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THE CAUCHY PROBLEM WITH IMPULSE ACTION AND DEGENERATION FOR PARABOLIC EQUATIONS

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We study the Cauchy problem for the second-order linear parabolic equations with impulse conditions in the time variable and power singularity in the coefficients of any order with respect to the time and space variables. By using the maximum principle and apriori estimates we establish the existence and uniqueness of solution of the problem in Hölder spaces with power weight.

The study of systems with discontinuous trajectories is stimulated by the development of technology, where impulse computing devices play an important role. Research of tasks in theory of automatic control, theory of nuclear reactors, dynamic systems, leads to solving of boundary value problems for differential equations with impulse action. Fundamental results in the theory of investigation of solutions of systems of ordinary differential equations with impulse action are given in monographs [1–3]. The existence of the periodic solutions of impulse action problems for equations of hyperbolic type has been studied in papers [4, 5]. In papers [6,7] there are presented results of research on classical solutions of momentum boundary value problems for parabolic degenerate equations. Papers [8,9] are devoted to the boundary value problems with nonlocal and integral conditions on time variable for parabolic equations with degeneration in coefficients on time and space variables.

This article suggests an algorithm for solving of the Cauchy problem for the second-order parabolic equation with degree singularities and degeneracy in coefficients of time and space variables of arbitrary order for some multiple points and impulse actions on time variable at certain points in time. The existence and uniqueness of a solution of the task is proven in Hölder spaces with a degree of weight.

1. Formulation of the Problem and Main Restrictions. Let η, t_0, t_1, $\ldots$, t_N, t_{N+1} be fixed positive numbers, $0 \le t_0 < t_1 < \cdots < t_N < t_{N+1}$, $t_0 < \eta < t_{N+1}$, $\eta \ne t_\lambda$, $\lambda \in \{1, 2, \ldots, N\}$, let Ω be a some bounded domain, $\overline{\Omega} \subset \mathbb{R}^{n-1}$, $\dim \Omega \le n-1$. Let us denote $\Pi_{(0)} = (\{(t,x) \,\big|\, t \in [t_0; t_{N+1}), x \in \Omega\} \cup \{(t,x) \,\big|\, t = \eta, x \in \mathbb{R}^n \setminus \overline{\Omega}\})$.

In the domain $\Pi = [t_0, t_{N+1}) \times \mathbb{R}^n$ we consider the problem of finding the function $u(t,x)$ which for $t \ne t_\lambda$, $(t,x) \notin \Pi_{(0)}$ satisfies the equation

$$(Lu)(t,x) = \Big[\partial_t - \sum_{ij=1}^{n} A_{ij}(t,x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} A_i(t,x)\partial_{x_i} + A_0(t,x)\Big]u(t,x) = f(t,x), \qquad (1)$$

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and conditions for the time variable

$$u(t_0+0,x)=\varphi_0(x), \tag{2}$$
$$u(t_\lambda+0,x)-u(t_\lambda-0,x)=b_\lambda(x)u(t_\lambda-0,x)+\varphi_\lambda(x). \tag{3}$$

The power singularities of coefficients of differential equation (1) at the point $P(t,x)\in \Pi\setminus\Pi_{(0)}$ are characterized by the functions $s_1(\beta_i^{(1)},t)$ and $s_2(\beta_i^{(2)},x)$: $s_1(\beta_i^{(1)},t)=|t-\eta|^{\beta_i^{(1)}}$ as $|t-\eta|\le 1$, $s_1(\beta_i^{(1)},t)\equiv 1$ as $|t-\eta|\ge 1$; $s_2(\beta_i^{(2)},x)=\rho^{\beta_i^{(2)}}(x)$ as $\rho(x)\le 1$, $s_2(\beta_i^{(2)},x)=1$ as $\rho(x)\ge 1$, $\rho(x)=\inf\limits_{z\in\overline{\Omega}}|x-z|$, $\beta_i^{(\nu)}\in(-\infty,\infty)$, $\nu\in\{1,2\}$, $\beta^{(\nu)}=(\beta_1^{(\nu)},\beta_2^{(\nu)},\dots,\beta_n^{(\nu)})$, $\beta=(\beta^{(1)},\beta^{(2)})$.

