УДК 517.524

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A TRUNCATION ERROR BOUND FOR SOME BRANCHED CONTINUED FRACTIONS OF A SPECIAL FORM

I. B. Bilanyk. *A truncation error bound for some branched continued fractions of a special form*, Mat. Stud. **52** (2019), 115–123.

By analogy with continued fractions, the angular domain of convergence of branched continued fraction of a special form is established. The truncation error bound for branched continued fractions of the special form is established. For this we have used the estimates for continued fractions obtained by J. Jensen, W. Gragg, D. Warner, the multidimensional analogue of the van Vleck Theorem, the analytic theory of continued fractions and branched continued fractions and elements of the theory of stability under perturbations. In comparison with already known results, in the paper, isolation conditions from zero for elements of branched continued fractions of the special form are weakened, but there are some requirements for the elements' rate of tending to zero. The obtained result is a multidimensional analogue of estimates for van Vleck continued fractions.

Introduction. One of the classical convergence criteria of continued fractions is the van Vleck Theorem [18, 19], which contains sufficient conditions of convergence of continued fractions with complex elements that lie in the angular domain. The proof is based on using the Stieltjes–Vitaly Theorem and it does not need obtaining truncation error bounds.

In the theory of continued fractions, there are some theorems where the fact of convergence is proved but the estimate of approximation is not. For example, it has not known yet the truncation error bound for continued fractions in the case when the elements satisfy the conditions of the Sleszynski–Pringsheim Theorem [5].

J. Jensen [14, 18] proved the estimates of convergence rate for van Vleck continued fractions. Another method to prove a similar estimate is showed by W. Gragg and D.Warner in the paper [13]. The method from that paper was also used for establishing the truncation error bounds for continued fractions in parabolic domains.

Theorem 1 (W. Gragg, D. Warner, [13])**.** *Let the elements of the continued fraction*

$$b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ddots}} = b_0 + \mathop{\mathrm{D}}_{n=1}^{\infty} \frac{1}{b_n}$$

satisfy the conditions

$$b_k \neq 0,\ |\arg b_k| < \theta,\ \theta < \frac{\pi}{2},\ k = 0, 1, 2, \dots \ . \tag{1}$$

2010 *Mathematics Subject Classification:* 11J70, 30B70, 31A55, 40A15.
Keywords: continued fraction; branched continued fraction of the special form; truncation error bound.
doi:10.30970/ms.52.2.115-123

Then

(i) there exist the finite limits of even and odd approximants;

(ii) the sequence of approximants $\{f_n\}$ *converges iff the series* $\sum_{n=1}^{\infty}|b_n|$ *diverges;*

(iii) the bound

$$|f_m - f_{n-1}| \le \frac{1}{d_n},\ m \ge n, \tag{2}$$

holds with

$$d_n \ge \frac{\operatorname{Re} b_1}{2+\operatorname{Re} b_1}\cos\theta \ln\left(1+(\operatorname{Re}(b_1))^2\min\left\{1,\frac{1}{|b_1|^2}\right\}\cos\theta\sum_{k=1}^{n}|b_k|\right),\ n\ge 1.$$

An analogue of the van Vleck Theorem is established for branched continued fractions (BCF) [5], two-dimensional continued fractions [10, 16], BCF of a special form [8]. By T. Antonova in the paper [1] the truncation error bound in an angular domain for BCF of the special form is obtained in case when elements are isolated from zero. The bound for two-dimensional continued fractions in an angular domain with putting on additional conditions is prowed by O. Sus' [17]. These conditions are requirements regarding the speed of divergence to zero of certain product that is constructed of elements of the two-dimensional continued fraction. In [8] there is a truncation error bound for BCF of the special form with elements isolated from zero. This result is similar to J. Jensen estimate.

In the paper, the truncation error bound is established for BCF of the special form

$$b_0 + \mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{i_{k-1}}\frac{1}{b_{i(k)}} = b_0 + \sum_{i_1=1}^{N}\cfrac{1}{b_{i(1)}+\sum_{i_2=1}^{i_1}\cfrac{1}{b_{i(2)}+\sum_{i_3=1}^{i_2}\cfrac{1}{b_{i(3)}+\ddots}}}, \tag{3}$$

where $b_0, b_{i(k)} \in \mathbb{C},\ \ i(k)\in\mathcal{I}$,

$$\mathcal{I} = \{i(k)=(i_1,i_2,\ldots,i_k)\ :1\le i_k\le i_{k-1}\le \ldots \le i_0;\ k\ge 1;\ i_0=N\},$$

N is a fixed natural number that determines the dimension of (3).

