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A. Z. Mokhon'ko, L. I. Kolyasa

SOME PROPERTIES OF MEROMORPHIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATION WITH MEROMORPHIC COEFFICIENTS

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Estimations of growth of the meromorphic solutions of linear differential equations with meromorphic coefficients in terms of Nevanlinna's characteristics have been obtained. Namely, it is proven that if in the equation $f^{(n)}+a_{n-1}(z)f^{(n-1)}+\ldots+a_{s+1}(z)f^{s+1}+\ldots+a_0(z)f=0$ the coefficients $a_j(z)$, $j=0,1,\ldots,n-1$, are meromorphic functions in $\mathbb{C}$, such that the coefficient $a_j(z)$, $j=s+1,s+2,\ldots,n-1$, grow slower than the coefficients a_s does, then the equation can have at most s linearly independent meromorphic solutions, the growth of which is restricted by the growth of the coefficient a_s.

1. Introduction. Let M be the field of meromorphic in $\mathbb{C}$ functions. Let $\mathcal{E}$ be the ring of entire functions, $\mathcal{E}\subset M$. Consider the equation

$$\begin{gathered} f^{(n)}+a_{n-1}f^{(n-1)}+\ldots+a_{s+1}f^{(s+1)}+\ldots+a_0f=0, \\ a_j\in M, j=0,1,\ldots,n-1. \end{gathered} \tag{1}$$

If P is the set of poles of all coefficients, then all solutions of (1) are analytic, usually multi-valued functions in $\mathbb{C}\setminus P$. There exist such equations (1) with coefficients $a_j\in M$, $j=0,1,\ldots,n-1$, the solutions of which $f\in M$. Thus, all the solutions of the Gauss equation $z(z-1)f''+(4z-2)f'+2f=0$ are single-valued meromorphic functions in $\mathbb{C}$.

We are interested in solutions $f\in M$ of the equation (1). Applications of Nevanlinna theory to analytic theory of differential equations are widely known, see [3], [9], [7]. In particular in the proof of Theorem 1 we follow the approach [3].

Let us use the standard notations of meromorphic functions theory [4], [5]. The Landau symbols $O(\ldots)$, $o(\ldots)$ are used in this article at $r\to+\infty$.

Growth rate of $f\in M$ is described by Nevanlinna's characteristics $m(r,f)$, $T(r,f)$ [4, p. 24–27]; remind

$$N(r,f)=\int\limits_0^r\frac{n(t,f)-n(0,f)}{t}dt+n(0,f)\ln r,\quad m(r,f)=\frac{1}{2\pi}\int\limits_0^{2\pi}\ln^+\left|f\left(re^{i\varphi}\right)\right|d\varphi,$$

$$T(r,f)=m(r,f)+N(r,f),\quad \ln^+x\stackrel{\text{def}}{=}\max(\ln x,0),x\geq 0; \tag{2}$$

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$$\ln^+\left|\sum_{\nu=1}^{n} x_\nu\right| \le \sum_{\nu=1}^{n} \ln^+|x_\nu| + \ln n.$$

If f is an entire function, then $T(r,f) = m(r,f)$. Let us denote by $D(r,f)$ any of the characteristics $T(r,f)$, $m(r,f)$. If $f, g \in M$, then [4, p. 44, 45]

$$D(r, f+g) \le D(r,f) + D(r,g) + \ln 2, \quad D(r, f\cdot g) \le D(r,f) + D(r,g),$$
$$T\Big(r, \frac{f}{g}\Big) \le T(r,f) + T(r,g) + O(1). \tag{3}$$

The function $f \in M$ has a finite order of growth $\rho[f]$, if

$$\rho[f] = \varlimsup_{r\to+\infty} \frac{\ln T(r,f)}{\ln r} < +\infty. \tag{4}$$

The symbol E stands for some sets of intervals on $[0, +\infty)$ with a finite sum of lengths (mes $E < +\infty$).