Denote by $(x_1^{(1)},\dots,x_i^{(1)},\dots,x_n^{(1)})$ the coordinates of the point $x^{(1)}\in\mathbb{R}^n$. Let $(x_1^{(2)},\dots, x_i^{(2)},\dots,x_n^{(2)})$ be the coordinates of the point $x^{(2)}\in\mathbb{R}^n$, l, α, $q^{(1)}$, $q^{(2)}$, $\gamma^{(1)}$, $\gamma^{(2)}$, $\mu_j^{(1)}$, $\mu_j^{(2)}$ be real numbers, $j\in\{0,1,\dots,n\}$, $q^{(\nu)}\ge 0$, $\gamma^{(\nu)}\ge 0$, $l\ge 0$, $\mu_j^{(\nu)}\ge 0$, $\alpha\in(0,1)$, $[l]$ be the integer part of l. Let D be an arbitrary closed domain, $\overline{D}\subset\mathbb{R}^n$, $Q^{(k)}=[t_k,t_{k+1})\times D$, $P_1^{(k)}(t^{(1)},x^{(1)})$, $R^{(k)}(t^{(2)},x^{(2)})$, $H^{(k)}(t^{(1)},x^{(2)})$ be arbitrary points of the domain $Q^{(k)}$, $\overline{Q}^{(k)}\subset \Pi^{(k)}=[t_k,t_{k+1})\times\mathbb{R}^n$, $k\in\{0,1,\dots,N\}$, $Q_{(0)}=Q^{(k)}\bigcap\Pi_{(0)}$.

Below we define the function space to study problem (1)–(3).

Let $H^l(\gamma;\beta;q;\Pi)$ be the set of functions $u(t,x)\in L_1(\Pi)$ having continuous partial derivatives in $Q^{(k)}\setminus Q_{(0)}$ of the form $\partial_t^j\partial_x^r u$, $2i+|r|\le[l]$ and a finite value of the norm

$$\|u;\gamma;\beta;q;\Pi\|_l=\sup_k\Big\{\sum_{2j+|r|\le[l]}\|u;\gamma;\beta;q,Q^{(k)}\|_{2j+|r|}+\langle u;\gamma;\beta;q;Q^{(k)}\rangle_l\Big\},$$

where, e.g.,

$$\|u;\gamma;\beta;0;Q^{(k)}\|_0=\sup_k\Big(\sup_{P_1^{(k)}\in\overline{Q}^{(k)}}|u(P_1^{(k)})|\Big)\equiv\sup_k\|u;Q^{(k)}\|_0,$$

$$\|u;\gamma;\beta;q;Q^{(k)}\|_{2j+|r|}=\sup_{P_1^{(k)}\in\overline{Q}^{(k)}}[s_1(q^{(1)}+2j\gamma^{(1)},t^{(1)})s_2(q^{(2)}+2j\gamma^{(2)},x^{(1)})\times$$
$$\times\prod_{i=1}^n s_1(r_i(\gamma^{(1)}-\beta_i^{(1)}),t^{(1)})s_2(r_i(\gamma^{(2)}-\beta_i^{(2)}),x^{(1)})|\partial_t^j\partial_x^r u(P_1^{(k)})|],$$

$$\langle u;\gamma;\beta;q;Q^{(k)}\rangle_l=\sum_{2j+|r|=[l]}\Big\{\sum_{\nu=1}^n\sup_{(P_1^{(k)},H^{(k)})\subset\overline{Q}^{(k)}}\Big[s_1(q^{(1)}+l\gamma^{(1)},t^{(1)})\times$$
$$\times s_2(q^{(2)}+2j\gamma^{(2)},\tilde{x})\prod_{i=1}^n s_1(-r_i\beta_i^{(1)},t^{(1)})s_2(r_i(\gamma^{(2)}-\beta_i^{(2)},\tilde{x}))s_1(-\{l\}\beta_\nu^{(1)},t^{(1)})\times$$
$$\times s_2(\{l\}(\gamma^{(2)}-\beta_\nu^{(2)},\tilde{x})|x_\nu^{(1)}-x_\nu^{(2)}|^{-\{l\}}|\partial_t^j\partial_x^r u(P_1^{(k)})-$$
$$-\partial_t^j\partial_x^r u(H^{(k)})|\Big]+\sup_{(R^{(k)},H^{(k)})\subset\overline{Q}^{(k)}}\Big[s_1(q^{(1)}+l\gamma^{(1)},\tilde{t})s_2(q^{(2)}+(l-|r|\gamma^{(2)},x^{(2)})\times$$
$$\times\prod_{i=1}^n s_1(-r_i\beta_i^{(1)},\tilde{t})s_2(r_i(\gamma^{(2)}-\beta_i^{(2)},x^{(2)}))|t^{(1)}-t^{(2)}|^{-\{\frac{l}{2}\}}|\partial_t^j\partial_x^r u(R^{(k)})-\partial_t^j\partial_x^r u(H^{(k)})|\Big]\Big\}.$$

Here: $s_1(a,\tilde{t})=\min(s_1(a,t^{(1)}),s_1(a,t^{(2)}))$, $\partial_x^r=\partial_{x_1}^{r_1}\partial_{x_2}^{r_2}\dots\partial_{x_n}^{r_n}$, $|r|=r_1+r_2+\dots+r_n$, $s_2(a,\tilde{x})=\min(s_2(a,x^{(1)}),s_2(a,x^{(2)}))$, $l=[l]+\{l\}$.

