УДК 517.98

T. V. Vasylyshyn

THE ALGEBRA OF SYMMETRIC POLYNOMIALS ON $(L_\infty)^n$

T. V. Vasylyshyn. *The algebra of symmetric polynomials on* $(L_\infty)^n$, Mat. Stud. **52** (2019), 71–85.

The paper deals with continuous symmetric (invariant under composition of the variable with any measure preserving bijection of $[0,1]$) complex-valued polynomials on the nth Cartesian power of the complex Banach space L_∞ of all Lebesgue measurable essentially bounded complex-valued functions on $[0,1]$. We construct an algebraic basis of the algebra of all such polynomials. Results of the paper can be used for investigations of algebras of symmetric continuous polynomials and of symmetric analytic functions on the nth Cartesian power of L_∞.

1. Introduction. Let X be a Banach space, which has a symmetric structure, like has a symmetric basis or is rearrangement invariant. It is natural to consider polynomials and analytic functions on X, which are invariant (symmetric) with respect to a group of operators $G(X)$ acting on X, which preserve this structure. In particular, if X is a rearrangement invariant Banach space of Lebesgue measurable functions on some Lebesgue measurable set $\Omega \subset \mathbb{R}$ of nonzero measure, then $G(X)$ used to be the group of operators $B_\sigma \colon X \ni x \mapsto x \circ \sigma \in X$, where σ is a bijection of Ω, which preserves the measure.

Firstly symmetric polynomials on real Banach spaces of Lebesgue measurable integrable in a power p functions on $[0,1]$ and $[0,+\infty)$, where $1 \le p < +\infty$, were studied by Nemirovskii and Semenov in [7]. Some of their results were generalized to real separable rearrangement invariant Banach spaces of Lebesgue measurable functions on $[0,1]$ and $[0,+\infty)$ by González, Gonzalo and Jaramillo in [3]. In particular, there were constructed algebraic bases (see definition below) of algebras of continuous symmetric real-valued polynomials on such spaces.

Symmetric polynomials and symmetric analytic functions on some non-separable Banach spaces were studied in [1, 2]. In particular, in [1] it was constructed an algebraic basis of the algebra of continuous symmetric polynomials on the complex Banach space L_∞ of all Lebesgue measurable essentially bounded complex-valued functions on $[0,1]$. In [2] it was shown that the trivial polynomial is the unique continuous symmetric polynomial on the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions on $[0,+\infty)$. Symmetric polynomials on Cartesian products of some Banach spaces were studied in [4, 5, 8–11]. In particular, in [9] and [10] there were constructed algebraic bases of algebras of continuous symmetric polynomials on Cartesian powers of complex Banach spaces of Lebesgue measurable integrable in a power p, where $1 \le p < +\infty$, complex-valued functions on $[0,1]$ and $[0,+\infty)$ respectively. In [8] and [11] Hamel bases of vector spaces

2010 *Mathematics Subject Classification:* 46G25, 46E15, 46E25, 46G20.
Keywords: polynomial; symmetric polynomial; algebraic combination; algebraic basis.
doi:10.30970/ms.52.1.71-85

of, respectively, 2- and 3-homogeneous continuous symmetric polynomials on the Cartesian square of L_∞ were constructed.

In this work, we construct an algebraic basis of the algebra of all continuous symmetric polynomials on the nth Cartesian power of L_∞. This result generalizes some results of [1,8,11] and can be used for investigations of algebras of continuous symmetric polynomials and of symmetric analytic functions on the nth Cartesian power of L_∞.

2. Preliminaries. We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{Z}_+$ the set of all nonnegative integers.

A mapping $P\colon X \to \mathbb{C}$, where X is a Banach space with norm $\|\cdot\|_X$, is called an *N-homogeneous polynomial*, where $N \in \mathbb{N}$, if there exists an N-linear symmetric mapping $A_P\colon X^N \to \mathbb{C}$ such that

$$P(x) = A_P(\underbrace{x, \ldots, x}_{N})$$

for every $x \in X$. Here "symmetric" means that $A_P(x_{\tau(1)}, \ldots, x_{\tau(N)}) = A_P(x_1, \ldots, x_N)$ for every permutation $\tau\colon \{1, \ldots, N\} \to \{1, \ldots, N\}$. The mapping A_P is called the N-linear symmetric mapping associated with P.

It is known (see, e.g., [6, Theorem 1.10]) that A_P can be recovered from P by means of the so-called Polarization Formula:

$$A_P(x_1, \ldots, x_N) = \frac{1}{N!2^N} \sum_{\varepsilon_1, \ldots, \varepsilon_N = \pm 1} \varepsilon_1 \ldots \varepsilon_N P(\varepsilon_1 x_1 + \ldots + \varepsilon_N x_N). \tag{1}$$

We shall use the Polynomial Formula (see [6, Theorem 1.8])

$$P(x_1 + \ldots + x_k) = \sum_{N_1 + \ldots + N_k = N} \frac{N!}{N_1! \ldots N_k!} A_P(\underbrace{x_1, \ldots, x_1}_{N_1}, \underbrace{x_2, \ldots, x_2}_{N_2}, \ldots, \underbrace{x_k, \ldots, x_k}_{N_k}), \tag{2}$$

where $N_1, \ldots, N_k \in \mathbb{Z}_+$, and its corollary, the Binomial Formula (see [6, Corollary 1.9])

$$P(x + y) = \sum_{m=0}^{N} \binom{N}{m} A_P(\underbrace{x, \ldots, x}_{m}, \underbrace{y, \ldots, y}_{N-m}). \tag{3}$$

It is known that an N-homogeneous polynomial $P\colon X \to \mathbb{C}$ is continuous if and only if

$$\|P\| = \sup_{\|x\|_X \le 1} |P(x)| < +\infty.$$

Consequently, if P is a continuous N-homogeneous polynomial, then

$$|P(x)| \le \|P\| \|x\|_X^N \tag{4}$$

for every $x \in X$.

A mapping $P = P_0 + P_1 + \ldots + P_N$, where $P_0 \in \mathbb{C}$ and P_j is a j-homogeneous polynomial for every $j \in \{1, \ldots, N\}$, is called a *polynomial* of degree at most N.

A mapping $f\colon X \to \mathbb{C}$ is called an *algebraic combination* of mappings $f_1, \ldots, f_k\colon X \to \mathbb{C}$ if there exists a polynomial $Q\colon \mathbb{C}^k \to \mathbb{C}$ such that

$$f(x) = Q(f_1(x), \ldots, f_k(x))$$

for every $x \in X$. Mappings $f_1, \ldots, f_k \colon X \to \mathbb{C}$ are called *algebraically independent* if $Q(f_1(x), \ldots, f_k(x)) = 0$ for every $x \in X$ if and only if the polynomial Q is identically equal to zero. If mappings $f_1, \ldots, f_k$ are algebraically independent and polynomials $Q_1, Q_2 \colon \mathbb{C}^k \to \mathbb{C}$ are such that

$$Q_1(f_1(x), \ldots, f_k(x)) = Q_2(f_1(x), \ldots, f_k(x))$$

for every $x \in X$, then the polynomial Q_1 is identically equal to the polynomial Q_2. Thus, every algebraic combination of algebraically independent mappings is unique. A set of mappings $\mathcal{B}$ is called an *algebraic basis* of some algebra of mappings $\mathcal{A}$, if every element of $\mathcal{A}$ can be uniquely represented as an algebraic combination of some elements of $\mathcal{B}$.

