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POLYNOMIAL COMPLEX GINZBURG-LANDAU EQUATIONS IN ZHIDKOV SPACES

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We consider the so-called complex Ginzburg-Landau equations with a polynomial nonlinearity in the real line. We prove existence results concerned with the initial value problem for these equations in Zhidkov spaces with a new approach using splitting methods.

1. Introduction. We consider the 1-dimensional autonomous system

$$\begin{cases} \partial_t u = (\alpha + i\beta)\partial_{xx} u + \gamma u + (a+ib)B(u), \\ u(0) = u_0 \end{cases} \tag{1}$$

where $u(x,t)$ is a complex valued function with $x \in \mathbb{R}$, $t > 0$, $\alpha > 0, \beta > 0, \gamma \geq 0$, $a > 0$, $b > 0$ and B a continuous map. The linear term represented by $(\alpha + i\beta)\partial_{xx}$ characterizes the complex Ginzburg-Landau equations. For $\beta = 0$ (1) reduces to a nonlinear heat equation and for $\alpha = 0$ to a nonlinear Schrödinger equation. A large amount of work has been done to prove well-posedness of (1) with different nonlinearities (See, for instance, [1, 14, 15]).

In this paper, we analyze well-posedness for the polynomial complex Ginzburg-Landau equation in Zhidkov spaces by applying splitting methods for abstract semilinear evolution equations [5, 8]. These techniques were used to achieve well-posedness results for the fractional reaction-diffusion equation [3]. Zhidkov spaces were introduced by P. Zhidkov in [19] which consist of functions defined on $\mathbb{R}$, bounded and uniformly continuous, with derivatives up to k order in L^2. These spaces are applied in nonlinear optics to model dark solitons, these are solutions of the form $u(x,t) = u_v(x - vt)$. For instance, in [9] dark soliton solutions are described for a complex Ginzburg-Landau equation. This type of solutions are important in many other problems concerning, for instance, Schrödinger equations [13] and KdV Dark equations [7, 18, 4]. A typical example of a function in Zhidkov spaces is described in [13, 16]:

$$u_v(x) = \sqrt{1 - \frac{v^2}{2}} \tanh\left(\sqrt{1 - \frac{v^2}{2}} \frac{x}{\sqrt{2}}\right) + i\frac{v}{\sqrt{2}}.$$

Our aim is to prove the existence of solutions in Zhidkov spaces with $k = 1$ in the context of polynomial nonlinearities, i.e. $B(u) = -u^n$. Such nonlinearities appear not only in complex

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Ginzburg-Landau equations but also in other models, such as FitzHugh-Nagumo equations [2, 12] and Fisher-Kolgomorov equations [11, 17].

The paper is organized as follows: In Section 2 we set notations and preliminary results. In Section 3 we analyze the nonlinear problem. Finally, in Section 4 using splitting methods, we combine results from Sections 2 and 3 to obtain that the solution of (1) is in a Zhidkov space.

2. Notations and Preliminaries. We introduce some definitions and preliminary results.

Definition 1. We define $C_{\mathrm{u}}(\mathbb{R})$ as the set of uniformly continuous and bounded functions on $\mathbb{R}$.

Definition 2. We denote the Zhidkov spaces as, for $k > d/2$,

$$X^k(\mathbb{R}^d) = \{u \in L^\infty(\mathbb{R}^d) \cap C_u(\mathbb{R}^d) : \ \partial_j \in L^2(\mathbb{R}^d), 1 \leq |j| \leq k\}$$

equipped with the norm:

$$\|u\|_{X^k} = \|u\|_{L^\infty} + \sum_{1 \leq |a| \leq k} \|\partial_a u\|_{L^2} \tag{2}$$

Remark 1. Zhidkov spaces are closed for the norm defined in (2) (see [13]).

The following definitions and proofs can be extended to $x \in \mathbb{R}^d$ (see [10]).

