\infty}^{+\infty} \left[c^2|u_n'(s)|^2 - c_0^2(|Au_n(s)|^2 + |Bu_n(s)|^2)\right] ds +$$

$$+ \int_{-\infty}^{+\infty} \left[\frac{1}{\mu}\{V'(Au_n(s))Au_n(s) + V'(Bu_n(s))Bu_n(s)\} - V(Au_n(s)) - V(Bu_n(s))\right] ds +$$

$$+ K \int_{-\infty}^{+\infty} \left[1 - \cos(u_n(s)) + f(u_n(s)) - \frac{1}{\mu}u_n(s)\sin(u_n(s)) - \frac{1}{\mu}u_n(s)f'(u_n(s))\right] \geq$$

$$\geq \frac{\mu-2}{2\mu}(c^2 - c_0^2)\|\tilde{u}'_n(s)\|^2_{L^2(\mathbb{R})} + \omega K\|u_n(s)\|_{L^2(\mathbb{R})} \geq \alpha\|u_n(s)\|^2,$$

where $\alpha = \min\{\frac{\mu-2}{2\mu}(c^2 - c_0^2), \omega K\}$. Thus, we have

$$M + 1 + \frac{1}{\mu}\|u_n\| - \alpha\|u_n\|^2 \geq 0,$$

wherefrom, by (9), we obtain

$$\|u_n\| \leq \frac{\frac{1}{\mu} + \sqrt{\frac{1}{\mu^2} + 4\alpha(M+1)}}{2\alpha} \leq \frac{1}{\alpha\mu} + \sqrt{\frac{M+1}{\alpha}} < \frac{\pi}{2}.$$

Hence, $\{u_n\}$ is bounded, and

$$\|u_n\|_{L^\infty(\mathbb{R})} \le \|u_n\| < \frac{\pi}{2}.$$

$\square$

Remark 3. Since $J(u) = \tilde{J}(u)$ for all $u \in E$ with $\|u_n\|_{L^\infty(\mathbb{R})} < \frac{\pi}{2}$, then $\{u_n\}$ is a Palais–Smale sequence of the functional J.

Lemma 5. *Assume* (h), (9) *and* $c^2 > c_0^2$. *Then the Palais–Smale sequence* $\{u_n\} \subset E$ *does not converge to zero in measure.*

Proof. Since $\{u_n\}$ is bounded in E, then $\{Au_n\}$ and $\{Bu_n\}$ are bounded in E and in $L^\infty(\mathbb{R})$. Furthermore, since

$$\max\{\|Au\|_{L^2(\mathbb{R})}, \|Au\|_{L^\infty(\mathbb{R})}\} \le \|u'\|_{L^2(\mathbb{R})}, \quad \max\{\|Bu\|_{L^2(\mathbb{R})}, \|Bu\|_{L^\infty(\mathbb{R})}\} \le \|u'\|_{L^2(\mathbb{R})},$$

there exists $\varepsilon_0 > 0$ (small enough) such that

$$\max\{\|Au_n\|_{L^2(\mathbb{R})}, \|Au_n\|_{L^\infty(\mathbb{R})}\} \le \sup_n \|u_n\| \le C_1 := \frac{\pi}{2} - \varepsilon_0,$$

$$\max\{\|Bu_n\|_{L^2(\mathbb{R})}, \|Bu_n\|_{L^\infty(\mathbb{R})}\} \le \sup_n \|u_n\| \le C_1.$$

By assumption (h), $r^{-2}\left(\frac{1}{2}V'(r)r - V(r)\right) \to 0$, as $r \to 0$. Also, $u^{-2}\left(1 - \cos r - \frac{1}{2}r\sin r\right) \to 0$ as $r \to 0$. Thus, there is constant $C_2 > 0$ such that

$$\sup_{|r| \le C_1} \frac{\frac{1}{2}V'(r)r - V(r)}{r^2} \le C_2, \quad \sup_{|r| \le C_1} \frac{1 - \cos r - \frac{1}{2}r\sin r}{r^2} \le C_2.$$

Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|r| < \delta$

$$\left|\frac{1}{2}V'(r)r - V(r)\right| \le \varepsilon r^2, \quad \left|1 - \cos r - \frac{1}{2}r\sin r\right| \le \varepsilon r^2.$$

Due to inequality $\frac{1}{2}V'(r)r - V(r) \ge 0$, we have

$$0 \le \int_{-\infty}^{+\infty} \left[\frac{1}{2}V'(Au_n(s))Au_n(s) - V(Au_n(s))\right] ds \le$$

$$\le |\{s \in \mathbb{R} : |Au_n(s)| > \delta\}|C_2\|Au_n\|_{L^\infty(\mathbb{R})}^2 + \varepsilon\|Au_n\|_{L^2(\mathbb{R})}^2 C_1^2\left(|\{s \in \mathbb{R} : |Au_n(s)| > \delta\}|C_2 + \varepsilon\right),$$

$$0 \le \int_{-\infty}^{+\infty} \left[\frac{1}{2}V'(Bu_n(s))Bu_n(s) - V(Bu_n(s))\right] ds \le$$

$$\le |\{s \in \mathbb{R} : |Bu_n(s)| > \delta\}|C_2\|Bu_n\|_{L^\infty(\mathbb{R})}^2 + \varepsilon\|Bu_n\|_{L^2(\mathbb{R})}^2 \le$$

$$\le C_1^2\left(|\{s \in \mathbb{R} : |Bu_n(s)| > \delta\}|C_2 + \varepsilon\right).$$