Assume that initial problem (1)–(3) satisfies the following conditions:

a) for an arbitrary vector $\xi=(\xi_1,\xi_2,\dots,\xi_n)$, $\forall(t,x)\in\Pi\setminus\Pi_{(0)}$ the following inequality holds

$$\pi_1|\xi|^2\le\sum_{ij=1}^{n}s_1(\beta_i^{(1)},t)s_1(\beta_j^{(1)},t)s_2(\beta_i^{(2)},x)s_2(\beta_j^{(2)},x)A_{ij}(t,x)\xi_i\xi_j\le\pi_2|\xi|^2,$$

where π_1, π_2 are fixed positive constants and $s_1(\mu_i^{(1)},t)s_2(\mu_i^{(2)},x)A_i\in H^\alpha(\gamma;\beta;0;\Pi\setminus\Pi_{(0)})$,

$$s_1(\beta_i^{(1)},t)s_1(\beta_j^{(1)},t)s_2(\beta_i^{(2)},x)s_2(\beta_j^{(2)},x)A_{ij}\in H^\alpha(\gamma;\beta;0;\Pi\setminus\Pi_{(0)}),$$
$$s_1(\mu_0^{(1)},t)s_2(\mu_0^{(2)},x)A_0\in H^\alpha(\gamma;\beta;0;\Pi\setminus\Pi_{(0)}),\ A_0\ge -a,\ a\ge 0,$$
$$\gamma^{(\nu)}=\max\Big\{\max_i(1+\beta_i^{(\nu)}),\max_i\Big(\mu_i^{(\nu)}-\beta_i^{(\nu)}\Big),\frac{\mu_0^{(\nu)}}{2}\Big\},\ \nu\in\{1,2\};$$

b) functions $f\in H^\alpha(\gamma;\beta;\mu_0;\Pi\setminus\Pi_{(0)})$, $\varphi_0\in H^{2+\alpha}(\tilde\gamma;\tilde\beta;0;\mathbb{R}^n\setminus\overline{\Omega})$, $\varphi_\lambda\in H^{2+\alpha}(\tilde\gamma;\tilde\beta;0;(\Pi\setminus\Pi_{(0)}\,|\cap(t=t_\lambda))$, $b_\lambda(x)\in C^{2+\alpha}((\Pi\setminus\Pi_{(0)})\cap(t=t_\lambda))$, $\tilde\gamma=(0,\gamma^{(2)})$, $\tilde\beta=(0,\beta^{(2)})$.

Let us formulate the main result of the paper.

Theorem 1. *Let conditions a), b) be satisfied for problem (1)–(3). Then there exists the unique solution of problem (1)–(3) in the space $H^{2+\alpha}(\gamma,\beta;0;\Pi\setminus\Pi_{(0)})$ and the following estimate is correct:*

$$\begin{aligned}\|u;\gamma;\beta;0;\Pi\setminus\Pi_{(0)}\|_{2+\alpha}\le c\Big\{&\sum_{k=1}^{N}\prod_{\lambda=k}^{N}(1+\|b_\lambda;\Pi\cap(t=t_\lambda)\|_0)\times\\ &\times(\|\varphi_{k-1};\tilde\gamma;\tilde\beta;0;\Pi\cap(t=t_{k-1})\|_{2+\alpha}+\|f;\gamma;\beta;\mu_0;\Pi^{(k-1)}\|_\alpha)+\\ &+\|\varphi_N;\tilde\gamma;\tilde\beta;0;\Pi\cap(t=t_N)\|_{2+\alpha}+\|f;\gamma;\beta;\mu_0;\Pi^{(N)}\|_\alpha\Big\}\equiv B.\end{aligned}\tag{4}$$

To prove Theorem 1 we construct a sequence of solutions of problems with smooth coefficients, for which the boundary value is the solution of problem (1)–(3).

2. Evaluation of solutions of problems with smooth coefficients. Let $\Pi_m=\Pi\cap\{(t,x)\in\Pi\,|\,s_1(1,t)\ge m_1^{-1},s_2(1,x)\ge m_2^{-1}\}$, $m_1>1$, $m_2>1$ be a sequence of domains converging to $\Pi\setminus\Pi_{(0)}$ as $m_1\to\infty$, $m_2\to\infty$.