Definition 3. We denote $U(t)$ as the one parameter semigroup that solves the underlying linear equation

$$\partial_t u = (\alpha + i\beta)\partial_{xx} u + \gamma u \tag{3}$$

The operator can be represented by the convolution in x

$$U(t) = (4\pi t(\alpha + i\beta))^{-1/2} e^{(-x^2/[4t(\alpha+i\beta)])+\gamma t} * u_0 = G_t(x) * u_0$$

and the kernel G_t satisfies

$$|G_t(x)| = (4\pi t(\alpha^2 + \beta^2))^{-1/2} e^{(-x^2/[4t(\alpha^2+\beta^2)])+\gamma t}.$$

Clearly, $G_t(x) \in L^1(\mathbb{R})$.

Proposition 1. *The one-parameter family* $\{U(t)\}_{t\geq 0}$ *of operators defined as* $U(t)u_0 = G_t * u_0$ *is a strongly continuous semigroup on* $C_u(\mathbb{R})$.

Proof. The proof is similar to Proposition 2.2 in [3]. □

Lemma 1. *If* $u_0 \in X^1(\mathbb{R})$ *then* $U(t)u_0 \in X^1(\mathbb{R})$ *for* $t > 0$

Proof. As $u_0 \in L^\infty(\mathbb{R})$ and $G_t(x) \in L^1(\mathbb{R})$ then using Young's inequality we have

$$\|G_t * u_0\|_{L^\infty} \leq \|G_t\|_{L^1} \|u_0\|_{L^\infty}.$$

On the other hand, we obtain

$$\|\partial_x (G_t * u_0)\|_{L^2} = \|G_t * \partial_x u_0\|_{L^2} \leq \|G_t\|_{L^1} * \|\partial_x u_0\|_{L^2}$$

As $G_t \in L^1(\mathbb{R})$ and $\partial_x u_0 \in L^2(\mathbb{R})$ we have the result. □

Remark 2. Similarly, if $x \in \mathbb{R}^n$ and we have k derivatives of $U(t)u_0$, a similar procedure proves that $U(t)u_0 \in X^k$.

Next, we consider integral solutions of the problem (1). We say that $u \in C([0,T], C_{\mathrm{u}}(\mathbb{R}))$ is a mild solution of (1) if and only if u verifies

$$u(t) = U(t)u_0 + \int_0^t U(t-t')B(u(t'))dt'. \tag{4}$$

If F is a locally Lipschitz map, for any $z_0 \in C_{\mathrm{u}}(\mathbb{R})$ there exists the unique solution of the equation

$$\begin{cases} \partial_t z = B(z), \\ z(0) = z_0, \end{cases} \tag{5}$$

defined in the interval $[0, T^*(z_0))$. Moreover, there exists a nonincreasing function $\bar{T}\colon [0,\infty) \to [0,\infty)$, such that $T^*(z_0) \geq \bar{T}(|z_0|)$. The solution of (5) is solution of the integral equation

$$z(t) = z_0 + \int_0^t B(z(t'))dt'. \tag{6}$$

Also, one of the following alternatives holds:

— $T^*(z_0) = \infty$;

— $T^*(z_0) < \infty$ and $|z(t)| \to \infty$ when $t \uparrow T^*(z_0)$.

We will denote by $\mathsf{N}(t,.)\colon C_{\mathrm{u}}(\mathbb{R}) \to C_{\mathrm{u}}(\mathbb{R})$ the flow generated by the ordinary equation, i.e., for any $x \in \mathbb{R}$, $\mathsf{N}(t,u_0)(x)$ is the solution of the problem (5) with initial data $z_0 = u_0(x)$. Therefore, if $u(t) = \mathsf{N}(t, u_0)$

$$u(x,t) = u_0(x) + \int_0^t B(u(x,t'))dt'$$

We recall well-known local existence results for evolution equations.

Theorem 1. *There exists a function $T^*\colon C_{\mathrm{u}}(\mathbb{R}) \to \mathbb{R}_+$ such that for $u_0 \in C_{\mathrm{u}}(\mathbb{R})$, there exists the unique $u \in C([0,T^*(u_0)), C_{\mathrm{u}}(\mathbb{R}))$ mild solution of (1) with $u(0) = u_0$. Moreover, one of the following alternatives holds:*

- $T^*(u_0) = \infty$;
- $T^*(u_0) < \infty$ *and* $\lim_{t\uparrow T^*(u_0)} |u(t)| = \infty$.