Let L_∞ be the complex Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0,1]$ with the norm

$$\|x\|_\infty = \operatorname{ess\,sup}_{t \in [0,1]} |x(t)|.$$

Let $n \in \mathbb{N}$. Let $(L_\infty)^n$ be the nth Cartesian power of L_∞ with the norm

$$\|y\|_{\infty,n} = \max_{1 \le s \le n} \|y_s\|_\infty,$$

where $y = (y_1, \ldots, y_n) \in (L_\infty)^n$.

Let Ξ be the set of all bijections $\sigma \colon [0,1] \to [0,1]$ such that both σ and σ^{-1} are measurable and preserve the Lebesgue measure. A function $f \colon (L_\infty)^n \to \mathbb{C}$ is called *symmetric* if

$$f(y \circ \sigma) = f(y)$$

for every $y = (y_1, \ldots, y_n) \in (L_\infty)^n$ and for every $\sigma \in \Xi$, where $y \circ \sigma = (y_1 \circ \sigma, \ldots, y_n \circ \sigma)$.

Formula (1) implies the following corollary.

Corollary 1. *Let $P \colon (L_\infty)^n \to \mathbb{C}$ be a symmetric N-homogeneous polynomial. Then*

$$A_P(z_1 \circ \sigma, \ldots, z_N \circ \sigma) = A_P(z_1, \ldots, z_N)$$

for every $z_1, \ldots, z_N \in (L_\infty)^n$ and $\sigma \in \Xi$.

For every multi-index $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ such that $|k| \ge 1$, where $|k| = k_1 + \ldots + k_n$, let us define a mapping $R_k \colon (L_\infty)^n \to \mathbb{C}$ by

$$R_k(y) = \int_{[0,1]} \prod_{\substack{s=1 \\ k_s > 0}}^{n} (y_s(t))^{k_s}\, dt, \tag{5}$$

where $y = (y_1, \ldots, y_n) \in (L_\infty)^n$. Note that R_k is a continuous symmetric $|k|$-homogeneous polynomial and $\|R_k\| = 1$.

For every $E \subset [0,1]$, let

$$1_E(t) = \begin{cases} 1, & \text{if } t \in E \\ 0, & \text{otherwise.} \end{cases}$$

We shall use the following result, proven in [1].

Lemma 1 ([1], Lemma 4.5). *Let $P\colon L_\infty \to \mathbb{C}$ be a continuous symmetric N-homogeneous polynomial, where $N \geq 2$. Let $1 \leq m < N$ and $[a,b], [c_1,d_1], [c_2,d_2] \subset [0,1]$ be such that $\mu\big([a,b]\cap[c_1,d_1]\big) = 0$, $\mu\big([a,b]\cap[c_2,d_2]\big) = 0$, $\mu\big([c_1,d_1]\cap[c_2,d_2]\big) = 0$ and $\mu\big([c_1,d_1]\cup[c_2,d_2]\big) \leq \mu\big([a,b]\big)$. Let $y_1,\ldots,y_{N-m} \in L_\infty$ be such that the restrictions of $y_1,\ldots,y_{N-m}$ to $[a,b]\cup[c_1,d_1]\cup[c_2,d_2]$ are constant. Then there exists a constant $C(m,a,b) > 0$ such that*

$$\Big|A_P\big(\underbrace{1_{[c_1,d_1]\cup[c_2,d_2]},\ldots,1_{[c_1,d_1]\cup[c_2,d_2]}}_{m}, y_1,\ldots,y_{N-m}\big)\Big| \leq$$
$$\leq \mu\big([c_1,d_1]\cup[c_2,d_2]\big)\|y_1\|_\infty \ldots \|y_{N-m}\|_\infty C(m,a,b),$$

where A_P is the N-linear symmetric mapping associated with P.

For $m \in \mathbb{N}$, let $c_{00}^{(m)}(\mathbb{C}^n)$ be the vector space of all sequences $x = (x_1,\ldots,x_m,\bar{0},\ldots)$, where $x_j = \big(x_j^{(1)},\ldots,x_j^{(n)}\big) \in \mathbb{C}^n$ for $j \in \{1,\ldots,m\}$ and $\bar{0} = (0,\ldots,0) \in \mathbb{C}^n$. Note that $c_{00}^{(m)}(\mathbb{C}^n)$ is isomorphic to $(\mathbb{C}^n)^m$. A function $f\colon c_{00}^{(m)}(\mathbb{C}^n) \to \mathbb{C}$ is called symmetric if

$$f((x_{\tau(1)},\ldots,x_{\tau(m)},\bar{0},\ldots)) = f((x_1,\ldots,x_m,\bar{0},\ldots))$$

for every $(x_1,\ldots,x_m,\bar{0},\ldots) \in c_{00}^{(m)}(\mathbb{C}^n)$ and for every bijection $\tau\colon \{1,\ldots,m\} \to \{1,\ldots,m\}$. For every $k = (k_1,\ldots,k_n) \in \mathbb{Z}_+^n$ such that $|k| \geq 1$, let us define the mapping $H_k^{(m)}\colon c_{00}^{(m)}(\mathbb{C}^n) \to \mathbb{C}$ by

$$H_k^{(m)}(x) = \sum_{j=1}^{m} \prod_{\substack{s=1\\ k_s>0}}^{n} \big(x_j^{(s)}\big)^{k_s}.$$

We shall use following results, proven in [5].

Corollary 2 ([5], Corollary 7). *Let $M = \{k^{(1)},\ldots,k^{(s)}\} \subset \mathbb{Z}_+^n$ be such that $|k^{(j)}| \geq 1$ for every $j \in \{1,\ldots,s\}$. Then there exists $m' \in \mathbb{N}$ such that for every $m \geq m'$ polynomials $H_{k^{(1)}}^{(m)},\ldots,H_{k^{(s)}}^{(m)}$ are algebraically independent of $c_{00}^{(m)}(\mathbb{C}^n)$.*

Theorem 1 ([5], Theorem 8). *Every symmetric N-homogeneous polynomial $P\colon c_{00}^{(m)}(\mathbb{C}^n) \to \mathbb{C}$, where m is an arbitrary positive integer, can be represented as an algebraic combination of polynomials $H_k^{(m)}$, where $k \in \mathbb{Z}_+^n$ such that $1 \leq |k| \leq N$.*

3. The main result. For $m \in \mathbb{N}$, let $J_m\colon c_{00}^{(m)}(\mathbb{C}^n) \to (L_\infty)^n$ be defined by

$$J_m(x) = \left(\sum_{j=1}^{m} x_j^{(1)} 1_{\left[\frac{j-1}{m},\frac{j}{m}\right]},\ldots,\sum_{j=1}^{m} x_j^{(n)} 1_{\left[\frac{j-1}{m},\frac{j}{m}\right]}\right)$$

for $x = (x_1,\ldots,x_m,\bar{0},\ldots) \in c_{00}^{(m)}(\mathbb{C}^n)$. Note that J_m is a linear injective operator. It can be checked that

$$(R_k \circ J_m)(x) = \frac{1}{m} H_k^{(m)}(x) \tag{6}$$

for every $m \in \mathbb{N}$ and $x \in c_{00}^{(m)}(\mathbb{C}^n)$.

