УДК 517.5

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PROPERTIES OF ANALYTIC SOLUTIONS OF A DIFFERENTIAL EQUATION

M. M. Sheremeta, Yu. S. Trukhan. *Properties of analytic solutions of a differential equation*, Mat. Stud. **52** (2019), 138–143.

For an analytic in the unit disk $\{z\colon |z|<1\}$ solution of the differential equation $z(z-1)w''+\beta_1 zw'+\gamma_2 w=0$ the close-to-convexity, possible growth and the l-index boundedness are investigated.

1. Introduction. An analytic univalent in $\mathbb{D}=\{z\colon |z|<1\}$ function

$$f(z)=\sum_{n=0}^{\infty} f_n z^n \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [1, p.203] that the condition $\operatorname{Re}\{1+zf''(z)/f'(z)\}>0$ $(z\in\mathbb{D})$ is necessary and sufficient for the convexity of f. By W. Kaplan [2] a function f is said to be close-to-convex in $\mathbb{D}$ (see also [1, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\operatorname{Re}(f'(z)/\Phi'(z))>0\,(z\in\mathbb{D})$. A close-to-convex function f has the characteristic property that the complement G to the domain $f(\mathbb{D})$ can be filled with rays L which go from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0)\neq 0$. Hence it follows that a function f is close-to-convex in $\mathbb{D}$ if and only if the function

$$g(z)=z+\sum_{n=2}^{\infty} g_n z^n \tag{2}$$

is close-to-convex in $\mathbb{D}$, where $g_n=f_n/f_1$. The result of J. W. Alexander ([3]) implies the following lemma for the function (2) (see also [4, p. 9]).

Lemma 1. *If coefficients of the function of form* (2) *satisfy the condition*

$$1\geq 2g_2\geq\cdots\geq kg_k\geq (k+1)g_{k+1}\geq\cdots>0$$

then the function f is close-to-convex in $\mathbb{D}$.

2010 *Mathematics Subject Classification:* 30C45.
Keywords: differential equation; close-to-convexity; l-index boundedness.
doi:10.30970/ms.52.2.138-143

Using this lemma S. Shah [5] indicated conditions on real parameters β_0, β_1, γ_0, γ_1, γ_2 of the differential equation

$$z^2w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0, \tag{3}$$

under which there exists an entire transcendental solution given by (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$ functions. Putting (1) into (3) it is easy to see that for a coefficients of series (1) recurrent formula $f_n = \xi_n f_{n-1} + \eta_n f_{n-2}$ $(n \geq 2)$ holds. S. Shah studied the cases when this two-term recurrent formula reduces to a one-term. For example, he proved that if $\beta_1 + \gamma_2 = 0$, $-1 \leq \beta_0 < 0$, $\beta_1 > 0$, $\gamma_0 = 0$ and $-\beta_1/2 < \gamma_1 \leq 0$, then equation (3) has an entire solution (2) such that all the derivatives $g^{(n)}$ $(n \geq 0)$ are close-to-convex in $\mathbb{D}$ functions and $\ln M_g(r) = (1 + o(1))|\beta_0|r$ as $r \to +\infty$, where $M_g(r) = \max\{|g(z)|\colon\ |z| = r\}$.

The investigations of S. Shah about close-to-convexity of an entire solutions of differential equation (3) with two-term recurrent formula for the coefficients are continued in papers [6–10], and in [11–14] this results are generalized for linear differential equation of n-th order with polynomial coefficients of the n-th degree.