Proof. See Theorem 4.3.4 in [6]. □

Proposition 2. *Under conditions of the theorem above, we have the following statements:*

1. $T^*\colon C_{\mathrm{u}}(\mathbb{R}) \to \mathbb{R}_+$ *is lower semi-continuous;*
2. *If $u_{0,n} \to u_0$ in $C_{\mathrm{u}}(\mathbb{R})$ and $0 < T < T^*(u_0)$, then $u_n \to u$ in the Banach space $C([0,T], C_{\mathrm{u}}(\mathbb{R}))$.*

Proof. See Proposition 4.3.7 in [6]. □

3. Nonlinear equation. In order to apply the Lie-Trotter method, we prove that the solution of the nonlinear ordinary equation $z(t) \in X^1(\mathbb{R})$. We analyze different related nonlinearities, a cubic nonlinearity $B(u) = -(a+ib)z^3$ (FitzHugh-Nagumo nonlinearity), a quadratic nonlinearity $B(u) = -(a+ib)z^2$ (Fisher-Kolmogorov nonlinearity), and an n-th degree nonlinearity $B(u) = -(a+ib)z^n$, generalizing the previous cases.

3.1. Cubic nonlinearity. We study the solution for the nonlinear equation (5) with a cubic nonlinearity, that is

$$\begin{cases} \partial_t z = -(a+ib)z^3, \\ z(0) = z_0, \end{cases} \tag{7}$$

Lemma 2. *If $u_0(x) = z_0 \in X^1(\mathbb{R})$ then the solution of the equation* (7), $z(t) \in X^1(\mathbb{R})$ *for* $t \in (0, T^*(z_0))$.

Proof. The solutions for this ODE are

$$z(t) = \pm \frac{1}{\sqrt{\frac{1}{z_0^2} + 2t(a+ib)}} \tag{8}$$

Clearly, if $z_0 \in L^\infty(\mathbb{R})$ then $z(t) \in L^\infty(\mathbb{R})$. The spatial derivative of $z(t)$ is

$$\partial_x z(t) = \pm \frac{\partial_x z_0}{z_0^3(\frac{1}{z_0^2} + 2t(a+ib))^{\frac{3}{2}}} \tag{9}$$

We are interested in the L^2 norm. Using Hölder inequality we obtain

$$\|\partial_x z(t)\|_{L^2} = \|\frac{\partial_x z_0}{z_0^3(\frac{1}{z_0^2} + 2t(a+ib))^{\frac{3}{2}}}\|_{L^2} \leq \|\partial_x z_0\|_{L^2} \|\frac{1}{z_0^3(\frac{1}{z_0^2} + 2t(a+ib))}\|_{L^\infty}$$

As $z_0 \in X^1(\mathbb{R}$ then $\partial_x z_0 \in L^2$. On the other hand $z_0 \in X^1(\mathbb{R})$, means that $z_0 \in L^\infty(\mathbb{R})$. Therefore $z(t) \in X^1(\mathbb{R})$. □

Remark 3. We observe that solution (8) and the spatial derivative (9) are not defined for all $x \in \mathbb{R}$ such that $z_0 = 0$. For these cases, it is proved in [3] that we have global well-posedness inside invariant convex regions.

3.2. Fisher-Kolmogorov nonlinearity. We study the solution for a Fisher-Komogorov nonlinearity, that is a quadratic nonlinearity,

$$\begin{cases} \partial_t z = -(a+ib)z^2, \\ z(0) = z_0. \end{cases} \tag{10}$$

Lemma 3. *If $u_0(x) = z_0 \in X^1(\mathbb{R})$ then the solution of the equation* (10), $z(t) \in X^1(\mathbb{R})$ *for* $t \in (0, T^*(z_0))$.