In the domain Π we consider the problem of finding the function $u_m(t,x)$, that satisfies the equation

$$(L_1u_m)(t,x)\equiv\Big(\partial_t-\sum_{ij=1}^{n}a_{ij}(t,x)\partial_{x_i}\partial_{x_j}+\sum_{i=1}^{n}a_i(t,x)\partial_{x_i}+a_0(t,x)\Big)u_m(t,x)=f_m(t,x),\tag{5}$$

and conditions concerning to the time variable

$$u_m(t_0+0,x)=\varphi_0^{(m)}(x),\tag{6}$$
$$u_m(t_\lambda+0,x)-u_m(t_\lambda-0,x)=b_\lambda^{(m)}(x)u_m(t_\lambda-0,x)+\varphi_\lambda^{(m)}(x).\tag{7}$$

Here the coefficients a_{ij}, a_i, a_0, $b_\lambda^{(m)}(x)$ and functions f_m, $\varphi_0^{(m)}$, $\varphi_\lambda^{(m)}$ for $(t,x)\in\Pi_m$ coincide with A_{ij}, A_i, A_0, b_λ and f, φ_0, φ_λ, respectively. For $(t,x)\in\Pi\setminus\Pi_m$ the coefficients a_{ij}, a_i, a_0, $b_\lambda^{(m)}$, and functions f_m, $\varphi_0^{(m)}$, $\varphi_\lambda^{(m)}$ are continuous extensions of the coefficients

A_{ij}, A_i, A_0, b_λ and functions f, φ_0, φ_λ from the domain Π_m into the domain $\Pi \setminus \Pi_m$ [10, p. 83], $m = (m_1, m_2)$.

Let us find the estimate of the solution of problem (5)–(7). In the space $C^{2+\alpha}(\Pi)$ we introduce the norm $\|u_m; \gamma; \beta; q; \Pi\|_l$, which for every m is equivalent to the Hölder norm determined by analogy as $\|u; \gamma; \beta; q; \Pi\|_l$. But instead of the functions $s_1(\beta_i^{(1)}, t)$, $s_2(\beta_i^{(2)}, x)$ we take $d_1(\beta_i^{(1)}, t)$, $d_2(\beta_i^{(2)}, x)$: $d_1(\beta_i^{(1)}, t) = \max(s_1(\beta_i^{(1)}, t), m_1^{-\beta_i^{(1)}})$ as $\beta_i^{(1)} \geq 0$ and $d_1(\beta_i^{(1)}, t) = \min(s_1(\beta_i^{(1)}, t), m_1^{-\beta_i^{(1)}})$ as $\beta_i^{(1)} < 0$; $d_2(\beta_i^{(2)}, x) = \max(s_2(\beta_i^{(2)}, x), m_2^{-\beta_i^{(2)}})$ as $\beta_i^{(2)} \geq 0$ and $d_2(\beta_i^{(2)}, x) = \min(s_2(\beta_i^{(2)}, x), m_2^{-\beta_i^{(2)}})$ as $\beta_i^{(2)} < 0$.

To solve problem (5)–(7), we have a valid theorem.

Theorem 2. *Let u_m be the classical solutions of problem (5)–(7) in the domain Π and let conditions a), b) be satisfied. Then for $u_m(t, x)$ the estimate is true*

$$|u_m| \leq \sum_{k=1}^{N} \prod_{\lambda=k}^{N} (1 + \|b_\lambda^{(m)}; \Pi \cap (t = t_\lambda)\|_0)(\|\varphi_{(k-1)}^{(m)}; \Pi \cap (t = t_{k-1})\|_0 +$$
$$+ t_k \|f_m; \Pi^{(k-1)}\|_0) e^{a t_k} + (\|\varphi_N^{(m)}; \Pi \cap (t = t_N)\|_0 + t_{N+1} \|f_m; \Pi^{(N)}\|_0) e^{a t_{N+1}}. \tag{8}$$

Proof. Inequality (8) can be proved by the scheme of proving in Theorem 2.5 from [11, p. 27]. In view of this theorem for the solution of problem (5)–(6) in $\Pi^{(0)}$ we estimate

$$\|u_m; \Pi^{(0)}\|_0 \leq (\|\varphi_0^{(m)}; \Pi^{(0)} \cap (t = t_0)\|_0 + t_1 \|f_m; \Pi^{(0)}\|_0) e^{a t_1}. \tag{9}$$

