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IDEMPOTENT MATRICES OVER A DUO RING

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In the present paper, we study diagonability of matrices over a duo ring. We prove that an idempotent matrix over such a ring is diagonalizable under a similarity transformation.

Introduction and main results. The problem of diagonalization of idempotent matrices is a classic one. The first results were obtained in 1946 by A. Foster [1]. He proved that each idempotent matrix over a commutative ring R is diagonalizable under a similarity transformation if and only if each idempotent matrix over R has a characteristic vector.

In 1966, Steger [4] used Foster's result to prove the following theorem. Let R be a commutative ring with identity and A be an $n \times n$ idempotent matrix over R. If there exist invertible matrices P and Q such that PAQ is a diagonal matrix then there is an invertible matrix U over R such that UAU^{-1} is a diagonal matrix.

Later, the results Foster's and Steger's were extended to different classes of commutative and non commutative rings [2], [3], [6], [7] and [8].

In the present paper, we prove that an idempotent matrix over such ring is diagonalizable under a similarity transformation.

Everywhere in this paper R is a duo ring. Recall that a ring R is said to be a right (left) duo-ring if every right (left) ideal of this ring is a 2-sided ideal. If a ring is both left and right duo-ring, then it is called a duo-ring. Examples of an duo-ring are abelian regular ring, directly sum fields.

We say that matrices A and B over a ring R are equivalent if there are unimodular matrices P and Q of appropriate size such that $B = PAQ$.

The matrices A and B are called similar, if there exists an invertible matrix U over ring R of an appropriate size such that $A = UBU^{-1}$. Remark that a square matrix is said to be diagonalizable if it is similar to a diagonal matrix.

The ring R is PT (projective trivial) ring if every idempotent matrix over R is similar to a diagonal matrix.

Recall that $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in R^n$ is called a right unimodular vector if there exists $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ such that $\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n = 1$. A right unimodular vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in R^n$ is completable if it is the row of some invertible matrix over R. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a characteristic vector of A if $\alpha \in R^n$ is a completable right unimodular vector and $\alpha A = \lambda\alpha$ for some $\lambda \in R$ (we call λ the characteristic value of α).

Two elements x and y of R are called right associates if $xR = yR$, left associates it $Rx = Ry$, and associates if they are both right and left associates.

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Lemma 1. *Let R be a duo ring and $e = e^2$ be an idempotent element in R. Then $ea = ae$, where $a \in R$.*

Proof. Since R is a duo ring, we have $ea = a'e$, where $a' \in R$. Then $ea = a'e = a'e^2 = eae$ and $ae = ea' = e^2a' = eae$. So $ea = ae$. □

Lemma 2 ([5], Lemma 2.1)**.** *Let R is duo ring and $b, e \in R$, where $e^2 = e$. If $bR = eR$, then b and e is right associates.*

Theorem 1. *Let R be a duo ring and A be an $n \times n$ n idempotent matrix over R. If there exist invertible matrices P and Q such that PAQ is a diagonal matrix, then there is an invertible matrix U such that UAU^{-1} is a diagonal matrix.*

Proof. Suppose that there exist invertible matrices P and Q such that $PAQ = \operatorname{diag}(b_1, b_2, \dots, b_n)$. Denote $B = PAQ$ and $U = Q^{-1}P^{-1} = (u_{ij})$. Then

$$(BU)^2 = (PAP^{-1})(PAQQ^{-1}P^{-1}) = PAQQ^{-1}P^{-1} = BU$$

and

$$BUB = PAP^{-1}PAQ = PA^2Q = PAQ = B.$$

Therefore we have $b_i = b_iu_{ii}b_i$, b_iu_{ii} and $u_{ii}b_i$ are idempotents of R. Set $e = b_iu_{ii}$. Then $b_i = eb_i$ and thus $b_i = e(1 - e + b_i)$. Note that

$$(1-e+b_i)(1-e+eu_{ii}) = 1-e+b_i-e+e^2-e^2u_{ii}+b_i-b_ie+b_ieu_{ii} = 1-e+b_i-b_ie+b_ieu_{ii}.$$

By Lemma 1, we obtain

$$1 - e + b_i - eb_i + eb_iu_{ii} = 1 - e + b_i - b_i + ee = 1, \quad (1 - e + eu_{ii})(1 - e + b_i) = 1.$$

Then $(1 - e + b_i)(1 - e + eu_{ii}) = 1 = (1 - e + eu_{ii})(1 - e + b_i)$. So $1 - e + b_i$ is a unit, b_i and b_iu_{ii} differs by a unit factor. Thus, we may assume that Q has been adjusted so that $b_i^2 = b_i$, $i = 1, 2, \dots, n$. The matrix equality $BUB = B$ means that

$$\begin{aligned} &b_iu_{ii} = b_i, \\ &b_iu_{ij}b_j = 0, i \neq j, i, j = 1, 2, \dots, n \end{aligned} \tag{1}$$