If $f \in M$ then the following relationships are satisfied ([4, p. 122–125])

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O(1), \text{ if } \rho[f] < 1, \quad k \in \mathbb{N}, \tag{5}$$

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O(\ln r), \text{ if } \rho[f] < +\infty, \quad k \in \mathbb{N}, \tag{6}$$

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O\left(\ln\left(T(r,f)\cdot r\right)\right), \ r \overline{\in} E, \ \text{mes} E < +\infty, \text{if } \rho[f] = +\infty. \tag{7}$$

If $R(z), z \in \mathbb{C}$ is a rational function of degree d, then

$$T(r, R(z)) = d \ln r + O(1). \tag{8}$$

It is known ([4, p.50]), that the function $f \in M$ is a transcendental function if and only if $\lim\limits_{r\to\infty} \frac{T(r,f)}{\ln r} = +\infty$.

For the function $f \in M$ the following inequality is true ([4, p. 131, theorem 2.3])

$$T(r, f') \le 2T(r,f) + m\Big(r, \tfrac{f'}{f}\Big), \tag{9}$$

(if f has finite order, then $T(r,f') \le 2T(r,f) + O(\ln(r))$).

Remind also [8] that if $F(z) = a_k f^k + \ldots + a_1 f + a_0$, where $a_j, f \in M$, $j = 0, 1, \ldots, k$; $a_k(z) \not\equiv 0$, $z \in \mathbb{C}$, then

$$T(r, F) = kT(r,f) + O(\sum_{j=0}^{k} T(r, a_j)). \tag{10}$$

If $f \in M$, $f \ne$ const, then $T(r,f) \nearrow +\infty$ (see [4, p.28] and [5, p.27]).

2. Statment of the problem. As we know, if in the equation (1) the coefficients $a_j \in \mathcal{E}$, $j = 0, 1, \ldots, n-1$, then all the solutions of this equation are entire functions; if moreover all the coefficients are polynomials, then all solutions are finite order of growth, if some coefficients a_j are transcendental functions, then among the solutions there are functions of infinite order [2]. There are several scales for measuring growth order of functions of the infinite growth rate. In [1] the concept of p-th iterated order $\rho_p(f)$ was used to measure the growth of entire solutions of linear differential equations. The notion of $[p, q]$-order was applied in [6] in a similar situation. The definitions of these orders do not described an arbitrary rate. There is no such a drawback in the scale used in article [2] to prove an analogue of such a theorem M. Frei [3]. Let a_s be the last transcendental function in the sequence of coefficients $a_0, a_1, \dots, a_{n-1}$ ($a_j \in \mathcal{E}, j = 0, 1, \dots, n-1$); then (1) has no more than s linearly independent solutions of finite order.

The major idea that was used in the proof by [3] was to reduce of the order of the equation. Having made this reduction, we obtain a linear equation with coefficients and solutions from the field M of meromorphic functions (see (22), (23)). In this paper, from the beginning it is assumed that the coefficients and solutions of the equation (1), which are considered, belong to the field M.

In particular, we consider the case when the only requirement for the coefficients is the assumption that the coefficient a_s grows faster than the coefficients a_j, $j = s+1, s+2, \dots, n-1$ do (in terms of Nevanlinna's characteristics $m(r, f), T(r, f)$). It is also shown that application of more precise estimates of logarithmic derivative (5)–(7) for important subclasses of meromorphic functions gives us the opportunity to obtain additional information about the growth rate of components of the fundamental system of solutions.

3. Main result.

The following theorem has been proven.

Theorem 1. *Let* $s \in \{0, 1, \dots, n-1\}$. *Let* $a_j \in M$, $j \in \{0, 1, \dots, n-1\}$.