Proof. The solution of this ODE is:

$$z(t) = \frac{z_0}{(a+ib)tz_0 + 1} \tag{11}$$

Clearly, if $z_0 \in L^\infty(\mathbb{R})$ then $z(t) \in L^\infty(\mathbb{R})$. The spatial derivative of $z(t)$ is:

$$\partial_x z(t) = \frac{\partial_x z_0}{((a+ib)tz_0^2 + 1)^2} \tag{12}$$

Applying the L^2 norm and using Hölder inequality we obtain

$$\|\partial_x z(t)\|_{L^2} = \|\frac{\partial_x z_0}{((a+ib)tz_0^2 + 1)^2}\|_{L^2} \leq \|\partial_x z_0\|_{L^2} \|\frac{1}{((a+ib)tz_0^2 + 1)^2}\|_{L^\infty}$$

As $z_0 \in X^1(\mathbb{R}$ then $\partial_x z_0 \in L^2$. On the other hand $z_0 \in X^1(\mathbb{R})$, means that $z_0 \in L^\infty(\mathbb{R})$. Therefore $z(t) \in X^1(\mathbb{R})$. □

3.3. General case. We analyze the ODE with an n-th degree nonlinearity for $n > 2$.

$$\begin{cases} \partial_t z = -(a+ib)z^n, \\ z(0) = z_0, \end{cases} \tag{13}$$

Lemma 4. *If $u_0(x) = z_0 \in X^1(\mathbb{R})$ then the solution of the equation* (13), *$z(t) \in X^1(\mathbb{R})$ for $t \in (0, T^*(z_0))$.*

Proof. The solution of this ODE is:

$$z(t) = \frac{1}{((n-1)(a+ib)t + z_0^{1-n})^{\frac{1}{(n-1)}}} \tag{14}$$

Clearly, if $z_0 \in L^\infty(\mathbb{R})$ then $z(t) \in L^\infty(\mathbb{R})$. The spatial derivative of $z(t)$ is:

$$\partial_x z(t) = \frac{\partial_x z_0}{z_0^n((n-1)(a+ib)t + z_0^{1-n})^{1+\frac{1}{n-1}}} \tag{15}$$

Taking the L^2 norm and using Hölder inequality we obtain

$$\begin{aligned} \Big\|\partial_x z(t)\|_{L^2} &= \|\frac{\partial_x z_0}{z_0^n((n-1)(a+ib)t + z_0^{1-n})^{1+\frac{1}{n-1}}}\Big\|_{L^2} \leq \\ &\leq \Big\|\partial_x z_0\|_{L^2} \|\frac{1}{z_0^n((n-1)(a+ib)t + z_0^{1-n})^{1+\frac{1}{n-1}}}\Big\|_{L^\infty} \end{aligned}$$

As $z_0 \in X^1(\mathbb{R})$ then $\partial_x z_0 \in L^2$. On the other hand $z_0 \in X^1(\mathbb{R})$, means that $z_0 \in L^\infty(\mathbb{R})$. Therefore $z(t) \in X^1(\mathbb{R})$. □

Remark 4. As we mentioned in Remark 3, the well-posedness result is limited by all $x \in \mathbb{R}$ where the solution does not blow-up.

4. Splitting method. This section is based on the splitting method developed in [8]. We apply the Lie-Trotter method to the linear and nonlinear problem. The temporal variable must be broken down into regular intervals and the evolution of the linear and nonlinear problems are considered alternately. This is described by two sequences $\{V_{h,k}\}$ for the linear equation and $\{W_{h,k}\}$ for the nonlinear equation. Using Theorem 3.9 from [8], this approximate solution converges to the solution of problem (1), when the time intervals $h = t/n \to 0$.

Let X a Banach space and we define $\alpha : \mathbb{R} \to \mathbb{R}$ a periodic function of period 1 as

$$\alpha(t) = \begin{cases} 2, & \text{if } k \le t < k + 1/2, \\ 0, & \text{if } k - 1/2 \le t < k, \end{cases}$$

for $k \in \mathbb{Z}$.

Given $h > 0$, we define the function $\alpha_h : \mathbb{R} \to \mathbb{R}$ as $\alpha_h(t) = \alpha(t/h)$. Clearly $0 \le \alpha_h \le 2$, α_h is h-periodic and its mean value is 1.

We consider $\tau_h : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\tau_h(t, t') = \int_{t'}^{t} \alpha_h(t'')dt'',$$

We define $\Omega = \{(t, t') \in \mathbb{R}^2 : 0 \le t' \le t\}$ and $U_h : \Omega \to \mathcal{B}(X)$ given by

$$U_h(t, t') = U(\tau_h(t, t')).$$

We consider the system,

$$\begin{cases} \partial_t u_h + \alpha_h(t)(-\Delta)u_h(x, t) = (2 - \alpha_h(t))F(u_h(x, t)), \\ u_h(x, 0) = u_{h0}(x) \end{cases}$$

where $u(x, t) \in X,\ t > 0$ and $F : X \to X$ is a continuous function.