For $r \in \mathbb{N}$, let

$$D_r = J_{2^r}(c_{00}^{(2^r)}(\mathbb{C}^n)).$$

Let

$$D = \bigcup_{r=1}^{\infty} D_r. \tag{7}$$

For every nonempty finite set $M \subset \mathbb{Z}_+^n$ and for every mapping $l\colon M \to \mathbb{Z}_+$, let

$$\varkappa(l, M) = \sum_{k\in M} |k| l(k). \tag{8}$$

For $N \in \mathbb{N}$, let

$$M_N = \big\{k \in \mathbb{Z}_+^n\colon\ 1 \le |k| \le N\big\}. \tag{9}$$

Proposition 1. *Let $M = \{k^{(1)}, \ldots, k^{(s)}\} \subset \mathbb{Z}_+^n$ be such that $|k^{(j)}| \ge 1$ for every $j \in \{1, \ldots, s\}$. Then polynomials $R_{k^{(1)}}, \ldots, R_{k^{(s)}}$ are algebraically independent of $(L_\infty)^n$.*

Proof. Let a polynomial $Q\colon \mathbb{C}^s \to \mathbb{C}$,

$$Q(z_1, \ldots, z_s) = \sum_{(l_1,\ldots,l_s)\in\Omega} \gamma_{(l_1,\ldots,l_s)} z_1^{l_1} \ldots z_s^{l_s},$$

where Ω is a finite subset of $\mathbb{Z}_+^s$ and $\gamma_{(l_1,\ldots,l_s)} \in \mathbb{C}$, be such that

$$Q\big(R_{k^{(1)}}(y), \ldots, R_{k^{(s)}}(y)\big) = 0 \tag{10}$$

for every $y \in (L_\infty)^n$. Let us show that Q is identically equal to zero. By Corollary 2, there exists $m \in \mathbb{N}$ such that polynomials $H_{k^{(1)}}^{(m)}, \ldots, H_{k^{(s)}}^{(m)}$ are algebraically independent of $c_{00}^{(m)}(\mathbb{C}^n)$. By (6) and (10),

$$Q\Big(\frac{1}{m} H_{k^{(1)}}^{(m)}(x), \ldots, \frac{1}{m} H_{k^{(s)}}^{(m)}(x)\Big) = 0$$

for every $x \in c_{00}^{(m)}(\mathbb{C}^n)$, i.e.,

$$\tilde{Q}\Big(H_{k^{(1)}}^{(m)}(x), \ldots, H_{k^{(s)}}^{(m)}(x)\Big) = 0$$

for every $x \in c_{00}^{(m)}(\mathbb{C}^n)$, where

$$\tilde{Q}(z_1, \ldots, z_s) = \sum_{(l_1,\ldots,l_s)\in\Omega} \gamma_{(l_1,\ldots,l_s)} \frac{1}{m^{l_1+\ldots+l_s}} z_1^{l_1} \ldots z_s^{l_s}.$$

Since polynomials $H_{k^{(1)}}^{(m)}, \ldots, H_{k^{(s)}}^{(m)}$ are algebraically independent, it follows that $\tilde{Q}$ is identically equal to zero. Therefore, $\gamma_{(l_1,\ldots,l_s)} \frac{1}{m^{l_1+\ldots+l_s}} = 0$ and, consequently, $\gamma_{(l_1,\ldots,l_s)} = 0$ for every $(l_1, \ldots, l_s) \in \Omega$. Thus, Q is identically equal to zero. This completes the proof of the proposition. □

Proposition 2. *Let $P\colon (L_\infty)^n \to \mathbb{C}$ be a symmetric N-homogeneous polynomial. Then there exist unique coefficients*

$$\big\{\alpha_l \in \mathbb{C}\colon \quad l\colon M_N \to \mathbb{Z}_+ \text{ such that } \varkappa(l, M_N) = N\big\}$$

such that

$$P(y) = \sum_{\substack{l\colon M_N\to\mathbb{Z}_+\\ \varkappa(l,M_N)=N}} \alpha_l \prod_{\substack{k\in M_N\\ l(k)>0}} (R_k(y))^{l(k)}$$

for every $y \in D$, where M_N is defined by (9), $\varkappa$ is defined by (8), and D is defined by (7).

Proof. Note that $P \circ J_m$ is a symmetric N-homogeneous polynomial on $c_{00}^{(m)}(\mathbb{C}^n)$ for every $m \in \mathbb{N}$. Therefore, by Theorem 1, $P \circ J_m$ can be represented as an algebraic combination of polynomials $\{H_k^{(m)}\colon\ k \in M_N\}$, i.e., there exist coefficients $\beta_l^{(m)} \in \mathbb{C}$ such that

$$(P \circ J_m)(x) = \sum_{\substack{l\colon M_N \to \mathbb{Z}_+ \\ \varkappa(l, M_N) = N}} \beta_l^{(m)} \prod_{\substack{k \in M_N \\ l(k) > 0}} \big(H_k^{(m)}(x)\big)^{l(k)} \tag{11}$$

for every $x \in c_{00}^{(m)}(\mathbb{C}^n)$. By Corollary 2, there exists $m' \in \mathbb{N}$ such that polynomials $\{H_k^{(m)}\colon k \in M_N\}$ are algebraically independent for every $m \geq m'$. Therefore, for $m \geq m'$, the representation (11) is unique.

By (6) and (11),

$$(P \circ J_m)(x) = \sum_{\substack{l\colon M_N \to \mathbb{Z}_+ \\ \varkappa(l, M_N) = N}} \alpha_l^{(m)} \prod_{\substack{k \in M_N \\ l(k) > 0}} \big((R_k \circ J_m)(x)\big)^{l(k)}$$

for every $x \in c_{00}^{(m)}(\mathbb{C}^n)$, where $\alpha_l^{(m)} = \beta_l^{(m)} m^{\sum_{k \in M_N} l(k)}$. Consequently,

$$P(y) = \sum_{\substack{l\colon M_N \to \mathbb{Z}_+ \\ \varkappa(l, M_N) = N}} \alpha_l^{(2^r)} \prod_{\substack{k \in M_N \\ l(k) > 0}} \big(R_k(y)\big)^{l(k)} \tag{12}$$

for every $r \in \mathbb{N}$ and $y \in D_r$. The representation (12) is unique, if $2^r \geq m'$.

Let $r' \in \mathbb{N}$ be such that $2^{r'} \geq m'$. Since $D_{r'} \subset D_{r'+1} \subset \ldots$, it follows that $\alpha_l^{(2^{r'})} = \alpha_l^{(2^{r'+1})} = \ldots$. Let $\alpha_l = \alpha_l^{(2^{r'})}$. Then

$$P(y) = \sum_{\substack{l\colon M_N \to \mathbb{Z}_+ \\ \varkappa(l, M_N) = N}} \alpha_l \prod_{\substack{k \in M_N \\ l(k) > 0}} \big(R_k(y)\big)^{l(k)} \tag{13}$$

for every $y \in D$, and the representation (13) is unique. □

Lemma 2. *The set of all functions of the form*

$$\Big(\sum_{j=1}^{\iota} h_j^{(1)} 1_{E_j}, \ldots, \sum_{j=1}^{\iota} h_j^{(n)} 1_{E_j}\Big), \tag{14}$$

where $\iota \in \mathbb{N}$, $\big(h_j^{(1)}, \ldots, h_j^{(n)}\big) \in \mathbb{C}^n$ *for every* $j \in \{1, \ldots, \iota\}$, *and* $E_1, \ldots, E_\iota$ *are disjoint Lebesgue measurable subsets of* $[0,1]$, *is dense in* $(L_\infty)^n$.