1) If
$$m(r, a_j) = O(1), \quad j = s+1, s+2, \dots, n-1, \quad m(r, a_s) \neq O(1), \tag{11}$$
then the equation (1) *can have no more than* s *linearly independent solutions* $f \in M$, *of order smaller than 1;*

2) If
$$m(r, a_j) = O(\ln r), \quad j = s+1, s+2, \dots, n-1, \quad m(r, a_s) \neq O(\ln r), \tag{12}$$
then the equation (1) *can have no more than* s *linearly independent solutions* $f \in M$, *of finite order;*

3) If
$$m(r, a_{s+1}), m(r, a_{s+2}), \dots, m(r, a_{n-1}) = o(m(r, a_s)), \quad r \overline{\in} E, \tag{13}$$
then the equation (1) *can have no more than* s *linearly independent solutions* $f \in M$ *such that*
$$\ln(r \cdot T(r, f)) = o(m(r, a_s)), \quad r \overline{\in} E \tag{14}$$
(the growth rate of which is limited by the rate of growth of coefficients).

The assumption (12) is satisfied, for example if a_j, $j = s+1, s+2, \dots, n-1$ are rational functions, a_s is a transcendental function. Indeed, a transcendental function grows faster than a rational [4, p. 49, 50].

Example 1. Consider the equation $f' - f(e^z - \frac{1}{z}) = 0$. This equation is of the form $f' + a_0(z)f = 0$, $a_0(z) = -e^z + \frac{1}{z}$. Here we have
$$|a_0(z)| = \left| e^{re^{i\varphi}} - \frac{1}{re^{i\varphi}} \right| = e^{r\cos\varphi} + o(1).$$

Therefore, taking into account the fact that $\ln^+ x = \max(\ln x, 0)$, $x \geq 0$, we have

$$m(r, a_0) = \frac{1}{2\pi} \int\limits_0^{2\pi} \ln^+ |a_0(re^{i\varphi})| d\varphi \sim \frac{1}{2\pi} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r \cos\varphi d\varphi = \frac{r}{\pi},$$

$r \to \infty$. Since in this example $s = 0$, then taking into consideration conclusion of a Theorem 1, this means that the equation which we consider has no such solutions $f \in M$ that $\ln(r \cdot T(r, f)) = o(m(r, a_0))$. Indeed, its solution is the function $f(z) = \frac{1}{z}\exp(\exp z)$. Taking into consideration (8), (10) we have $T(r, f) = T(r, \frac{1}{z}e^{e^z}) = T(r, e^{e^z}) + O(\ln r)$. But [5, p. 26] for the entire function $\exp(\exp z)$ the expression $T(r, e^{e^z}) = m(r, e^{e^z}) \sim \frac{e^r}{(2\pi^3 r)^{1/2}},\ r \to \infty$ is true. Thus, $T(r, f) \sim \frac{e^r}{(2\pi^3 r)^{1/2}},\ r \to \infty$; $\ln(r \cdot T(r, f)) \sim r,\ r \to \infty$. Therefore, $r \sim \ln(r \cdot T(r, f)) \neq o(m(r, a_0)),\ m(r, a_0) \sim \frac{r}{\pi},\ r \to \infty$.

Example 2. The functions $f_1(z) = \exp(\exp z)$ and $f_2(z) \equiv 1,\ z \in \mathbb{C}$, are linearly independent solutions of the equation $f'' - (e^z + 1)f' = 0$. This equation is of the form $f'' + a_1(z)f' = 0$, $a_1(z) = -e^z - 1$. Here we have

$$|a_1(re^{i\varphi})| = |e^{re^{i\varphi}} + 1| = e^{r\cos\varphi} + O(1),\ 0 \leq \varphi \leq 2\pi.$$

Therefore analogically to the previous example the equality $m(r, a_1) = (1 + o(1))\frac{r}{\pi}$ is true. Since in this example $s = 1$, taking into consideration conclusions of the theorem the equation which we consider has at most one solution $f \in M$ such that $\ln(r \cdot T(r, f)) = o(m(r, a_1)),\ r \overline{\in} E$. As it is indicated above, such a solution is $f_2(z) \equiv 1,\ z \in \mathbb{C}$. For the solution $f_1(z) = \exp(\exp z)$, as indicated in Example 1,

$$T(r, f_1) = T(r, e^{e^z}) \sim \frac{e^r}{(2\pi^3 r)^{1/2}},\ r \to \infty;\ \ln(r \cdot T(r, f_1)) \sim r,\ r \to \infty.$$

Therefore, $r \sim \ln(r \cdot T(r, f_1)) \neq o(m(r, a_1)), m(r, a_1) = (1 + o(1))\frac{r}{\pi},\ r \to \infty$.