Also, since $\|u_n\|_{L^\infty(\mathbb{R})} < \frac{\pi}{2}$,

$$0 \le \int_{-\infty}^{+\infty} \left[1 - \cos(u_n(s)) - \frac{1}{2}u_n(s)\sin(u_n(s))\right] ds \le$$

$$\leq |\{s \in \mathbb{R}\colon |u_n(s)| > \delta\}|C_2\|u_n\|_{L^\infty(\mathbb{R})}^2 + \varepsilon\|u_n\|_{L^2(\mathbb{R})}^2 \leq C_1^2\left(|\{s \in \mathbb{R}\colon |u_n(s)| > \delta\}|C_2 + \varepsilon\right).$$

For all n large enough, $|\langle J'(u_n), u_n\rangle| < \frac{\beta}{4}$ and $J(u_n) - \frac{3}{4}\beta > 0$. Then for $\varepsilon > 0$ small enough we have

$$0 < \frac{\beta}{2} \leq J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle = \int\limits_{-\infty}^{+\infty}\left[\frac{1}{2}V'(Au_n(s))Au_n(s) - V(Au_n(s))\right]ds+$$

$$+ \int\limits_{-\infty}^{+\infty}\left[\frac{1}{2}V'(Bu_n(s))Bu_n(s) - V(Bu_n(s))\right]ds+$$

$$+ K\int\limits_{-\infty}^{+\infty}\left[1 - \cos(u_n(s)) - \frac{1}{2}u_n(s)\sin(u_n(s))\right]ds \leq$$

$$\leq C_1^2\left[(|\{s \in \mathbb{R}\colon |Au_n(s)| > \delta\}| + |\{s \in \mathbb{R}\colon |Bu_n(s)| > \delta\}|)\,C_2 + \varepsilon\right]+$$
$$+KC_1^2\left(|\{s \in \mathbb{R}\colon |u_n(s)| > \delta\}|C_2 + \varepsilon\right) =$$
$$= C_1^2 C_2\left[(|\{s \in \mathbb{R}\colon |Au_n(s)| > \delta\}| + |\{s \in \mathbb{R}\colon |Bu_n(s)| > \delta\}|)+\right.$$
$$\left.+K|\{s \in \mathbb{R}\colon |u_n(s)| > \delta\}|\right] + (1 + K)\varepsilon C_1^2.$$

Thus, if $u_n \to 0$ in measure, then $Au_n \to 0$ and $Bu_n \to 0$ in measure. Therefore for n large enough right-hand part of last inequality can be arbitrary small, in particular, less than $\frac{\beta}{2}$. This is a contradiction. $\square$

The following theorem is the main result of the paper.

Theorem 1. *Assume* (h), (9) *and* $c^2 > c_0^2$. *Then equation* (5) *has a nontrivial solution* u *satisfying* (6).

Proof. Let $\{u_n\} \subset E$ be a Palais–Smale sequence of J constructed above. Since, due to Lemma 4 and Lemma 5, $\{u_n\}$, is bounded and does not converge to zero in measure, there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and a sequence $\{\eta_n\} \subset \mathbb{R}$ such that $w_n := u_n(\cdot + \eta_n) \to u \neq 0$ weakly in E (see Lemma 6 in [22]).

Let $g \in C_0^\infty(\mathbb{R})$, then weak convergence $w_n \to u$ in E implies weak convergence $w_n \to u$ in $L^2(\mathbb{R})$, hence

$$\int\limits_{-\infty}^{+\infty} c^2 w_n'(s)g'(s)ds \to \int\limits_{-\infty}^{+\infty} c^2 u'(s)g'(s)ds,$$

as $n \to \infty$. Since the convergences $Aw_n \to Au$ and $Bw_n \to Bu$ $(n \to \infty)$ is strong in $C^0(\text{supp}(Ag))$ and $C^0(\text{supp}(Bg))$, respectively, and U' is uniformly continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the second and third terms in the expression for J' converge as well, i.e.,

$$\lim_{n\to\infty}\int\limits_{-\infty}^{+\infty} U'(Aw_n(s))Ag(s)ds = \lim_{n\to\infty}\int\limits_{\text{supp}(Ag)} U'(Aw_n(s))Ag(s)ds =$$

$$= \lim_{n\to\infty}\int\limits_{\text{supp}(Ag)} U'(Au(s))Ag(s)ds = \int\limits_{-\infty}^{+\infty} U'(Au(s))Ag(s)ds,$$

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} U'(Bw_n(s))Bg(s)ds = \lim_{n\to\infty} \int_{\text{supp}(Bg)} U'(Bw_n(s))Bg(s)ds =$$

$$= \lim_{n\to\infty} \int_{\text{supp}(Bg)} U'(Bu(s))Bg(s)ds = \int_{-\infty}^{+\infty} U'(Bu(s))Bg(s)ds.$$

Similarly

$$\lim_{n\to\infty} \int_{-\infty}^{+\infty} \sin(w_n(s))g(s)ds = \int_{-\infty}^{+\infty} \sin(u(s))g(s)ds.$$

Thus, for all $g \in C_0^\infty(\mathbb{R})$ as $n \to \infty$,

$$\langle J'(w_n), g\rangle \to \langle J'(u), g\rangle,$$

so

$$|\langle J'(u), g\rangle| = \lim_{n\to\infty} |\langle J'(w_n), g\rangle| = \lim_{n\to\infty} |\langle J'(u_n(\cdot + \eta_n)), g\rangle| =$$

$$= \lim_{n\to\infty} |\langle J'(u_n), g(\cdot - \eta_n)\rangle| \leq \lim_{n\to\infty} \|J'(u_n)\| \cdot \|g\| = 0.$$

And this implies that $\langle J'(u), g\rangle = 0$ for all $g \in C_0^\infty(\mathbb{R})$, so $u \in E$ is a critical point of J:

$$0 < J(u) := b < +\infty.$$

By Lemma 2, u is a solution of equation (5) satisfying (6). $\qquad\square$

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