Proof. The set of all functions of the form

$$\sum_{m=1}^{\mu} d_m 1_{F_m},$$

where $\mu \in \mathbb{N}$, $d_m \in \mathbb{C}$ for every $m \in \{1, \ldots, \mu\}$, and sets $F_1, \ldots, F_\mu$ are disjoint Lebesgue measurable subsets of $[0,1]$, is dense in L_∞. Therefore, the set of all functions of the form

$$\Big(\sum_{m_1=1}^{\mu_1} d_{m_1}^{(1)} 1_{F_{m_1}^{(1)}}, \ldots, \sum_{m_n=1}^{\mu_n} d_{m_n}^{(n)} 1_{F_{m_n}^{(n)}}\Big), \tag{15}$$

where $\mu_1, \ldots, \mu_n \in \mathbb{N}$, $d_m^{(s)} \in \mathbb{C}$ for every $s \in \{1, \ldots, n\}$ and $m \in \{1, \ldots, \mu_s\}$, and sets $F_1^{(s)}, \ldots, F_{\mu_s}^{(s)}$ are disjoint Lebesgue measurable subsets of $[0,1]$ for every $s \in \{1, \ldots, n\}$, is dense in $(L_\infty)^n$.

Let $y \in (L_\infty)^n$ be the function of the form (15). Let

$$\Omega = \{1, \ldots, \mu_1\} \times \ldots \times \{1, \ldots, \mu_n\}.$$

Let β be a bijection between Ω and $\{1, \ldots, \iota\}$, where $\iota = |\Omega|$. For every $\omega = (\omega_1, \ldots, \omega_n) \in \Omega$, let

$$E_{\beta(\omega)} = \bigcap_{s=1}^{n} F_{\omega_s}^{(s)}.$$

Then $E_1, \ldots, E_\iota$ are disjoint Lebesgue measurable subsets of $[0,1]$. For every $s \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, \iota\}$, let us define the number $h_j^{(s)} \in \mathbb{C}$ in the following way. If the sets E_j and $\bigcup_{m=1}^{\mu_s} F_m^{(s)}$ are disjoint, then we set $h_j^{(s)} = 0$. Otherwise, there exists unique $m' \in \{1, \ldots, \mu_s\}$ such that $E_j \subset F_{m'}^{(s)}$. In this case we set $h_j^{(s)} = d_{m'}^{(s)}$. Then

$$y = \left(\sum_{j=1}^{\iota} h_j^{(1)} 1_{E_j}, \ldots, \sum_{j=1}^{\iota} h_j^{(n)} 1_{E_j} \right).$$

Thus, every function of the form (15) belongs to the set of all functions of the form (14). Consequently, the set of all functions of the form (14) is dense in $(L_\infty)^n$. This completes the proof of the lemma. □

For $h = (h_1, \ldots, h_n) \in \mathbb{C}^n$ and $A \subset [0,1]$, let

$$h * 1_A = (h_1 1_A, \ldots, h_n 1_A).$$

If A is Lebesgue measurable, then $h * 1_A \in (L_\infty)^n$ and

$$\|h * 1_A\|_{\infty,n} = \max_{1 \le j \le n} \|h_j 1_A\|_\infty = \max_{1 \le j \le n} |h_j|.$$

Lemma 3. *Let $P\colon (L_\infty)^n \to \mathbb{C}$ be a continuous symmetric N-homogeneous polynomial, where $N \ge 2$. Let $1 \le m < N$ and $[a,b], [c_1,d_1], [c_2,d_2] \subset [0,1]$ be such that $\mu\big([a,b] \cap [c_1,d_1]\big) = 0, \mu\big([a,b] \cap [c_2,d_2]\big) = 0, \mu\big([c_1,d_1] \cap [c_2,d_2]\big) = 0$ and $\mu\big([c_1,d_1] \cup [c_2,d_2]\big) \le \mu\big([a,b]\big)$. Let $y_1, \ldots, y_{N-m} \in (L_\infty)^n$ be such that the restrictions of $y_1, \ldots, y_{N-m}$ to $[a,b] \cup [c_1,d_1] \cup [c_2,d_2]$ are constant. Then there exists a constant $C(m,a,b) > 0$ such that*

$$\Big| A_P\big(\underbrace{h * 1_{[c_1,d_1] \cup [c_2,d_2]}, \ldots, h * 1_{[c_1,d_1] \cup [c_2,d_2]}}_{m}, y_1, \ldots, y_{N-m}\big) \Big| \le$$

$$\le n^N \Big(\max_{1 \le j \le n} \big| h^{(j)} \big| \Big)^m \mu\big([c_1,d_1] \cup [c_2,d_2]\big) \|y_1\|_{\infty,n} \ldots \|y_{N-m}\|_{\infty,n} C(m,a,b)$$

for every $h = \big(h^{(1)}, \ldots, h^{(n)}\big) \in \mathbb{C}^n$.

Proof. For $k \in \{1, \ldots, n\}$, let $v_k\colon L_\infty \to (L_\infty)^n$ be defined by

$$v_k(x) = \big(\underbrace{0, \ldots, 0}_{k-1}, x, \underbrace{0, \ldots, 0}_{n-k}\big),$$

where $x \in L_\infty$. Note that v_k is a continuous linear mapping.

For $(k_1, \ldots, k_N) \in \{1, \ldots, n\}^N$, let $B_{(k_1,\ldots,k_N)} \colon \underbrace{L_\infty \times \ldots \times L_\infty}_{N} \to \mathbb{C}$ be defined by

$$B_{(k_1,\ldots,k_N)}(x_1, \ldots, x_N) = A_P\big(v_{k_1}(x_1), \ldots, v_{k_N}(x_N)\big),$$

where $x_1, \ldots, x_N \in L_\infty$. By the continuity and the linearity of v_k and by the continuity and the N-linearity of A_P, the mapping $B_{(k_1,\ldots,k_N)}$ is continuous and N-linear. By the symmetry of A_P, the mapping $B_{(k_1,\ldots,k_N)}$ is symmetric. Therefore, the mapping $\widehat{B}_{(k_1,\ldots,k_N)} \colon L_\infty \to \mathbb{C}$, defined by

$$\widehat{B}_{(k_1,\ldots,k_N)}(x) = B_{(k_1,\ldots,k_N)}(x, \ldots, x),$$

where $x \in L_\infty$, is a continuous N-homogeneous polynomial. Note that $v_k(x \circ \sigma) = v_k(x) \circ \sigma$ for every $k \in \{1, \ldots, n\}$ and $\sigma \in \Xi$. Therefore, by Corollary 1,