If $k \geq 1$, then from given inequality (7) we obtain inequalities

$$\|u_m; \Pi^{(k)}\|_0 \leq [(1 + \|b_k^{(m)}; \Pi^{(k)} \cap (t = t_k)\|_0)(\|u_m; \Pi^{(k-1)}\|_0 +$$
$$+ \|\varphi_k^{(m)}; \Pi^{(k)} \cap (t = t_k)\|_0 + t_{k+1} \|f_m; \Pi^{(k)}\|_0)] e^{a t_{k+1}}, \quad k \in \{1, 2, \ldots, N\}. \tag{10}$$

Combining inequalities (9), (10) we obtain estimate (8). □

Theorem 3. *If conditions a), b) are satisfied, then the inequality solving problem (5)–(7) is*

$$\|u_m; \gamma; \beta; 0; \Pi\|_{2+\alpha} \leq c \Bigg\{ \sum_{k=1}^{N} \Big[\prod_{\lambda=k}^{N} (1 + \|b_\lambda^{(m)}; \Pi \cap (t = t_\lambda)\|_0) \times$$
$$\times (\|\varphi_{k-1}^{(m)}; \tilde{\gamma}; \tilde{\beta}; 0; \Pi \cap (t = t_{k-1})\|_{2+\alpha} + \|f_m; \gamma; \beta; \mu_0; \Pi^{(k-1)}\|_\alpha) + \tag{11}$$
$$+ \|\varphi_N^{(m)}; \tilde{\gamma}; \tilde{\beta}; 0; \Pi \cap (t = t_N)\|_{2+\alpha} + \|f_m; \gamma; \beta; \mu_0; \Pi^{(N)}\|_\alpha \Big] \Bigg\}.$$

Proof. Using the definition of the norm and interpolation inequalities from [12 p. 38], we have

$$\|u_m; \gamma; \beta; 0; \Pi^{(k)}\|_{2+\alpha} \leq (1 + \varepsilon^\alpha) \langle u_m; \gamma; \beta; 0; \Pi^{(k)} \rangle_{2+\alpha} + c(\varepsilon) \|u_m; \Pi^{(k)}\|_0,$$

where ε is an arbitrary real number, $\varepsilon \in (0, 1)$. It is sufficient to evaluate the half norm $\langle u_m; \gamma; \beta; 0; \Pi^{(k)} \rangle_{2+\alpha}$. From the definition of the half norm it follows that there exist the points $P_1^{(k)}$, $R^{(k)}$, $H^{(k)}$ in $\Pi^{(k)}$ for which one of the following inequalities holds

$$\frac{1}{2} \|u_m; \gamma; \beta; 0; \Pi^{(k)}\|_{2+\alpha} \leq E_\delta, \quad \delta \in \{1, 2\}, \tag{12}$$

where

$$E_1 = \sum_{2j+|r|=2} \sum_{\nu=1}^{n} d_1((2+\alpha)\gamma^{(1)}, t^{(1)}) \prod_{i=1}^{n} d_1(-r_i\beta_i^{(1)}, t^{(1)}) d_2(r_i(\gamma^{(2)} - \beta_i^{(2)}), \tilde{x}) \times$$
$$\times d_1(-\alpha\beta_\nu^{(1)}, t^{(1)}) d_2(-\alpha(\gamma^{(2)} - \beta_\nu^{(2)}, \tilde{x}) |x_\nu^{(1)} - x_\nu^{(2)}|^{-\alpha} |\partial_t^j \partial_x^r u(P_1^{(k)}) - \partial_t^j \partial_x^r u(H^{(k)})|,$$
$$E_2 = \sum_{2j+|r|=2} d_1((2+\alpha)\gamma^{(1)}; \tilde{t}) d_2((2-|r|+\alpha)\gamma^{(2)}; x^{(2)}) \times$$
$$\times \prod_{i=1}^{n} d_1(-r_i\beta_i^{(1)}, \tilde{t}) d_2(r_i(\gamma^{(2)} - \beta_i^{(2)}, x^{(2)}) |t^{(1)} - t^{(2)}|^{-\frac{\alpha}{2}} |\partial_t^j \partial_x^r u(R^{(k)}) - \partial_t^j \partial_x^r u(H^{(k)})|,$$

$d_1(a, \tilde{t}) = \min(d_1(a, t^{(1)}), d_1(a, t^{(2)}))$, $d_2(a, \tilde{x}) = \min(d_2(a, x^{(1)}), d_2(a, x^{(2)}))$.