The nth approximant of (3) is the expression

$$f_n = b_0 + \mathop{\mathrm{D}}_{k=1}^{n}\sum_{i_k=1}^{i_{k-1}}\frac{1}{b_{i(k)}}.$$

The BCF (3) converges if there exists the finite limit $\lim\limits_{n\to\infty} f_n = f$. The number f is called the value of (3).

Tails of nth approximant of (3) are following finite BCF

$$Q_{i(n)}^{(n)} = b_{i(n)},\ Q_{i(p)}^{(n)} = b_{i(p)} + \mathop{\mathrm{D}}_{r=p+1}^{n}\sum_{i_r=1}^{i_{r-1}}\frac{1}{b_{i(r)}}, \tag{4}$$

where $n = 1, 2, \ldots$; $p = \overline{1, n-1}$; $i(n) \in \mathcal{I}$; $i(p) \in \mathcal{I}$.

The convergence problem and researching truncation error bounds of BCF of the special form in different domains are considered in a large number of papers, in particular, by D. Bodnar [6, 9], O. Baran [3, 4], T. Antonova [1, 2], R. Dmytryshyn [11, 12], etc.

Henceforward we will use the following formulas that gives a possibility to estimate absolute errors of calculating a finite continued fraction.

Let

$$f_n = b_0 + \mathop{\mathrm{D}}_{k=1}^{n} \frac{1}{b_k}, \ \widetilde{f}_n = \widetilde{b}_0 + \mathop{\mathrm{D}}_{k=1}^{n} \frac{1}{\widetilde{b}_k}, \ b_k, \widetilde{b}_k \in \mathbb{C}, \ k = 0, 1, \ldots, n; \ n \geq 1.$$

Then ([7, 15])

$$f_n - \widetilde{f}_n = b_0 - \widetilde{b}_0 + (-1)^n \sum_{k=1}^{n} \frac{b_k - \widetilde{b}_k}{\prod\limits_{r=1}^{k} Q_r^{(n)} \widetilde{Q}_r^{(n)}}, \tag{5}$$

where

$$Q_r^{(n)} = b_r + \mathop{\mathrm{D}}_{m=r+1}^{n} \frac{1}{b_m}, \ \widetilde{Q}_r^{(n)} = \widetilde{b}_r + \mathop{\mathrm{D}}_{m=r+1}^{n} \frac{1}{\widetilde{b}_m}$$

are the tails of the corresponding finite continued fractions.

Main result. We consider the subsets $\mathcal{I}^{(m)}$ of the set of multiindices $\mathcal{I}$,

$$\mathcal{I}^{(m)} = \{ i(n) = (i_1, i_2, \ldots, i_n) \ : m \leq i_n \leq i_{n-1} \leq \ldots \leq i_0; \ n \geq 1; \ i_0 = N \}, \ m = \overline{2, N},$$

and we set

$$s[k] = (\underbrace{s, s, \ldots, s}_{k}), \ 1 \leq s \leq N, \ k = 1, 2, \ldots.$$

The main result of the paper is the following theorem that is an analog of Theorem 1 for BCF of the special form.

Theorem 2. *Let there exist constants θ, δ, β such that $0 < \theta < \frac{\pi}{4}$, $0 < \delta < 1$, $0 \leq \beta \leq \frac{1}{2}$ and the elements of the BCF of the special form* (3) *satisfy the conditions*

$$|\arg b_0| \leq \theta, \ \left|\arg b_{i(k)}\right| \leq \theta, \ i(k) \in \mathcal{I}, \tag{6}$$

$$\operatorname{Re}(b_0) \geq \delta, \ \operatorname{Re}(b_{i(n)}) \geq \delta, \ \operatorname{Re}(b_{1[s]}) \geq \frac{\delta}{s^{\beta}}, \ \operatorname{Re}(b_{i(n)1[s]}) \geq \frac{\delta}{s^{\beta}}, \ s \geq 1, \ i(n) \in \mathcal{I}^{(2)}. \tag{7}$$