$$\begin{aligned} B_{(k_1,\ldots,k_N)}(x_1 \circ \sigma, \ldots, x_N \circ \sigma) = A_P\big(v_{k_1}(x_1 \circ \sigma), \ldots, v_{k_N}(x_N \circ \sigma)\big) = \\ = A_P\big(v_{k_1}(x_1) \circ \sigma, \ldots, v_{k_N}(x_N) \circ \sigma\big) = A_P\big(v_{k_1}(x_1), \ldots, v_{k_N}(x_N)\big) = \\ = B_{(k_1,\ldots,k_N)}(x_1, \ldots, x_N) \end{aligned}$$

for every $x_1, \ldots, x_N \in L_\infty$ and $\sigma \in \Xi$. Consequently, $\widehat{B}_{(k_1,\ldots,k_N)}(x \circ \sigma) = \widehat{B}_{(k_1,\ldots,k_N)}(x)$ for every $x \in L_\infty$ and $\sigma \in \Xi$. Thus, $\widehat{B}_{(k_1,\ldots,k_N)}$ is symmetric. Therefore, by Lemma 1, there exists a constant $C_{(k_1,\ldots,k_N)}(m, a, b) > 0$ such that

$$\begin{aligned} \Big|B_{(k_1,\ldots,k_N)}\big(\underbrace{1_{[c_1,d_1]\cup[c_2,d_2]}, \ldots, 1_{[c_1,d_1]\cup[c_2,d_2]}}_{m}, z_1, \ldots, z_{N-m}\big)\Big| \leq \\ \leq \mu\big([c_1, d_1] \cup [c_2, d_2]\big)\|z_1\|_\infty \ldots \|z_{N-m}\|_\infty C_{(k_1,\ldots,k_N)}(m, a, b) \end{aligned} \tag{16}$$

for every $z_1, \ldots, z_{N-m} \in L_\infty$ such that the restrictions of $z_1, \ldots, z_{N-m}$ to $[a, b] \cup [c_1, d_1] \cup [c_2, d_2]$ are constant.

Let $h = \big(h^{(1)}, \ldots, h^{(n)}\big) \in \mathbb{C}^n$ and the functions $y_j = \big(y_j^{(1)}, \ldots, y_j^{(n)}\big) \in (L_\infty)^n$, where $j \in \{1, \ldots, N - m\}$, are such that the restriction of y_j to $[a, b] \cup [c_1, d_1] \cup [c_2, d_2]$ is constant for every $j \in \{1, \ldots, N - m\}$. Since

$$h * 1_{[c_1,d_1]\cup[c_2,d_2]} = \sum_{k=1}^{n} h^{(k)} v_k(1_{[c_1,d_1]\cup[c_2,d_2]}), \quad y_j = \sum_{k=1}^{n} v_k\big(y_j^{(k)}\big)$$

for every $j \in \{1, \ldots, N - m\}$, by the N-linearity of A_P,

$$A_P\big(\underbrace{h * 1_{[c_1,d_1]\cup[c_2,d_2]}, \ldots, h * 1_{[c_1,d_1]\cup[c_2,d_2]}}_{m}, y_1, \ldots, y_{N-m}\big) =$$

$$= A_P\Bigg(\sum_{k_1=1}^{n} h^{(k_1)} v_{k_1}(1_{[c_1,d_1]\cup[c_2,d_2]}), \ldots, \sum_{k_m=1}^{n} h^{(k_m)} v_{k_m}(1_{[c_1,d_1]\cup[c_2,d_2]}), \sum_{k_{m+1}=1}^{n} v_{k_{m+1}}\big(y_1^{(k_{m+1})}\big), \ldots$$

$$\ldots, \sum_{k_N=1}^{n} v_{k_N}\big(y_{N-m}^{(k_N)}\big)\Bigg) = \sum_{k_1=1}^{n} \ldots \sum_{k_N=1}^{n} h^{(k_1)} \ldots h^{(k_m)} B_{(k_1,\ldots,k_N)}\Big(\underbrace{1_{[c_1,d_1]\cup[c_2,d_2]}, \ldots, 1_{[c_1,d_1]\cup[c_2,d_2]}}_{m},$$

$$y_1^{(k_{m+1})}, \ldots, y_{N-m}^{(k_N)}\Big).$$

Therefore, by (16),

$$\Big|A_P\big(\underbrace{h*1_{[c_1,d_1]\cup[c_2,d_2]},\dots,h*1_{[c_1,d_1]\cup[c_2,d_2]}}_{m},y_1,\dots,y_{N-m}\big)\Big|\le$$

$$\le\sum_{k_1=1}^n\cdots\sum_{k_N=1}^n\big|h^{(k_1)}\big|\dots\big|h^{(k_m)}\big|\mu\big([c_1,d_1]\cup[c_2,d_2]\big)\big\|y_1^{(k_{m+1})}\big\|_\infty\dots\big\|y_{N-m}^{(k_N)}\big\|_\infty C_{(k_1,\dots,k_N)}(m,a,b).$$

Let

$$C(m,a,b)=\max_{(k_1,\dots,k_N)\in\{1,\dots,n\}^N}C_{(k_1,\dots,k_N)}(m,a,b).$$

Since $\big|h^{(k)}\big|\le\max_{1\le j\le n}\big|h^{(j)}\big|$ for every $k\in\{1,\dots,n\}$ and $\big\|y_j^{(k)}\big\|_\infty\le\|y_j\|_{\infty,n}$ for every $j\in\{1,\dots,N-m\}$ and $k\in\{1,\dots,n\}$, it follows that

$$\sum_{k_1=1}^n\cdots\sum_{k_N=1}^n\big|h^{(k_1)}\big|\dots\big|h^{(k_m)}\big|\mu\big([c_1,d_1]\cup[c_2,d_2]\big)\big\|y_1^{(k_{m+1})}\big\|_\infty\dots\big\|y_{N-m}^{(k_N)}\big\|_\infty\times$$

$$\times C_{(k_1,\dots,k_N)}(m,a,b)\le\sum_{k_1=1}^n\cdots\sum_{k_N=1}^n\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^m\mu\big([c_1,d_1]\cup[c_2,d_2]\big)\|y_1\|_{\infty,n}\dots\|y_{N-m}\|_{\infty,n}\times$$

$$\times C(m,a,b)=n^N\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^m\mu\big([c_1,d_1]\cup[c_2,d_2]\big)\|y_1\|_{\infty,n}\dots\|y_{N-m}\|_{\infty,n}C(m,a,b).$$

This completes the proof of the lemma. □

Lemma 4. *Let $P\colon(L_\infty)^n\to\mathbb{C}$ be a continuous symmetric N-homogeneous polynomial, where $N\ge 2$. Then there exists a sequence $\{s_k\}_{k=1}^\infty\subset[0,1]$ such that $\lim_{k\to\infty}s_k=0$ and*

$$\big|P\big(h*1_{[0,r_k]}\big)\big|\le\frac{1}{k}\big(\|P\|+1\big)\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N$$

for every $h=\big(h^{(1)},\dots,h^{(n)}\big)\in\mathbb{C}^n$ and for every sequence $\{r_k\}_{k=1}^\infty$ such that $0\le r_k\le s_k$ for each $k\in\mathbb{N}$.

Proof. Let $h=\big(h^{(1)},\dots,h^{(n)}\big)\in\mathbb{C}^n$. By (4),

$$\big|P\big(h*1_{[0,a]}\big)\big|\le\|P\|\big\|h*1_{[0,a]}\big\|_{\infty,n}^N\le\|P\|\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N\tag{17}$$

for every $a\in[0,1]$.