If $|x_\nu^{(1)} - x_\nu^{(2)}| \geq n^{-1} d_1(\gamma^{(1)} - \beta_\nu^{(1)}, t^{(1)}) d_2(\gamma^{(2)} - \beta_\nu^{(2)}, \tilde{x}) \frac{\varepsilon_1}{4} \equiv T_1$, then

$$E_1 \leq 2\varepsilon_1^{-\alpha} \|u_m; \gamma; \beta; 0; \Pi^{(k)}\|_2. \tag{13}$$

If $|t^{(1)} - t^{(2)}| \geq d_1(2\gamma^{(1)}, \tilde{t}) d_2(2\gamma^{(2)}, x^{(2)}) \frac{\varepsilon_1}{16} \equiv T_2$, then

$$E_2 \leq 2\varepsilon_1^{-\alpha} \|u_m; \gamma; \beta; 0; \Pi^{(k)}\|_2. \tag{14}$$

Applying the interpolation inequalities to (13), (14), we obtain

$$E_\delta \leq \varepsilon^\alpha \|u_m; \gamma; \beta; 0; \Pi^{(k)}\|_{2+\alpha} + c(\varepsilon) \|u_m; \Pi^{(k)}\|_0. \tag{15}$$

Let $|x_\nu^{(1)} - x_\nu^{(2)}| \leq T_1$ and $|t^{(1)} - t^{(2)}| \leq T_2$. Assume that

$$d_1(2\gamma^{(1)}, \tilde{\tau}) = d_1(2\gamma^{(1)}, t^{(1)}),$$

$d_2(2\gamma^{(2)}, \tilde{x}) = d_2(2\gamma^{(2)}, x^{(2)})$, $(t^{(1)}, x^{(2)}) \in \Pi^{(k)}$. In the domain $\Pi^{(k)}$ we write problem (5)–(7) in the form

$$(L_2 u_m) \equiv \Big(\partial_t - \sum_{ij=1}^{n} a_{ij}(t^{(1)}, x^{(2)}) \partial_{x_i} \partial_{x_j}\Big) u_m = \sum_{ij=1}^{n} [a_{ij}(t,x) - a_{ij}(t^{(1)}, x^{(2)})] \partial_{x_i} \partial_{x_j} u_m -$$
$$- \sum_{i=1}^{n} a_i(t,x) \partial_{x_i} u_m - a_0(t,x) u_m + f_m(t,x) \equiv F_m(t,x;u_m) + f_m(t,x), \tag{16}$$
$$u_m(t_k + 0, x) = \psi_m^{(k)}(t_k, x), \tag{17}$$

where $\psi_m^{(0)}(t_0, x) = \varphi_0^{(m)}(x)$, $x \in \mathbb{R}^n$;

$$\psi_m^{(k)}(t_k, x) = [1 + b_k^{(m)}(x)] u_m(t_k - 0, x) + \varphi_k^{(m)}(x), x \in \Pi^{(k)} \cap (t = t_k), \quad k \in \{1, 2, \ldots, N\}.$$

Let V_{ε_2} be a domain from $\Pi^{(k)}$, $V_{\varepsilon_2} = \{(t,x) \in \Pi^{(k)}, |t - t^{(1)}| \leq \varepsilon_2^2 T_2,\ |x_i - x_i^{(2)}| \leq \varepsilon_2 T_1,\ i \in \{1, \ldots, n\}\}$. In problem (16), (17) we make the substitution $u_m(t,x) = v_m(t,y)$, $y_i = d_1(\beta_i^{(1)}, t^{(1)}) d_2(\beta_i^{(2)}, x^{(2)}) x_i$. As a result, we deduce

$$(L_2 v_m)(t,y) = \Big[\partial_t - \sum_{ij=1}^{n} a_{ij}(t^{(1)}, x^{(2)}) d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)}) \times$$

$$\times d_2(\beta_i^{(2)}, x^{(2)}) d_2(\beta_j^{(2)}, x^{(2)}) \partial_{y_i} \partial_{y_j} \Big] v_m = F_m(t, \tilde{y}; v_m) + f_m(t, \tilde{y}), \tag{18}$$

$$v_m(t_k + 0, y) = \psi_m^{(k)}(t_k, \tilde{y}), \tag{19}$$

where $\tilde{y} = (d_1(-\beta_1^{(1)}, t^{(1)}) d_2(-\beta_1^{(2)}, x^{(2)}) y_1, \ldots, d_1(-\beta_n^{(1)}, t^{(1)}) d_2(-\beta_n^{(2)}, x^{(2)}) y_n)$.