Let $0 < R \leq +\infty$, and l be positive continuous on $[0, R)$ function such that $l(r) > \beta/(R - r)$, where $\beta = \mathrm{const} > 1$. Analytic in $\mathbb{D}_R = \{z\colon\ |z| < R\}$ function f is said to be of bounded l-index [15, p.71], if there exists $N \in \mathbb{Z}_+$ such that for all $n \in \mathbb{Z}_+$ and $z \in \mathbb{D}_R$

$$\frac{|f^{(n)}(z)|}{n!l^n(|z|)} \leq \max\left\{\frac{|f^{(k)}(z)|}{k!l^k(|z|)}\colon\ 0 \leq k \leq N\right\}. \tag{4}$$

The least such integer N is called l-index and is denoted by $N(f, l)$. If $G \subset \mathbb{D}_R$ and there exists $N \in \mathbb{Z}_+$ such that inequality (4) holds for all $n \in \mathbb{Z}_+$ and $z \in G$ f is said to be function of bounded l-index on (or in) G, and l-index is denoted by $N(f, l;\, G)$.

Boundedness of l-index of an entire solutions of differential equation (3) are studied in papers [16–21].

Instead of equation (3) here we consider differential equation

$$z(z-1)w'' + (\beta_0 z^2 + \beta_1 z)w' + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0 \tag{5}$$

and study properties of its analytic in $\mathbb{D}$ solutions.

2. Recurrent formulas for the coefficients. By elementary calculus we establish that in order that an analytic in the neighborhood of the origin function of form (1) be the solution of equation (5) it is necessary and sufficient that

$$\gamma_2 f_0 = 0, \quad 2f_2 = (\beta_1 + \gamma_2)f_1 + \gamma_1 f_0 \tag{6}$$

and

$$f_{n+1} = \frac{n(n + \beta_1 - 1) + \gamma_2}{n(n+1)} f_n + \frac{\beta_0(n-1) + \gamma_1}{n(n+1)} f_{n-1} + \frac{\gamma_0}{n(n+1)} f_{n-2}, \quad (n \geq 2). \tag{7}$$

To use Lemma 1 it is necessary that all the coefficients f_n for $n \geq 1$ are positive. That is why we assume like in [5] parameters β_0, β_1, γ_2, γ_1, γ_0 are real. In view of (6) we may assume that $f_0 = 0$. Remark also that if $\beta_0 = \gamma_1 = \gamma_0 = 0$, then (6) and (7) implies $\gamma_2 f_0 = 0$ and one-term recurrent formula

$$f_{n+1} = \frac{n(n + \beta_1 - 1) + \gamma_2}{n(n+1)} f_n \quad (n \geq 1). \tag{8}$$

Like in [5] here we consider only the case when the coefficients of the solution of equation (5) are determined by one-term recurrent formula, i.e. formula (8).

3. Close-to-convexity and growth. As $f_0 = 0$, so (6) implies $2f_2 = (\beta_1+\gamma_2)f_1$. It implies that f_1 can be arbitrary and f_2 depends on the choice of f_1. We choose $f_1 = 1$. Then $2f_2 = \beta_1 + \gamma_2$ and $1 \ge 2f_2 > 0$ if and only if

$$0 < \beta_1 + \gamma_2 \le 1. \tag{9}$$

For such a choice of f_0 and f_1 the solution is of the form

$$f(z) = z + \sum_{n=2}^{\infty} f_n z^n, \tag{10}$$

where coefficients f_n are defined by recurrent formula (8).

The following theorem is true.

Theorem 1. *If* $\beta_0 = \gamma_1 = \gamma_0 = 0$, $-2 \le \beta_1 \le 1$ *and* $0 < \beta_1+\gamma_2 \le 1$ *then differential equation* (5) *has an analytic in* $\mathbb{D}$ *solution* f *of form* (10) *which is close-to-convex in* $\mathbb{D}$ *function such that* $M_f(r) := \max\{|f(z)|\colon |z| = r\} \asymp 1$ $(r \to 1)$ *in the case* $\beta_1 < 1$ *and* $M_f(r) \asymp -\ln(1-r)$ $(r \to 1)$ *in the case* $\beta_1 = 1$.