We set $s_1=1$. By (17),

$$\big|P\big(h*1_{[0,r_1]}\big)\le\|P\|\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N\le\big(\|P\|+1\big)\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N$$

for every $r_1\in[0,s_1]$.

Let $k\ge 2$. Let $t\ge 0$ be such that $kt<1/2$. Since, almost everywhere on $[0,1]$,

$$h*1_{[0,kt]}=\sum_{j=1}^k h*1_{[(j-1)t,jt]},$$

by (2),

$$P\big(h*1_{[0,kt]}\big)=\sum_{N_1+\ldots+N_k=N}\frac{N!}{N_1!\ldots N_k!}\times$$

$$\times A_P\Big(\underbrace{h*1_{[0,t]},\ldots,h*1_{[0,t]}}_{N_1},\underbrace{h*1_{[t,2t]},\ldots,h*1_{[t,2t]}}_{N_2},\ldots,\underbrace{h*1_{[(k-1)t,kt]},\ldots,h*1_{[(k-1)t,kt]}}_{N_k}\Big),$$

where $N_1,\ldots,N_k\in\mathbb{Z}_+$. For every multi-index $(N_1,\ldots,N_k)\in\mathbb{Z}_+^k$, let

$$\mathcal{V}((N_1,\ldots,N_k))=\big\{j\in\{1,\ldots,k\}\colon N_j>0\big\}.$$

Then

$$P\big(h*1_{[0,kt]}\big)=\sum_{j=1}^k P\big(h*1_{[(j-1)t,jt]}\big)+\sum_{\substack{N_1+\ldots+N_k=N\\ |\mathcal{V}((N_1,\ldots,N_k))|\geq 2}}\frac{N!}{N_1!\ldots N_k!}\times$$

$$\times A_P\Big(\underbrace{h*1_{[0,t]},\ldots,h*1_{[0,t]}}_{N_1},\underbrace{h*1_{[t,2t]},\ldots,h*1_{[t,2t]}}_{N_2},\ldots,\underbrace{h*1_{[(k-1)t,kt]},\ldots,h*1_{[(k-1)t,kt]}}_{N_k}\Big).$$

Since P is symmetric, it follows that

$$P\big(h*1_{[(j-1)t,jt]}\big)=P\big(h*1_{[0,t]}\big)$$

for every $j\in\{1,\ldots,k\}$. Therefore,

$$\sum_{j=1}^k P\big(h*1_{[(j-1)t,jt]}\big)=kP\big(h*1_{[0,t]}\big).$$

Thus,

$$kP\big(h*1_{[0,t]}\big)=P\big(h*1_{[0,kt]}\big)-\sum_{\substack{N_1+\ldots+N_k=N\\ |\mathcal{V}((N_1,\ldots,N_k))|\geq 2}}\frac{N!}{N_1!\ldots N_k!}\times \tag{18}$$

$$\times A_P\Big(\underbrace{h*1_{[0,t]},\ldots,h*1_{[0,t]}}_{N_1},\underbrace{h*1_{[t,2t]},\ldots,h*1_{[t,2t]}}_{N_2},\ldots,\underbrace{h*1_{[(k-1)t,kt]},\ldots,h*1_{[(k-1)t,kt]}}_{N_k}\Big). \tag{19}$$

Let $(N_1,\ldots,N_k)\in\mathbb{Z}_+^k$ be such that $N_1+\ldots+N_k=N$ and $|\mathcal{V}((N_1,\ldots,N_k))|\geq 2$. Let $\nu=|\mathcal{V}((N_1,\ldots,N_k))|$. Then $\mathcal{V}((N_1,\ldots,N_k))=\{l_1,\ldots,l_\nu\}$ for some $l_1,\ldots,l_\nu\in\{1,\ldots,k\}$ such that $l_1<\ldots<l_\nu$. By Lemma 3, in which we set $m=N_{l_1},[a,b]=[1/2,1],[c_1,d_1]=[(l_1-1)t,l_1t],c_2=d_2=0,y_1=\ldots=y_{N_{l_2}}=h*1_{[(l_2-1)t,l_2t]},y_{N_{l_2}+1}=\ldots=y_{N_{l_2}+N_{l_3}}=h*1_{[(l_3-1)t,l_3t]},\ldots,y_{N_{l_2}+\ldots+N_{l_{\nu-1}}+1}=\ldots=y_{N_{l_2}+\ldots+N_{l_\nu}}=h*1_{[(l_\nu-1)t,l_\nu t]}$,

$$\Big|A_P\Big(\underbrace{h*1_{[(l_1-1)t,l_1t]},\ldots,h*1_{[(l_1-1)t,l_1t]}}_{N_{l_1}},\ldots,\underbrace{h*1_{[(l_\nu-1)t,l_\nu t]},\ldots,h*1_{[(l_\nu-1)t,l_\nu t]}}_{N_{l_\nu}}\Big)\Big|\leq$$

$$\leq n^N\Big(\max_{1\leq j\leq n}\big|h^{(j)}\big|\Big)^{N_{l_1}}\mu\big([(l_1-1)t,l_1t]\big)\big\|h*1_{[(l_2-1)t,l_2t]}\big\|_{\infty,n}^{N_{l_2}}\ldots\big\|h*1_{[(l_\nu-1)t,l_\nu t]}\big\|_{\infty,n}^{N_{l_\nu}}\times$$

$$\times C(N_{l_1},1/2,1) \le n^N\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N tC(N_{l_1},1/2,1) \le n^N\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N t\tilde{C},$$

where

$$\tilde{C} = \max_{1\le j\le N} C(j,1/2,1).$$

Therefore, by (18), taking into account (17),

$$k\big|P\big(h*1_{[0,t]}\big)\big| \le \|P\|\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N + n^N\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N t\tilde{C}\sum_{\substack{N_1+\ldots+N_k=N\\ |\mathcal{V}((N_1,\ldots,N_k))|\ge 2}}\frac{N!}{N_1!\ldots N_k!} \le$$

$$\le \|P\|\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N + n^N\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N t\tilde{C}\sum_{N_1+\ldots+N_k=N}\frac{N!}{N_1!\ldots N_k!} =$$

$$= \big(\|P\| + \tilde{C}n^N k^N t\big)\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N.$$

We set $s_k = 1/\big(\tilde{C}n^N k^N\big)$. Then, for every $r_k\in[0,s_k]$, $\tilde{C}n^Nk^Nr_k\le 1$ and, consequently,

$$k\big|P\big(h*1_{[0,r_k]}\big)\big| \le (\|P\|+1)\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^N.$$