Proof. Let, firstly, in the equation (1) $s = 0$, i.e.

$$m(r, a_{n-1}), m(r, a_{n-2}), \ldots, m(r, a_1) = O(1), \tag{15}$$

$$m(r, a_0) \neq O(1). \tag{16}$$

and $f \in M$ is a solution of the equation (1) of order smaller than 1. Then taking into account (5) we obtain $m(r, \frac{f^{(k)}}{f}) = O(1),\ k \in \mathbb{N}$. From (1) it follows that

$$-a_0(z) \equiv \frac{f^{(n)}(z)}{f(z)} + a_{n-1}(z)\frac{f^{(n-1)}(z)}{f(z)} + \ldots + a_1(z)\frac{f'(z)}{f(z)},\ z \in \mathbb{C}. \tag{17}$$

In view of (17), (3), (15) we have

$$m(r, a_0) \leq \sum_{j=1}^{n} m\Big(r, \frac{f^{(j)}}{f}\Big) + \sum_{j=1}^{n-1} m(r, a_j) + O(1) = O(1),$$

which contradicts the conditions (16). Thus, if $s = 0$ and $n \in \mathbb{N}$, the equation (1) has no solutions $f \in M$ of order smaller than 1.

Let for any $n \in \mathbb{N}$ it is proven that if in the equation (1)

$$m(r, a_{n-1}), m(r, a_{n-2}), \dots, m(r, a_m) = O(1),\ \ m(r, a_{m-1}) \neq O(1),$$

then the equation (1) has no more than $m-1$ linearly independent solutions $f \in M$ of order smaller than 1. Let us prove this assertion for $s = m \geq 1$ and $n \in \mathbb{N}$.

Let in (1)

$$m(r, a_{n-1}), m(r, a_{n-2}), \dots, m(r, a_{m+1}) = O(1),\ \ m(r, a_m) \neq O(1), \tag{18}$$

but this equation has $m+1$ linearly independent solutions $w_1, w_2, \dots, w_{m+1} \in M$, of order $\rho[w_j] < 1$, $j = 1, 2, \dots, m+1$. Let us make the substitution $f(z) = u(z)w_1(z)$. Then the equation (1) can be written in the form (we take $w_1 = w$, $f^{(0)} \overset{\text{def}}{=} f$; $f^{(k)} = \sum_{j=0}^{k} C_k^j u^{(j)} w^{(k-j)}$, $k = 0, 1, \dots, n$)

$$\sum_{k=1}^{n} a_k \sum_{j=1}^{k} C_k^j u^{(j)} w^{(k-j)} + u \sum_{k=0}^{n} a_k w^{(k)} = 0,\ \ a_n = 1.$$

Since $w = w_1$ is the solution of (1), we have $\sum_{k=0}^{n} a_k w^{(k)} \equiv 0$, and the previous equation takes the form

$$\sum_{k=1}^{n} a_k \sum_{j=1}^{k} C_k^j u^{(j)} w^{(k-j)} = 0. \tag{19}$$

Let us divide both sides of the equation (19) by w and group the addends which contain $u^{(s)}$, $s = 1, \dots, n$, we obtain

$$\sum_{s=1}^{n} \left(u'\right)^{(s-1)} \sum_{k=s}^{n} a_k C_k^s \frac{w^{(k-s)}}{w} = 0, \tag{20}$$

or ($u' = v$, $t = s-1$)

$$\sum_{t=0}^{n-1} v^{(t)} \sum_{k=t+1}^{n} a_k C_k^{t+1} \frac{w^{(k-t-1)}}{w} = 0.$$

The last equation can be written in the form

$$v^{(n-1)} + b_{n-2} v^{(n-2)} + \ldots + b_0 v = 0, \tag{21}$$

where

$$b_t = \sum_{k=t+1}^{n} a_k C_k^{t+1} \frac{w^{(k-t-1)}}{w}, \quad t = 0, 1, \dots, n-1;\ a_n = b_{n-1} = 1. \tag{22}$$