Then BCF (3) *converges and the following truncation error bound holds*

$$|f_m - f_{Nn}| < \frac{M_N}{\ln\left(1 + \frac{\alpha}{1-\beta}\left((n+1)^{1-\beta} - 1\right)\right)}, \ m \geq Nn, \ n \in \mathbb{N}, \tag{8}$$

where M_N and α are positive numbers that do not depend on n and m,

$$M_N = \frac{2+\delta}{\delta \cos\theta} K_N, \ K_1 = 1, \ K_j = 4AK_{j-1} + 2, \ 2 \leq j \leq N,$$

$$A = \left(1 + \frac{1}{\delta^2}\right)\left(1 + \frac{1}{\delta^4 \cos^2 2\theta + 2\delta^2 \cos 2\theta}\right). \tag{9}$$

Proof. The analogue of the van Vleck Theorem for BCF of the special form [8] proves that the BCF (3) converges.

We shall prove estimate (8) by mathematical induction in N, the dimension of the BCF of the special form (3).

If $N = 1$ the BCF (3) becomes the continued fraction

$$b_0 + \frac{1}{b_1} + \frac{1}{b_{11}} + \frac{1}{b_{111}} + \dots . \tag{10}$$

Let f_k be its kth approximant, $k \geq 0$. From condition (6) it follows that the assumptions of Theorem 1 for the continued fraction (10) hold. In consequence, we obtain the estimate

$$|f_m - f_n| \leq \frac{1}{d_{n+1}} < \frac{1}{\mu_n}, \; m \geq n,$$

where

$$\mu_n = \frac{\operatorname{Re} b_1}{2 + \operatorname{Re} b_1} \cos\theta \ln\left(1 + (\operatorname{Re} b_1)^2 \min\left\{1, \frac{1}{|b_1|^2}\right\} \cos\theta \sum_{k=1}^{n} \left|b_{1[k]}\right|\right), \; n \geq 1.$$

It is obvious that

$$\frac{\operatorname{Re} b_1}{2 + \operatorname{Re} b_1} \geq \frac{\delta}{2 + \delta}.$$

Considering the conditions (6), (7) of Theorem 2, we get

$$(\operatorname{Re} b_1)^2 \min\left\{1, \frac{1}{|b_1|^2}\right\} = \min\left\{(\operatorname{Re} b_1)^2, \cos^2(\arg b_1)\right\} \geq \min\left\{\delta^2, \cos^2\theta\right\}$$

and

$$\sum_{k=1}^{n} \left|b_{1[k]}\right| \geq \sum_{k=1}^{n} \frac{\delta}{k^\beta} \geq \delta \int_1^{n+1} \frac{dx}{x^\beta} = \delta \frac{(n+1)^{1-\beta} - 1}{1 - \beta},$$

because the function $y = \dfrac{1}{x^\beta}$ monotonically decreases in $x, \; x > 0$.

Thus,

$$|f_m - f_n| < \frac{2+\delta}{\delta \cos\theta} \frac{1}{\ln\left(1 + \frac{\alpha}{1-\beta}\left((n+1)^{1-\beta} - 1\right)\right)}, \; m \geq n,$$

where $\alpha = \min\{\delta^3 \cos\theta, \delta \cos^3\theta\}$, that is, the inequality (8) is true when $N = 1$ where $M_1 = \frac{2+\delta}{\delta \cos\theta}$.

Suppose that the estimate (8) holds for $N = r - 1$. Let us now prove it for $N = r$. Consider the r-dimensional BCF of the special form (3) where $i_0 = r$. Write down its nth approximant in the form

$$f_n = b_0^{(r-1,n)} + \mathop{\mathrm{D}}_{k=1}^{n} \frac{1}{b_{r[k]}^{(r-1,n-k)}}, \; n \geq 1,$$

where

$$b_0^{(r-1,n)} = b_0 + \mathop{\mathrm{D}}_{l=1}^{n} \sum_{i_l=1}^{i_{l-1}} \frac{1}{b_{i(l)}}, \; b_{r[n]}^{(r-1,0)} = b_{r[n]}, \; b_{r[k]}^{(r-1,n-k)} = b_{r[k]} + \mathop{\mathrm{D}}_{l=1}^{n-k} \sum_{i_l=1}^{i_{l-1}} \frac{1}{b_{r[k]i(l)}}, \tag{11}$$

$k=1,2,\dots,n-1,\ i_0=r-1$, are the approximants of respective $(r-1)$-dimensional BCF that are contained in the structure of BCF (3).