Denote

$$D = \begin{pmatrix} 1 & b_1u_{12} & \dots & b_1u_{1n} \\ b_2u_{21} & 1 & \dots & b_2u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_nu_{n1} & b_nu_{n2} & \dots & 1 \end{pmatrix}$$

Then $D^2 = 2D - I$, so D is invertible. From 1 we obtain

$$BD = \begin{pmatrix} b_1 & b_1^2u_{12} & \dots & b_1^2u_{1n} \\ b_2^2u_{21} & b_2 & \dots & b_2^2u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n^2u_{n1} & b_n^2u_{n2} & \dots & b_n \end{pmatrix} = \begin{pmatrix} b_1 & b_1u_{12} & \dots & b_1u_{1n} \\ b_2u_{21} & b_2 & \dots & b_2u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_nu_{n1} & b_nu_{n2} & \dots & b_n \end{pmatrix};$$

$$DBU = \begin{pmatrix} b_1 & b_1u_{12} & \dots & b_1u_{1n} \\ b_2u_{21} & b_2 & \dots & b_2u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_nu_{n1} & b_nu_{n2} & \dots & b_n \end{pmatrix}.$$

Then $DBU = BD$. Thus,

$$(DP)A(DP)^{-1} = D(PAQQ^{-1}P^{-})D^{-1} = DBUD^{-1} = BDD^{-1} = B.$$

□

Theorem 2. *Let R be an PT ring. Then any right unimodular row $a = (a_1, a_2, \dots, a_n) \in R^n$ can be completed to an invertible matrix.*

Proof. Suppose $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is right unimodular row. Then there exists such a right unimodular row $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, that $\alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_n\beta_n = 1$. Denote $A = (\beta\alpha) = \beta_i\alpha_j$. Then $A^2 = A$.

Since R is a PT ring, there exists an invertible matrix P with $PAP^{-1}{=}\mathrm{diag}(e_1, e_2, \dots, e_n)$. Let $X = (x_1, x_2, \dots, x_n) = \alpha P^{-1}$, $Y = (y_1, y_2, \dots, y_n)^t = P\beta$. Then $XY = 1$ and $YX = \mathrm{diag}(e_1, e_2, \dots, e_n)$. Since

$$\sum_{i=1}^{n} x_iy_i = 1,\ y_ix_i = e_i, 1 \le i \le n$$

and $y_ix_j = 0, i \neq j, 1 \le i, j \le n$, one has $y_ix_iy_i = y_i, x_iy_ix_i = x_i, 1 \le i \le n$. So $e_i = y_ix_i$ and $f_i = x_iy_i$ are idempotents. From Lemma 1 it follows

$$\begin{aligned} e_i &= e_i^2 = y_ix_iy_ix_i = y_if_ix_i = f_iy_ix_i = f_ie_i; \\ f_i &= f_i^2 = x_iy_ix_iy_i = x_ie_iy_i = x_iy_ie_i = f_ie_i. \end{aligned}$$

So $e_i = f_i$ and $\sum_{i=1}^{n} x_iy_i = \sum_{i=1}^{n} y_ix_i = 1$. This means that $x = \sum_{i=1}^{n} x_i$ is a unit of R. Denote

$$D = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Then

$$DP = \begin{pmatrix} x & x_2 & x_3 & \dots & x_n \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

So $X = (x_1, x_2, \dots, x_n) = \alpha P^{-1}$ is the first row of some invertible matrix A. This implies that α is completable. □

Theorem 3. *Let R be a duo ring. The following are equivalent*

1) R is PT ring;

2) Each idempotent matrix over R has a characteristic vector.

Proof. Suppose that R is PT ring and A be an $n \times n$ idempotent matrix. Then there is an invertible matrix Q with $QAQ^{-1} = \operatorname{diag}(x_1, x_2, \ldots, x_n)$. If $\alpha = (1, 0, 0, \ldots, 0)$ is any right unimodular row, then $\alpha QAQ^{-1} = (x_1, 0, 0, \ldots, 0) = x_1\alpha$. And if $\beta = \alpha Q$ is a right unimodular row then $\beta A = \alpha QA = (\alpha QAQ^{-1})Q = x_1\alpha Q = x_1\beta$.