Denote $y_i^{(2)} = d_1(\beta_i^{(1)}, t^{(1)}) d_2(\beta_i^{(2)}, x^{(2)}) x_i^{(2)}$, $W_{\varepsilon_2}^{(k)} = \{(t, y), |t - t^{(1)}| \leq \varepsilon_2^2 T_2, |y_i - y_i^{(2)}| \leq \varepsilon_2 T_2^{1/2}\}$ and take the triply differentiable function $\mu(t, y)$, that satisfies the conditions

$$\mu(t,y) = \begin{cases} 1, & (t,y) \in W_{1/2}^{(k)}, 0 \leq \mu(t,y) \leq 1; \\ 0, & (t,y) \notin W_{3/4}^{(k)}, |\partial_t^j \partial_x^r \mu| \leq \\ & \leq c_{jr} d_1(-2j\gamma^{(1)}; t^{(1)}) \prod_{i=1}^n d_2(-r_i \gamma^{(2)}, x^{(2)}). \end{cases}$$

In view of (18), (19), the function $Z_m(t, y) = v_m(t, y)\mu(t, y)$ is a solution of the Cauchy problem

$$(L_2 Z_m)(t, y) = \sum_{ij=1}^n a_{ij}(t^{(1)}, x^{(2)}) d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)}) d_2(\beta_i^{(2)}, x^{(2)}) d_2(\beta_j^{(2)}, x^{(2)}) \times$$

$$\times [\partial_{y_i} \mu \partial_{y_j} v_m + \partial_{y_j} \mu \partial_{y_i} v_m] + v_m \Big[\sum_{ij=1}^n a_{ij}(t^{(1)}, x^{(2)}) d_1(\beta_i^{(1)}, t^{(1)}) d_1(\beta_j^{(1)}, t^{(1)}) \times$$

$$\times d_2(\beta_j^{(2)}, t^{(2)}) d_2(\beta_j^{(2)}, x^{(2)}) \partial_{y_i} \partial_{y_j} \mu - \partial_t \mu \Big] + \mu [F_m + f_m] \equiv F_m^{(1)} + \mu f_m, \tag{20}$$

$$Z_m(t_k + 0, x) = \psi_m^{(k)}(t_k, \tilde{y}) \mu(t_k, y). \tag{21}$$

By Theorem 5.1 from [11, p. 364], for the solution of problem (20), (21) inequalities hold

$$|y^{(1)} - y^{(2)}|^{-\alpha} |\partial_t^j \partial_y^r Z_m(t, y^{(1)}) - \partial_t^j \partial_y^r Z_m(t, y^{(2)})| \leq c(\|F_m^{(1)} + \mu F_m\|_{C^\alpha(W_{3/4}^{(k)})} + \tag{22}$$

$$+ \|\mu \psi_m^{(k)}\|_{C^{2+\alpha}(W_{3/4}^{(k)} \cap (t = t_k))}) \equiv B_m,$$

$$|t^{(1)} - t^{(2)}|^{-\alpha/2} |\partial_t^j \partial_y^r Z_m(t^{(1)}, y) - \partial_t^j \partial_y^r Z_m(t^{(2)}, y)| \leq c_1 B_m, \tag{23}$$

where $\{(t, y^{(1)}), (t, y^{(2)}), (t^{(1)}, y), (t^2, y)\} \subset W_{1/4}^{(k)}$, $2j + |r| = 2$.

Taking into account the properties of the function $\mu(t, y)$, we find

$$\|F_m^{(1)} + \mu f_m\|_{C^\alpha(W_{3/4}^{(k)})} \leq$$

$$\leq c_2 d_1(-(2+\alpha)\gamma^{(1)}; t^{(1)}) d_2(-(2+\alpha)\gamma^{(2)}; x^{(2)}) (\|v_m; \gamma; 0; 0; W_{3/4}^{(k)}\|_2 +$$

$$+ \|v_m; W_{3/4}^{(k)}\|_0 + \|f_m; \gamma; 0; 2\gamma; W_{3/4}^{(k)}\|_\alpha + \|F_m; \gamma; 0; 2\gamma; W_{3/4}^{(1)}\|_\alpha), \tag{24}$$

$$\|\mu \psi_m^{(k)}\|_{C^{2+\alpha}(W_{3/4}^{(k)} \cap (t = t_k))} \leq c_3 d_1(-(2+\alpha)\gamma^{(1)}, t^{(1)}) d_2(-(2+\alpha)\gamma^{(2)}, x^{(2)}) \times$$

$$\times \|\psi_m^{(k)}; \tilde{\gamma}; 0; 0; W_{3/4}^{(k)} \cap (t = t_k)\|_{2+\alpha}, \tag{25}$$

Substituting (24), (25) into (22), (23) and returning to the variables (t, x), we obtain

$$E_\delta \leq c_4(\|F_m; \gamma; \beta; 2\gamma; V_{3/4}^{(k)}\|_\alpha + \|f_m; \gamma; \beta; 2\gamma; V_{3/4}^{(k)}\|_\alpha +$$

$$+ \|u_m; V_{3/4}^{(k)}\|_0 + \|u_m; \gamma; \beta; 0; V_{3/4}^{(k)}\|_2 + \|\psi_m^{(k)}; \tilde{\gamma}; \tilde{\beta}; 0; V_{3/4}^{(k)} \cap (t = t_k)\|_{2+\alpha}). \tag{26}$$