Proof. As $\beta_1 \ge -2$ the sequence $(n(n + \beta_1 - 1) + \gamma_2)$ is nondecreasing and thus, in view of (9), $n(n+\beta_1-1)+\gamma_2 \ge \beta_1+\gamma_2 > 0$ for all $n \ge 1$, and as $\beta_1 \le 1$, so $n(n+\beta_1-1)+\gamma_2 \le n^2$ for all $n \ge 2$. Therefore (8) implies $f_n > 0$ and

$$(n+1)f_{n+1} = \frac{n(n+\beta_1-1)+\gamma_2}{n^2} n f_n \le n f_n.$$

Formula (8) implies also that $f_{n+1} = (1 + o(1))f_n$ $(n \to \infty)$, i.e. the radius of convergence of the series (10) equals 1. Therefore, in view of Lemma 1, the first part of Theorem 1 is proved.

As $f_1 = 1$ equality (8) implies that

$$f_{n+1} = \prod_{j=1}^{n} \frac{j(j+\beta_1-1)+\gamma_2}{j(j+1)}, \ n \ge 1,$$

therefore

$$\ln f_{n+1} = \sum_{j=1}^{n} \ln\left(1 + \frac{\beta_1-2}{j+1} + \frac{\gamma_2}{j(j+1)}\right) = \sum_{j=1}^{n} \frac{\beta_1-2}{j+1} + O(1) = (\beta_1-2)\ln(n+1) + O(1),$$

as $n \to +\infty$, that is $f_n \asymp n^{\beta_1-2}$ as $n \to +\infty$. Therefore if $\beta_1 < 1$ then $f(r) = O(1)$ as $r \uparrow 1$, and if $\beta_1 = 1$ then $f(r) \asymp \ln\frac{1}{1-r}$ as $r \uparrow 1$. In view of the equality $M_f(r) = f(r)$ for the function of form (10), the proof of Theorem 1 is complete. □

Let us note that if $\beta_1 = 1$ and $\beta_0 = \gamma_1 = \gamma_0 = \gamma_2 = 0$ equation (5) is of the form $(z-1)w'' + w' = 0$, and its solution $w = f(z)$ satisfying conditions $f(0) = 0$ and $f'(0) = 1$ is the function $f(z) = \ln\frac{1}{1-z}$. This function is convex, and thus close-to-convex in $\mathbb{D}$.

4. *l*-index boundedness. For investigation of the l-index boundedness of the solution (10) of differential equation (5) we will use the following lemma proved in [22].

Lemma 2. *If a function* (10) *is analytic in* $\overline{\mathbb{D}}_\varrho = \{z\colon\ |z| \le \varrho\}$ *and*

$$\sum_{n=1}^{\infty}(n+1)|f_{n+1}|\varrho^n \le a(\varrho) < 1, \tag{11}$$

then $N(f, l;\, \mathbb{D}_\varrho) \le 1$ *with* $l(|z|) = (1 + a(\varrho))/\big((1 - a(\varrho))(\varrho - |z|)\big)$.

The conclusion of Lemma 2 means that for a function (10) if (11) holds then

$$\frac{|f^{(n)}(z)|}{n!}\left(\frac{1-a_j(\varrho)}{1+a_j(\varrho)}(\varrho-|z|)\right)^n \le \max\left\{\frac{|f'(z)|}{1!}\frac{1-a_j(\varrho)}{1+a_j(\varrho)}(\varrho-|z|),\ |f(z)|\right\}$$

for all $z \in \mathbb{D}_\varrho$ and $n \ge 2$.

If $0 < \eta < 1$ and $z \in \mathbb{D}_{\eta\varrho}$, then $\varrho - |z| \ge (1-\eta)\varrho$ and last inequality implies $N(f, l;\, \mathbb{D}_{\eta\varrho}) \le 1$ with $l(|z|) = (1 + a(\varrho))/\big((1 - a(\varrho))(1-\eta)\varrho\big)$, because if $N(f, l_*, G) \le N$ and $l_*(r) \le l^*(r)$ it is easy to show [15, p. 23] that $N(f, l^*, G) \le N$. Therefore the following corollary is true.