Let

$$\widehat{f}_n = b_0^{(r-1)} + \mathop{\mathrm{D}}_{k=1}^{n} \frac{1}{b_{r[k]}^{(r-1)}}$$

are the nth approximant of continued fraction that is constructed of all calculated $(r-1)$-dimensional BCF in the BCF (3), that is,

$$b_0^{(r-1)} = b_0 + \mathop{\mathrm{D}}_{l=1}^{\infty} \sum_{i_l=1}^{i_{l-1}} \frac{1}{b_{i(l)}},\ b_{r[k]}^{(r-1)} = b_{r[k]} + \mathop{\mathrm{D}}_{l=1}^{\infty} \sum_{i_l=1}^{i_{l-1}} \frac{1}{b_{r[k]i(l)}}, \tag{12}$$

$$i_0 = r-1,\ k=1,2,\dots,n.$$

These BCF converge in accordance with the analogue of the van Vleck Convergence Theorem for BCF of the special form [8].

Then

$$|f_m - f_{rn}| \le \left|f_m - \widehat{f}_n\right| + \left|f_{rn} - \widehat{f}_n\right|,\ m \ge rn.$$

Let us now estimate the values $|f_p - \widehat{f}_n|,\ p \ge rn$.

Consider the next finite continued fraction

$$h_{p,n} = b_0^{(r-1)} + \frac{1}{b_{r[1]}^{(r-1)}} + \dots + \frac{1}{b_{r[n]}^{(r-1)}} + \frac{1}{b_{r[n+1]}^{(r-1,p-n-1)}} + \dots + \frac{1}{b_{r[p]}^{(r-1,0)}},$$

with elements that are defined in accordance with (11), (12). Obviously,

$$\left|f_p - \widehat{f}_n\right| \le |f_p - h_{p,n}| + \left|h_{p,n} - \widehat{f}_n\right|. \tag{13}$$

Let us now obtain estimations from above for each addend in the right-hand side of inequality (13). From the formula (5) it follows that

$$|f_p - h_{p,n}| \le \left|b_0^{(r-1)} - b_0^{(r-1,p)}\right| + \sum_{k=1}^{n} \frac{\left|b_{r[k]}^{(r-1)} - b_{r[k]}^{(r-1,p-k)}\right|}{\prod_{s=1}^{k} \left|\widetilde{Q}_{r[s]}^{(p)} Q_{r[s]}^{(p)}\right|}, \tag{14}$$

where $\widetilde{Q}_{r[s]}^{(p)}$, $Q_{r[s]}^{(p)}$ are sth tails of the continued fractions $h_{p,n}$ and f_p, respectively, $s=\overline{1,k}$.

Consider the products in the denominators of the previous relation. If $k=2l$, then

$$\prod_{s=1}^{2l} \left|\widetilde{Q}_{r[s]}^{(p)} Q_{r[s]}^{(p)}\right| = \prod_{s=1}^{l} \left(\left|\widetilde{Q}_{r[2s-1]}^{(p)} \widetilde{Q}_{r[2s]}^{(p)}\right| \left|Q_{r[2s-1]}^{(p)} Q_{r[2s]}^{(p)}\right| \right).$$