This completes the proof of the lemma. □

Lemma 5. *Let $P\colon (L_\infty)^n\to\mathbb{C}$ be a continuous symmetric N-homogeneous polynomial, where $N\ge 2$. Let*

$$x = \sum_{j=1}^{\iota} h_j * 1_{[a_j,b_j]},$$

where $\iota\in\mathbb{N}, h_j = \big(h_j^{(1)},\ldots,h_j^{(n)}\big)\in\mathbb{C}^n$ for $j\in\{1,\ldots,\iota\}$, and $a_1,\ldots,a_\iota,b_1,\ldots,b_\iota\in[0,1]$ are such that $a_1<b_1\le a_2<\ldots<b_\iota$. Let $l\in\{1,\ldots,\iota\}$ and

$$\varepsilon_k = \frac{1}{2}\min\{s_k, b_l-a_l\}$$

for $k\in\mathbb{N}$, where the sequence $\{s_k\}_{k=1}^\infty$ is given by Lemma 4. Then, for every sequences $\big\{a_l^{(k)}\big\}_{k=1}^\infty, \big\{b_l^{(k)}\big\}_{k=1}^\infty\subset[0,1]$ such that $a_l^{(k)}\in[a_l,a_l+\varepsilon_k]$ and $b_l^{(k)}\in[b_l-\varepsilon_k,b_l]$ for every $k\in\mathbb{N}$,

$$\lim_{k\to\infty}P(x-\delta_k) = P(x),$$

where

$$\delta_k = h_l * 1_{\left[a_l,a_l^{(k)}\right]\cup\left[b_l^{(k)},b_l\right]}.$$

Proof. By (3),

$$P(x-\delta_k) = P(x) + P(-\delta_k) + \sum_{m=1}^{N-1}\binom{N}{m}A_P\big(\underbrace{-\delta_k,\ldots,-\delta_k}_{m},\underbrace{x,\ldots,x}_{N-m}\big).$$

Since P is symmetric, it follows that

$$P(-\delta_k) = P\Big(-h_l * 1_{\left[0,a_l^{(k)}-a_l+b_l-b_l^{(k)}\right]}\Big).$$

Since $a_l^{(k)} - a_l + b_l - b_l^{(k)} \le 2\varepsilon_k \le s_k$, by Lemma 4,

$$|P(-\delta_k)| \le \frac{1}{k}\big(\|P\|+1\big)\Big(\max_{1\le j\le n}\big|h_l^{(j)}\big|\Big)^N.$$

For every $m \in \{1,\ldots,N-1\}$, by Lemma 3, in which we set $[a,b] = [a_l,b_l], [c_1,d_1] = \big[a_l,a_l^{(k)}\big], [c_2,d_2] = \big[b_l^{(k)},b_l\big]$ and $y_1 = \ldots = y_{N-m} = x$, there exists a constant $C(m,a_l,b_l) > 0$ such that

$$\Big|A_P\big(\underbrace{-\delta_k,\ldots,-\delta_k}_{m},\underbrace{x,\ldots,x}_{N-m}\big)\Big| \le n^N\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^m \mu\big([a_l,a_l^{(k)}] \cup \big[b_l^{(k)},b_l\big]\big)\|x\|_{\infty,n}^{N-m}\times$$
$$\times C(m,a,b) \le s_k n^N\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^m \|x\|_{\infty,n}^{N-m}C(m,a_l,b_l).$$

Therefore,

$$|P(x-\delta_k) - P(x)| \le \frac{1}{k}\big(\|P\|+1\big)\Big(\max_{1\le j\le n}\big|h_l^{(j)}\big|\Big)^N +$$
$$+s_k n^N\sum_{m=1}^{N-1}\binom{N}{m}\Big(\max_{1\le j\le n}\big|h^{(j)}\big|\Big)^m\|x\|_{\infty,n}^{N-m}C(m,a_l,b_l) \to 0$$

as $k\to\infty$, since $\lim_{k\to\infty} s_k = 0$. Thus, $\lim_{k\to\infty} P(x-\delta_k) = P(x)$. This completes the proof of the lemma. □

Theorem 2. *Every symmetric continuous N-homogeneous polynomial $P\colon (L_\infty)^n \to \mathbb{C}$ can be uniquely represented as*

$$P(y) = \sum_{\substack{l\colon M_N\to\mathbb{Z}_+\\ \varkappa(l,M_N)=N}} \alpha_l \prod_{\substack{k\in M_N\\ l(k)>0}} (R_k(y))^{l(k)},$$

where $y \in (L_\infty)^n$, $\alpha_l \in \mathbb{C}$, M_N is defined by (9), *and $\varkappa$ is defined by* (8).

Proof. Consider the case $N=1$. In this case, P is a continuous symmetric linear functional. Therefore, for every $y = \big(y^{(1)},\ldots,y^{(n)}\big) \in (L_\infty)^n$,

$$P(y) = \sum_{j=1}^n P\Big(\big(\underbrace{0,\ldots,0}_{j-1},y^{(j)},\underbrace{0,\ldots,0}_{n-j}\big)\Big).$$

Evidently, the mapping

$$L_\infty \ni x \mapsto P\Big(\big(\underbrace{0,\ldots,0}_{j-1},x,\underbrace{0,\ldots,0}_{n-j}\big)\Big) \in \mathbb{C}$$

is a continuous symmetric linear functional for every $j \in \{1,\ldots,n\}$. By [1, Theorem 4.3], every continuous symmetric linear functional $f\colon L_\infty \to \mathbb{C}$ can be represented as

$$f(x) = \alpha\int_{[0,1]} x(t)\,dt,$$

where $x \in L_\infty$ and $\alpha \in \mathbb{C}$. Therefore, there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that

$$P(y) = \sum_{j=1}^{n} \alpha_j \int_{[0,1]} y^{(j)}(t)\,dt = \sum_{j=1}^{n} \alpha_j R_{(\underbrace{0,\ldots,0}_{j-1},1,\underbrace{0,\ldots,0}_{n-j})}(y) \tag{20}$$

for every $y = \big(y^{(1)}, \ldots, y^{(n)}\big) \in (L_\infty)^n$. By Proposition 1, polynomials

$$R_{(1,\underbrace{0,\ldots,0}_{n-1})}, R_{(0,1,\underbrace{0,\ldots,0}_{n-2})}, \ldots, R_{(\underbrace{0,\ldots,0}_{n-1},1)}$$

are algebraically independent. Therefore, the representation (20) is unique. This completes the proof of the theorem for the case $N = 1$.

Consider the case $N \geq 2$. By Proposition 2, there exist unique coefficients

$$\big\{\alpha_l \in \mathbb{C}\colon \quad l\colon M_N \to \mathbb{Z}_+ \text{ such that } \varkappa(l, M_N) = N\big\}$$

such that

$$P(y) = \sum_{\substack{l\colon M_N \to \mathbb{Z}_+ \\ \varkappa(l,M_N)=N}} \alpha_l \prod_{\substack{k \in M_N \\ l(k)>0}} (R_k(y))^{l(k)} \tag{21}$$

for every $y \in D$.