Lemma 3. *If a function f of form* (10) *is analytic in* $\overline{\mathbb{D}}_\varrho = \{z\colon\ |z| \le \varrho\}$ *and satisfies* (11), *then* $N(f, l;\, \mathbb{D}_{\eta\varrho}) \le 1$ *for every* $\eta \in (0, 1)$ *with*

$$l(|z|) \equiv \frac{(1 + a(\varrho))}{(1-\eta)\varrho(1 - a(\varrho))}.$$

If the conditions of Theorem 1 hold then (8) implies for $\varrho \in (0,\ 1/2)$

$$\sum_{n=1}^{\infty}(n+1)|f_{n+1}|\varrho^n = (\beta_1+\gamma_2)\varrho + \varrho\sum_{n=1}^{\infty}\frac{(n+1)n + n\beta_1 + \beta_1 + \gamma_2}{(n+1)^2}(n+1)|f_{n+1}|\varrho^n \le$$

$$\le \varrho + \varrho\sum_{n=1}^{\infty}\frac{(n+1)n + n + 1}{(n+1)^2}(n+1)|f_{n+1}|\varrho^n = \varrho + \varrho\sum_{n=1}^{\infty}(n+1)|f_{n+1}|\varrho^n,$$

thus

$$\sum_{n=1}^{\infty}(n+1)|f_{n+1}|\varrho^n \le a(\varrho) = \frac{\varrho}{1-\varrho} < 1.$$

Therefore Lemma 3 implies such proposition.

Proposition 1. *If conditions of Theorem 1 hold, for the solution f of form* (10) *of differential equation* (5) $N(f, l;\, \mathbb{D}_{\eta\varrho}) \le 1$ *holds with* $l(|z|) \equiv 1/\big((1-\eta)\varrho(1-2\varrho)\big)$ *for every* $\varrho \in (0,\ 1/2)$ *and* $\eta \in (0,\ 1)$.

Let us note that if the assumptions of Theorem 1 are valid, equation (5) implies that equality

$$z(z-1)f''(z) + \beta_1 z f'(z) + \gamma_2 f(z) \equiv 0 \tag{12}$$

holds identically for the function of form (10). Let us also note that $-2 \le \beta_1 \le 1$ and (9) implies $-1 < \gamma_2 \le 3$. Therefore from (12) for $\eta\varrho \le |z| \le 1$ we get

$$(1-|z|)|f''(z)| \le |\beta_1||f'(z)| + \frac{|\gamma_2|}{|z|}|f(z)| \le |f'(z)| + \frac{3}{\eta\varrho}|f(z)|,$$

and if $A > 1$ then

$$\frac{|f''(z)|}{2!}\left(\frac{1-|z|}{A}\right)^2 \le \frac{|f'(z)|}{1!}\left(\frac{1-|z|}{A}\right)\frac{1}{2A}+\frac{3(1-|z|)}{2A^2\eta\varrho}|f(z)| \le$$

$$\le \frac{|f'(z)|}{1!}\left(\frac{1-|z|}{A}\right)\frac{1}{2A}+\frac{3}{2A\eta\varrho}|f(z)| \le \max\left\{\frac{|f'(z)|}{1!}\left(\frac{1-|z|}{A}\right),\ |f(z)|\right\} \tag{13}$$

if the condition $A \ge (3+\eta\varrho)/(2\eta\varrho)$ holds.