Taking into account

$$\mathrm{Re}(b_{r[k]}^{(r-1)}) = \mathrm{Re}(b_{r[k]}) + \mathrm{Re}\left(\mathop{\mathrm{D}}_{l=1}^{\infty} \sum_{i_l=1}^{i_{l-1}} \frac{1}{b_{r[k]i(l)}} \right) > \mathrm{Re}(b_{r[k]}) \ge \delta,\ k=\overline{1,n},$$

we estimate separately each factor of product

$$\left|\widetilde{Q}^{(p)}_{r[2s-1]}\widetilde{Q}^{(p)}_{r[2s]}\right| = \left|\widetilde{Q}^{(p)}_{r[2s]}\left(b^{(r-1)}_{r[2s-1]}+\frac{1}{\widetilde{Q}^{(p)}_{r[2s]}}\right)\right| \geq \operatorname{Re}\left(b^{(r-1)}_{r[2s-1]}\widetilde{Q}^{(p)}_{r[2s]}+1\right) =$$

$$= \operatorname{Re}\left(b^{(r-1)}_{r[2s-1]}\left(b^{(r-1)}_{r[2s]}+\frac{1}{\widetilde{Q}^{(p)}_{r[2s+1]}}\right)\right)+1 > \operatorname{Re}\left(b_{r[2s-1]}b_{r[2s]}\right)+1 \geq \delta^2\cos 2\theta+1.$$

For factors $\left|Q^{(p)}_{r[2s-1]}Q^{(p)}_{r[2s]}\right|$ the analogous estimations are valid.

Therefore,

$$\prod_{s=1}^{2l}\left|\widetilde{Q}^{(p)}_{r[s]}Q^{(p)}_{r[s]}\right| > \left(\delta^2\cos 2\theta+1\right)^{2l}.$$

If $k=2l+1$, then

$$\prod_{s=1}^{2l+1}\left|\widetilde{Q}^{(p)}_{r[s]}Q^{(p)}_{r[s]}\right| = \left|\widetilde{Q}^{(p)}_{r[1]}Q^{(p)}_{r[1]}\right|\prod_{s=1}^{l}\left(\left|\widetilde{Q}^{(p)}_{r[2s]}\widetilde{Q}^{(p)}_{r[2s+1]}\right|\left|Q^{(p)}_{r[2s]}Q^{(p)}_{r[2s+1]}\right|\right) >$$

$$> \left(\operatorname{Re}\left(b_{r[1]}\right)\right)^2\prod_{s=1}^{l}\left(\operatorname{Re}\left(b_{r[2s]}b_{r[2s+1]}\right)+1\right)^2 \geq \delta^2\left(\delta^2\cos 2\theta+1\right)^{2l}.$$

Hence,

$$\prod_{s=1}^{k}\left|\widetilde{Q}^{(p)}_{r[s]}Q^{(p)}_{r[s]}\right| > \delta^{1-(-1)^k}\left(\delta^2\cos 2\theta+1\right)^{2\left[\frac{k}{2}\right]},\ 1\leq k\leq n.$$

We shall estimate the values of numerators

$$\left|b^{(r-1)}_{r[k]}-b^{(r-1,p-k)}_{r[k]}\right|,$$

$1\leq k\leq n$, of the inequality (14) as the difference between the value of an $(r-1)$-dimensional BCF of the special form and its $(p-k)$th approximant, $p-k\geq(r-1)n$. By the inductive hypothesis, taking into account Theorem 1, we get

$$\left|b^{(r-1)}_{r[k]}-b^{(r-1,p-k)}_{r[k]}\right| < \frac{2+\delta}{\delta\cos\theta}\frac{2K_{r-1}}{\ln\left(1+\frac{\alpha}{1-\beta}\left((n+1)^{1-\beta}-1\right)\right)},\ k=1,2,\ldots,n,\ p\geq rn.$$

The analogous estimation is also true for the first addend of the right-hand side of the inequality (14).

In these terms,

$$|f_p-h_{p,n}| < \frac{2+\delta}{\delta\cos\theta}\frac{2K_{r-1}}{\ln\left(1+\frac{\alpha}{1-\beta}\left((n+1)^{1-\beta}-1\right)\right)}$$

$$\left(1+\frac{1}{\delta^2}+\frac{1}{\left(\delta^2\cos 2\theta+1\right)^2}+\frac{1}{\delta^2\left(\delta^2\cos 2\theta+1\right)^2}+\cdots+\frac{1}{\delta^{1-(-1)^n}\left(\delta^2\cos 2\theta+1\right)^{2\left[\frac{n}{2}\right]}}\right) <$$

$$< \frac{2+\delta}{\delta\cos\theta}\frac{2K_{r-1}}{\ln\left(1+\dfrac{\alpha}{1-\beta}\left((n+1)^{1-\beta}-1\right)\right)}\left(1+\frac{1}{\delta^2}\right)\sum_{k=0}^{\infty}\left(\delta^2\cos 2\theta+1\right)^{-2k} =$$

$$= \frac{2+\delta}{\delta\cos\theta}\frac{2K_{r-1}A}{\ln\left(1+\dfrac{\alpha}{1-\beta}\left((n+1)^{1-\beta}-1\right)\right)},\ p\geq rn,$$

where A is defined according to the formula (9).