In particular,

$$b_{m-1} = \sum_{k=m}^{n} a_k C_k^m \frac{w^{(k-m)}}{w} = a_m + \sum_{k=m+1}^{n} a_k C_k^m \frac{w^{(k-m)}}{w},$$

or

$$a_m = b_{m-1} - \sum_{k=m+1}^{n} a_k C_k^m \frac{w^{(k-m)}}{w}. \tag{23}$$

From (22), (23), (5) it follows that

$$m(r,b_t) \le \sum_{k=t+1}^{n} m(r,a_k) + \sum_{k=t+1}^{n} m\Big(r,\frac{w^{(k-t-1)}}{w}\Big) + O(1) \overset{(5)}{=}$$

$$= \sum_{k=t+1}^{n} m(r,a_k) + O(1), \quad t = m, m+1, \dots, n-1; \tag{24}$$

$$m(r,a_m) \overset{(23),(3)}{\le} m(r,b_{m-1}) + \sum_{k=m+1}^{n} m(r,a_k) +$$

$$+ \sum_{k=m+1}^{n} m\Big(r,\frac{w^{(k-m)}}{w}\Big) + O(1) \overset{(18),(5)}{=} m(r,b_{m-1}) + O(1), \tag{25}$$

Taking into account (24), (18) we have ($b_{n-1} = a_n = 1$)

$$m(r,b_{n-1}), \dots, m(r,b_{m+1}), m(r,b_m) = O(1). \tag{26}$$

From (25), (18) it follows that

$$m(r,b_{m-1}) \ne O(1). \tag{27}$$

The meromorphic linearly independent solutions $w_1(z), w_2(z), \dots, w_{m+1}(z)$ of the equation (1) in the case of the substitution $f = u \cdot w_1$ are transformed into the meromorphic linearly independent solutions $u_1 = \frac{w_2}{w_1}, \dots, u_m = \frac{w_{m+1}}{w_1}$ of the equation (19). These solutions after the substitution $u' = v$ become transformed into the meromorphic linearly independent solutions

$$v_j = u_j' = \Big(\frac{w_{j+1}}{w_1}\Big)', \quad j = 1, \dots, m,$$

of the equation (21).

By the assumption, $\rho[w_j] < 1$, $j = 1, 2, \dots, m+1$. Then the order $\rho[u_i]$ is

$$\rho[u_i] = \rho\Big[\frac{w_{i+1}}{w_1}\Big] < 1, \ i = 1, \dots, m.$$

Taking into account (9) we obtain

$$T(r,v_i) = T(r,u_i') \le 2T(r,u_i) + m\Big(r,\frac{u_i'}{u_i}\Big) + O(1) \overset{(5)}{=} 2T(r,u_i) + O(1), \ i = 1, \dots, m.$$

Hence, in view of the definition of the order (4) it follows that the solutions v_i, $i = 1, \dots, m$ of equation (21) have the order of growth $\rho[v_i] < 1$, $i = 1, \dots, m$.

Therefore, with the assumption that the equation (1) for $s = m$ has $m+1$ linearly independent solutions $w_1, \dots, w_m, w_{m+1}$ of order $\rho[w_j] < 1$, $j = 1, \dots, m+1$, it follows that equation (21) has m linearly independent meromorphic solutions $v_1, \dots, v_m$ of order $\rho[v_i] < 1$, $i = 1, \dots, m$, which contradicts the induction hypothesis. Cases 2), 3) of Theorem 1 are proved similarly, only estimates (6) and (7) are used instead of estimate (5). □

Conclusions. The relation between the rate of growth of meromorphic solutions and coefficients of n-th order linear differential equation has been established. Namely, the assertion of Frei's theorem order of growth of entire solutions of the equation (1) is expanded to the case when the coefficients of the equation and the solutions belong to the field of meromorphic functions, without restrictions for the rate of growth of the coefficients.

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Lviv Polytechnic National University
Lviv, Ukraine
kolyasa.lubov@gmail.com

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