Differentiating (12) $n \ge 1$ times we obtain

$$z(z-1)f^{(n+2)}(z)+((2n+\beta_1)z-n)f^{(n+1)}(z)+(n(n-1)+\beta_1+\gamma_2)f^{(n)}(z)\equiv 0.$$

Therefore at $A \ge 4/\eta\varrho$ for $|z| \ge \eta\varrho$ we get

$$\frac{|f^{(n+2)}(z)|}{(n+2)!}\left(\frac{1-|z|}{A}\right)^{n+2} \le \frac{(2n+|\beta_1|)\eta\varrho+n}{A(n+2)\eta\varrho}\frac{|f^{(n+1)}(z)|}{(n+1)!}\left(\frac{1-|z|}{A}\right)^{n+1}+$$

$$+\frac{n(n-1)+\beta_1+\gamma_2}{A^2(n+2)(n+1)\eta\varrho}\frac{|f^{(n)}(z)|}{n!}\left(\frac{1-|z|}{A}\right)^{n} \le$$

$$\le \frac{(2n+2)\eta\varrho+n}{A(n+2)\eta\varrho}\frac{|f^{(n+1)}(z)|}{(n+1)!}\left(\frac{1-|z|}{A}\right)^{n+1}+\frac{n(n-1)+1}{A(n+2)(n+1)\eta\varrho}\frac{|f^{(n)}(z)|}{n!}\left(\frac{1-|z|}{A}\right)^{n} \le$$

$$\le \frac{4}{A\eta\varrho}\max\left\{\frac{|f^{(n+1)}(z)|}{(n+1)!}\left(\frac{1-|z|}{A}\right)^{n+1},\ \frac{|f^{(n)}(z)|}{n!}\left(\frac{1-|z|}{A}\right)^{n}\right\} \le$$

$$\le \max\left\{\frac{|f^{(n+1)}(z)|}{(n+1)!}\left(\frac{1-|z|}{A}\right)^{n+1},\ \frac{|f^{(n)}(z)|}{n!}\left(\frac{1-|z|}{A}\right)^{n}\right\}. \tag{14}$$

Since the inequality $A \ge 4/(\eta\varrho)$ implies the inequality $A \ge (3+\eta\varrho)/(2\eta\varrho)$, in the case of $A \ge 4/(\eta\varrho)$ inequalities (13) and (14) imply the following inequality

$$\frac{|f^{(k)}(z)|}{k!}\left(\frac{1-|z|}{A}\right)^{k} \le \max\left\{\frac{|f'(z)|}{1!}\left(\frac{1-|z|}{A}\right),\ |f(z)|\right\}$$

for all $k \ge 2$. Therefore the following proposition is true.

Proposition 2. *If conditions of Theorem 1 are satisfied, for the solution* (10) *of differential equation* (5) $N(f, l; \mathbb{D}\setminus\mathbb{D}_{\eta\varrho}) \le 1$ *holds with* $l(|z|) \equiv 4/\big(\eta\varrho(1-|z|)\big)$ *for every* $\varrho \in (0,\ 1/2)$ *and* $\eta \in (0,\ 1)$.

Combining the conclusions of Propositions 1 and 2, we obtain the following theorem.

Theorem 2. *If the hypothesis of Theorem 1 hold, an analytic in* $\mathbb{D}$ *solution* f *of form* (10) *of differential equation* (5) *is of bounded* l*-index* $N(f, l) \le 1$ *with*

$$l(|z|) \equiv \max\big\{4/\big(\eta\varrho(1-|z|)\big),\ 1/\big((1-\eta)\varrho(1-2\varrho)\big)\big\}$$

for every $\varrho \in (0,\ 1/2)$ *and* $\eta \in (0,\ 1)$.

If we choose $\varrho = 3/8$ and $\eta = 1/2$ then $4/(\eta\varrho) = 1/((1-\eta)\varrho(1-2\varrho)) = 64/3$ and Theorem 2 implies the following corollary.

Corollary. *If the conditions of Theorem 1 hold, an analytic in* $\mathbb{D}$ *solution* f *of form* (10) *of differential equation* (5) *is of bounded* l*-index* $N(f, l) \le 1$ *with* $l(|z|) \equiv 64/3\big((1-|z|)\big)$.

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Received 12.04.2019
Revised 30.08.2019