Taking into account the interpolation inequalities and estimates of the norm of each addend in the expressions F_m, $\psi_m^{(k)}$, we obtain the inequalities

$$E_\delta \leq (\varepsilon_1^\alpha(n+2) + \varepsilon^2 n^2)\|u_m; \gamma; \beta; 0; V_{3/4}^{(k)}\|_{2+\alpha} + c_4\|u_m; V_{3/4}^{(k)}\|_0 +$$
$$+ c_5(\|f_m; \gamma; \beta; \mu_0; V_{3/4}^{(k)}\|_\alpha + \|\psi_m^{(k)}; \tilde{\gamma}; \tilde{\beta}; 0; V_{3/4}^{(k)} \cap (t = t_k)\|_{2+\alpha}), \tag{27}$$

Where

$$\|\psi_m^{(0)}; \tilde{\gamma}; \tilde{\beta}; 0; V_{3/4}^{(0)} \cap (t = t_0)\|_2 \leq c_6\|\varphi_0^{(m)}; \tilde{\gamma}; \tilde{\beta}; 0; \mathbb{R}^n\|_{2+\alpha},$$
$$\|\psi_m^{(k)}; \tilde{\gamma}; \tilde{\beta}; 0; V_{3/4}^{(k)} \cap (t = t_k)\|_{2+\alpha} \leq c_7(1 + \|b_k^{(m)}; \Pi \cap (t = t_k)\|_0) \times$$
$$\times (\|u_m; \gamma; \beta; 0; \Pi^{(k-1)}\|_{2+\alpha} + \|\varphi_k^{(m)}; \tilde{\gamma}; \tilde{\beta}; 0; \Pi^{(k)} \cap (t = t_k)\|_{2+\alpha}), \tag{28}$$

for $k \in \{1, 2, \ldots, N\}$.

Using inequalities (8), (12), (15), (26)–(28) and choosing ε and ε_1 sufficiently small, we obtain the estimate (11). □

Proof of Theorem 1. Since

$$\|f_m; \gamma; \beta; \mu_0; \Pi^{(k)}\|_\alpha \leq c\|f; \gamma; \beta; \mu_0; \Pi^{(k)}\|_\alpha,$$
$$\|\varphi_m^{(k)}; \tilde{\gamma}; \tilde{\beta}; 0; \Pi^{(k)} \cap (t = t_k)\|_{2+\alpha} \leq c\|\varphi_k; \tilde{\gamma}; \tilde{\beta}; 0; \Pi^{(k)} \cap (t = t_k)\|_{2+\alpha},$$

and estimate (11) holds, for the solution of problem (5)–(7) the estimate is true

$$\|u_m; \gamma; \beta; 0; \Pi\|_{2+\alpha} \leq B. \tag{29}$$

The right-hand side of inequality (29) does not depend on $m = (m_1, m_2)$. Moreover, the sequences
$\{U_m^{(0)}\} = \{u_m\}$,
$\{U_m^{(1)}\} = \{d_1(\gamma^{(1)}, t)d_2(\gamma^{(2)} - \beta_i^{(2)}, x)\partial_{x_i} u_m(t, x)\}$,
$\{U_m^{(2)}\} = \{d_1(2\gamma^{(1)}; t)d_2(\gamma^{(2)} - \beta_i^{(2)}, x)d_2(\gamma^{(2)} - \beta_j^{(2)}, x)\partial_{x_i}\partial_{x_j} u_m(t, x)\}$,
$\{U_m^{(3)}\} = \{d_1(2\gamma^{(1)}; t)d_2(2\gamma^{(2)}, x)\partial_t u_m(t, x)\}$
are uniformly bounded and equicontinuous in the domain $\overline{Q}^{(k)}$. According to the Arzela theorem, there exist subsequences $\{U_{\tilde{m}_k}^{(\nu)}\}$, uniformly convergent in $\overline{Q}^{(k)}$ to $\{U_0^{(\nu)}\}$, $\nu \in \{0, 1, 2, 3\}$. Since $Q^{(k)}$ is an arbitrary domain, $\overline{Q}^{(k)} \subset \Pi^{(k)}$, then passing to the limit for $m_{1k} \to \infty$, $m_{2k} \to \infty$ in problem (5)–(7), we find that $u(t, x) = U_0^{(0)}$ is the unique solution of problem (1)–(3) in the space $H^{2+\alpha}(\gamma; \beta; 0; \Pi)$ and its estimate (4) holds. □

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