Let

$$x = \sum_{j=1}^{\iota} h_j * 1_{[a_j,b_j]}, \tag{22}$$

where $\iota \in \mathbb{N}, h_j = \big(h_j^{(1)}, \ldots, h_j^{(n)}\big) \in \mathbb{C}^n$ for $j \in \{1, \ldots, \iota\}$, and $a_1, \ldots, a_\iota, b_1, \ldots, b_\iota \in [0,1]$ are such that $a_1 < b_1 \leq a_2 < \ldots < b_\iota$. For every $l \in \{1, \ldots, \iota\}$, we choose sequences

$$\big\{a_l^{(k)}\big\}_{k=1}^{\infty}, \big\{b_l^{(k)}\big\}_{k=1}^{\infty} \subset \bigcup_{m=1}^{\infty} \Big\{\frac{j}{2^m}\colon\ j \in \{1, \ldots, 2^m\}\Big\}$$

such that

$$a_l \leq a_l^{(k)} \leq a_l + \frac{1}{2}\min\{s_k, b_l - a_l\}, \quad b_l - \frac{1}{2}\min\{s_k, b_l - a_l\} \leq b_l^{(k)} \leq b_l,$$

where the sequence $\{s_k\}_{k=1}^{\infty}$ is given by Lemma 4. Then, the function

$$x_\eta = \sum_{j=1}^{\iota} h_j * 1_{\left[a_j^{(\eta_j)}, b_j^{(\eta_j)}\right]}$$

belongs to D for every multi-index $\eta = (\eta_1, \ldots, \eta_\iota) \in \mathbb{N}^\iota$. By using Lemma 5 ι times,

$$P(x) = \lim_{\eta_1\to\infty} \lim_{\eta_2\to\infty} \ldots \lim_{\eta_\iota\to\infty} P\big(x_{(\eta_1,\ldots,\eta_\iota)}\big) =$$

$$= \lim_{\eta_1\to\infty} \lim_{\eta_2\to\infty} \ldots \lim_{\eta_\iota\to\infty} \sum_{\substack{l\colon M_N \to \mathbb{Z}_+ \\ \varkappa(l,M_N)=N}} \alpha_l \prod_{\substack{k \in M_N \\ l(k)>0}} \big(R_k\big(x_{(\eta_1,\ldots,\eta_\iota)}\big)\big)^{l(k)} = \sum_{\substack{l\colon M_N \to \mathbb{Z}_+ \\ \varkappa(l,M_N)=N}} \alpha_l \prod_{\substack{k \in M_N \\ l(k)>0}} (R_k(x))^{l(k)}.$$

Thus, the equality (21) holds for every $x \in (L_\infty)^n$ of the form (22).

Let

$$z=\sum_{j=1}^{\iota} h_j * 1_{E_j}, \tag{23}$$

where $\iota\in\mathbb{N}$, $h_j=\big(h_j^{(1)},\ldots,h_j^{(n)}\big)\in\mathbb{C}^n$ for every $j\in\{1,\ldots,\iota\}$, and $E_1,\ldots,E_\iota$ are disjoint Lebesgue measurable subsets of $[0,1]$. By [1, Proposition 2.2], there exists $\sigma_{E_1,\ldots,E_\iota}\in\Xi$ such that

$$1_{E_j}=1_{\left[\sum_{m=1}^{j-1}\mu(E_m),\sum_{m=1}^{j}\mu(E_m)\right]}\circ\sigma_{E_1,\ldots,E_\iota}$$

for every $j\in\{1,\ldots,\iota\}$ almost everywhere on $[0,1]$. Consequently, $z=\hat z\circ\sigma_{E_1,\ldots,E_\iota}$, where

$$\hat z=\sum_{j=1}^{\iota} h_j * 1_{\left[\sum_{m=1}^{j-1}\mu(E_m),\sum_{m=1}^{j}\mu(E_m)\right]}.$$

Therefore, by the symmetry of P, $P(z)=P(\hat z)$. Since $\hat z$ is the function of the form (22), it follows that

$$P(\hat z)=\sum_{\substack{l\colon M_N\to\mathbb{Z}_+\\ \varkappa(l,M_N)=N}}\alpha_l\prod_{\substack{k\in M_N\\ l(k)>0}}(R_k(\hat z))^{l(k)}.$$

Since polynomials R_k are symmetric, it follows that $R_k(\hat z)=R_k(z)$ for every $k\in M_N$. Thus,

$$P(z)=\sum_{\substack{l\colon M_N\to\mathbb{Z}_+\\ \varkappa(l,M_N)=N}}\alpha_l\prod_{\substack{k\in M_N\\ l(k)>0}}(R_k(z))^{l(k)}.$$

By Lemma 2, the set of all functions of the form (23) is dense in $(L_\infty)^n$. Therefore, by the continuity of P,

$$P(y)=\sum_{\substack{l\colon M_N\to\mathbb{Z}_+\\ \varkappa(l,M_N)=N}}\alpha_l\prod_{\substack{k\in M_N\\ l(k)>0}}(R_k(y))^{l(k)} \tag{24}$$

for every $y\in(L_\infty)^n$. By Proposition 1, polynomials R_k, where $k\in M_N$, are algebraically independent. Therefore, the representation (24) is unique. This completes the proof. □

Corollary 3. *The set of polynomials* $\big\{R_k\colon\ k\in\mathbb{Z}_+^n, |k|\ge 1\big\}$ *is an algebraic basis of the algebra of all continuous symmetric complex-valued polynomials on* $(L_\infty)^n$.

Acknowledgement. The publication contains the results of studies conducted by President's of Ukraine grant for competitive projects 0119U103204.

REFERENCES

1. P. Galindo, T. Vasylyshyn, A. Zagorodnyuk, *The algebra of symmetric analytic functions on* L_∞, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, **147** (2017), №4, 743–761. doi:10.1017/S0308210516000287.
2. P. Galindo, T. Vasylyshyn, A. Zagorodnyuk, *Symmetric and finitely symmetric polynomials on the spaces* ℓ_∞ *and* $L_\infty[0,+\infty)$, Mathematische Nachrichten, **291** (2018), №11–12, 1712–1726. doi:10.1002/mana.201700314.

3. M. González, R. Gonzalo, J.A. Jaramillo, *Symmetric polynomials on rearrangement invariant function spaces*, J. London Math. Soc., **59** (1999), №2, 681–697. doi:10.1112/S0024610799007164.
4. V. Kravtsiv, *Algebraic basis of the algebra of block-symmetric polynomials on $\ell_1 \oplus \ell_\infty$*, Carpathian Math. Publ., **11** (2019), №1, 89–95. doi:10.15330/cmp.11.1.89-95.
5. V. Kravtsiv, T. Vasylyshyn, A. Zagorodnyuk, *On algebraic basis of the algebra of symmetric polynomials on $\ell_p(\mathbb{C}^n)$*, Journal of Function Spaces, **2017** (2017), Article ID 4947925, 8 p. doi:10.1155/2017/4947925.
6. J. Mujica, Complex analysis in Banach spaces, North Holland, 1986.
7. A.S. Nemirovskii, S.M. Semenov, *On polynomial approximation of functions on Hilbert space*, Mat. USSR Sbornik, **21** (1973), №2, 255–277. doi:10.1070/SM1973v021n02ABEH002016
8. T. Vasylyshyn, *Continuous block-symmetric polynomials of degree at most two on the space $(L_\infty)^2$*, Carpathian Math. Publ., **8** (2016), №1, 38–43. doi:10.15330/cmp.8.1.38-43.
9. T. Vasylyshyn, *Symmetric polynomials on $(L_p)^n$*, European Journal of Math., doi:10.1007/s40879-018-0268-3.
10. T.V. Vasylyshyn, *Symmetric polynomials on the Cartesian power of L_p on the semi-axis*, Mat. Stud., **50** (2018), №1, 93–104.
11. T. Vasylyshyn, A. Zagorodnyuk, *Continuous symmetric 3-homogeneous polynomials on spaces of Lebesgue measurable essentially bounded functions*, Methods of Functional Analysis and Topology, **24** (2018), №4, 381–398.

Vasyl Stefanyk Precarpathian National University, Ukraine
taras.v.vasylyshyn@gmail.com

Received 26.04.2019