Let us estimate the second addend in the right-hand side inequality (13) $|h_{p,n}-\widehat{f}_n|$, which is a difference of approximants of some continued fractions with elements from the domain (1). Similarly with (2) we obtain

$$\left|h_{p,n}-\widehat{f}_n\right| < \frac{1}{\nu_n},\ n\geq 1,$$

where

$$\nu_n = \frac{\operatorname{Re}\left(b_r^{(r-1)}\right)}{2+\operatorname{Re}\left(b_r^{(r-1)}\right)}\cos\theta\ln\left(1+\left(\operatorname{Re}\left(b_r^{(r-1)}\right)\right)^2\min\left\{1,\frac{1}{|b_r^{(r-1)}|^2}\right\}\cos\theta\sum_{s=1}^{n}|b_{r[s]}^{(r-1)}|\right).$$

Since $\operatorname{Re} b_{r[k]}^{(r-1)} > \delta,\ k=1,2,\ldots,$ taking into account the conditions of Theorem 2, we have

$$\nu_n > \frac{\delta\cos\theta}{2+\delta}\ln\left(1+\alpha n\right),\ n\geq 1.$$

Then

$$\left|h_{p,n}-\widehat{f}_n\right| < \frac{2+\delta}{\delta\cos\theta}\frac{1}{\ln\left(1+\alpha n\right)}.$$

Using the previous estimates of the addends in the right-hand side the equation (13), we obtain

$$\left|f_p-\widehat{f}_n\right| < \frac{2+\delta}{\delta\cos\theta}\frac{2K_{r-1}A+1}{\ln\left(1+\dfrac{\alpha}{1-\beta}\left((n+1)^{1-\beta}-1\right)\right)},\ p\geq rn,$$

because

$$n > \frac{1}{1-\beta}\left((n+1)^{1-\beta}-1\right).$$

For $m\geq rn$, putting in the last inequality $p=m$, $p=rn$, we get

$$|f_m-f_{rn}| \leq \left|f_m-\widehat{f}_n\right| + \left|f_{rn}-\widehat{f}_n\right| <$$

$$< \frac{2+\delta}{\delta\cos\theta}\frac{2\left(2K_{r-1}A+1\right)}{\ln\left(1+\dfrac{\alpha}{1-\beta}\left((n+1)^{1-\beta}-1\right)\right)} = \frac{M_r}{\ln\left(1+\dfrac{\alpha}{1-\beta}\left((n+1)^{1-\beta}-1\right)\right)},$$

where

$$M_r = \frac{2+\delta}{\delta\cos\theta}K_r,\ K_r = 2\left(2K_{r-1}A+1\right).$$

Consequently, for approximants f_k of N-dimensional BCF (3), where N is an arbitrary fixed natural number, the following estimation is true

$$|f_m - f_{Nn}| < \frac{M_N}{\ln\left(1 + \frac{\alpha}{1-\beta}\left((n+1)^{1-\beta} - 1\right)\right)}, \ m \geq Nn,$$

where

$$M_N = \frac{2+\delta}{\delta \cos\theta} K_N, \ N \geq 1, \quad K_1 = 1, \ K_j = 4AK_{j-1} + 2, \ 2 \leq j \leq N.$$

□

Conclusion. The obtained truncation error bound for some BCF of a special form in the domain that is narrower than its convergence domain. Thereby, the problem of establishing estimations in the whole convergence domain is still unsolved. The results of Theorem 2 can be used for establishing truncation error bounds for functional BCF with independent variables, in particular, multidimensional J-fractions with independent variables.

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Received 27.09.2019