We assume $A \neq 0$. Let α be a characteristic vector of A and $\alpha_1 A = x_1\alpha_1$. In a coordinate system $x_1, x_2, \ldots, x_n$ in which x_1 lies along e_1, the matrix A assumes the half reduced form

$$A' = \begin{pmatrix} x_1 & a'_{12} & a'_{13} & \ldots & a'_{1n} \\ 0 & a'_{22} & a'_{23} & \ldots & a'_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a'_{n2} & a'_{n3} & \ldots & a'_{nn} \end{pmatrix} \tag{2}$$

Since $(A')^2 = A'$, we have $x_1^2 = x_1$ and

$$\begin{pmatrix} a'_{22} & a'_{23} & \ldots & a'_{2n} \\ a'_{23} & a'_{33} & \ldots & a'_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n2} & a'_{n3} & \ldots & a'_{nn} \end{pmatrix}^2 = \begin{pmatrix} a'_{22} & a'_{23} & \ldots & a'_{2n} \\ a'_{23} & a'_{33} & \ldots & a'_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n2} & a'_{n3} & \ldots & a'_{nn} \end{pmatrix} \tag{3}$$

We proceed by induction. The theorem is trivially true for $n = 1$. Assume it is valid for $n - 1$. We will show its validity for n. In view of (2), (3) and our induction hypothesis, in a coordinate system determined by a suitably chosen base $e_1, e_2, \ldots, e_n$ the matrix A' (and hence A) is equivalent to a matrix A'' of the form

$$A'' = \begin{pmatrix} x_1 & a'_{12} & a'_{13} & \ldots & a'_{1n} \\ 0 & x_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & x_n \end{pmatrix}, \tag{4}$$

where $(A'')^2 = A''$ implies

$$\begin{array}{l} x_1^2 = x_1, x_2^2 = x_2, \ldots, x_n^2 = x_n \\ a'_{12}(x_1 + x_2 - 1) = 0 \\ a'_{13}(x_1 + x_3 - 1) = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a'_{1n}(x_1 + x_n - 1) = 0. \end{array} \tag{5}$$

In the $e_1, e_2, \ldots, e_n$ coordinate system, the vector $e_1 = (1, 0, \ldots, 0)$ is a characteristic vector of A'' belonging to the characteristic value x_1. Now we show that: to each of the characteristic values $x_2, x_3, \ldots, x_n$ of A'' there exists a corresponding basal characteristic vector $e'_2, e'_3, \ldots, e'_n$ respectively, where

$$\begin{array}{l} e'_2 = (r_2, 1, 0, 0 \ldots, 0) \\ e'_3 = (r_3, 0, 1, 0 \ldots, 0) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e'_n = (r_n, 0, 0, 0 \ldots, 1) \end{array} \tag{6}$$

Since $A''e_2 = (x_1r_2+a'_{12}, x_2, 0, 0, \ldots, 0)$ this is equivalent to show that Λ: there exist elements $r_2, r_3, \ldots, r_n$ in R such that

$$r_i(x_1 - x_i) = a'_{1i}, \quad (i = 2, 3, \ldots, n) \tag{7}$$

The proof of Λ is given in three cases. In each case $i = 2, 3, \ldots, n$.

Case 1. If a_1+a_i-1 is a unity element of R, then $a'_{1i} = 0$, and any value of r_i is a solution of (7).

Case 2. If $a_1 + a_i - 1 = 0$, then $a_1 - a_i$ is a unity, and $r_i = a'_{1i}(a_1 - a_i)^{-1} = a'_{1i}(a_1 - a_i)$ is a solution of (7).

Case 3. If $a_1+a_i-1 \neq 0$ and a_1+a_i-1 is not a unity element of R. Then a_1-a_i divides a'_{1i}, and $r_i = a_{1i}(2a_i - 1)$ is a solution of (7).

Take into account (5), (6) and (7), we conclude that there exists an invertible matrix

$$P = \begin{pmatrix} 1 & r_2 & r_3 & \ldots & r_n \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}$$

such that $PA''P^{-1} = \mathrm{diag}(x_1, x_2, \ldots, x_n)$. □

REFERENCES

1. Foster A. *Maximal idempotent sets in a ring with unit*// Duke Mathematical Journal. – 1946. – V.13, №2. – P. 247–258.
2. Laksov D. *Diagonalization of matrices over rings*// Journal of Algebra. – 2013. – V.376. – P. 123–138.
3. Richter R.B., Wardlaw W.P. *Diagonalization over commutative rings*// The American Mathematical Monthly. – 1990. – V.97, №3. – P. 223–227.
4. Steger A. *Diagonability of idempotent matrices*// Pac. J. Math. – 1966. – V.19, №3. – P. 535–542.
5. Steger A. *Elementary factorization in π-regular rings*// Canad. J. Math., – 1966. – V.18. – P. 307–313.
6. Shchedryk V.P. *Factorization of matrices over elementary divisor rings.* – Lviv: Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of the NAS of Ukraine. – 2017. – 304 p.
7. Tan Y.J. Diagonability of matrices over commutative semirings, Linear and Multilinear Algebra. – 2018.
8. Zabavsky B.V. *Diagonal reduction of matrices over rings*// Mathematical Studies, Monograph Series. – 2012. – V.16. – 251p.

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