Similarly, we define the integral equation:

$$u_h(t) = U_h(t, 0)u_{h0} + \int_0^t (2 - \alpha_h(t'))U_h(t, t')F(u_h(t'))dt' \tag{16}$$

The following two theorems are a consequence of Sections 2 and 3 of [8].

Theorem 2. *Let u_h the solution of* (16), *if $W_{h,k} = u_h(kh)$ y $V_{h,k} = u_h(kh - h/2)$, then*

$$V_{h,k+1} = U(h)U_{h,k}, \tag{17a}$$
$$W_{h,k+1} = N(kh + h, kh + h/2, V_{h,k+1}), \tag{17b}$$

where N is the flux asociated to $2F$, that is:

$$\begin{cases} \dot{w} = 2F(w(t)), \\ w(0) = w_0, \end{cases}$$

Proof. For $t_1 \in (0, t)$ it verifies

$$u_h(t) = U_h(t, t_1)u_{h0}(t_1) + \int_{t_1}^t (2 - \alpha_h(t'))U_h(t, t')F(u_h(t'))dt'$$

using that $t_1 = kh$ y $t = kh + h/2$, we have

$$V_{h,k+1} = U_h(kh + h/2, kh)W_{h,k} + \int_{kh}^{kh+h/2} (2 - \alpha_h(t'))U_h(kh + h/2, t')F(u_h(t'))dt',$$

given that $\alpha_h(t) = 2$ for $t \in [kh, kh + h/2)$, we have $\tau_h(kh + h/2, kh) = h$ and therefore (17a). Similarly, $\alpha_h(t) = 0$ for $t \in [kh + h/2, kh + h)$, then $\tau_h(t, kh + h/2) = 0$ and therefore

$$u_h(t) = V_{h,k+1} + 2\int_{kh+h/2}^{t} F(u_h(t'))dt',$$

evaluating in $t = kh + h$, we obtain (17b). □

Theorem 3. *Let $u \in C([0, T^*), X)$ the solution of the integral problem* (4)

$$u(t) = U(t)u_0 + \int_0^t U(t - t')F(u(t'))dt',$$

$T \in (0, T^*)$ and $\varepsilon > 0$. There exists $h^* > 0$ such that if $0 < h < h^*$, then u_h the solution of (16) *with $u_h(x, 0) = u_0(x)$, is defined in the interval $[0, T]$ and verifies $\|u(t) - u_h(t)\|_X \leq \varepsilon$ for $t \in [0, T]$.*

Proof. See Theorem 3.9 from [8]. □

We apply Lemma 1 from Section 2 related to the linear equation and Lemmas 2, 3 and 4 from Section 3 related to the nonlinear equations. In order to obtain well-posedness results for the solution $u(t)$ of equation (1), we use Theorem 3 to join the linear and nonlinear results. The following theorem is proved for the cubic case but the other cases are similar.

Theorem 4. *Let $u_0 \in X^1(\mathbb{R})$, then the solution of* (1) *$u(t) \in X^1(\mathbb{R})$ for $t \in (0, T^*(u_0))$.*

Proof. For $t \in [0, \min\{T^*(u_0)\})$, let $n \in \mathbb{N}$, $h = t/n$ and $\{W_{h,k}\}_{0\leq k\leq n}, \{V_{h,k}\}_{1\leq k\leq n}$ be the sequences given by $W_{h,0} = u_0$,

$$V_{h,k+1} = U(h)W_{h,k}, \tag{18a}$$
$$W_{h,k+1} = \mathsf{N}(h, V_{h,k+1}), \quad k = 0, \ldots, n - 1. \tag{18b}$$

We claim that $W_{h,k} \in X^1(\mathbb{R})$ for $k = 0, \ldots, n$. Clearly, the assertion is true for $k = 0$. If $W_{h,k-1} \in X^1(\mathbb{R})$, from Lemma 1, we have $U(h)W_{h,k-1} \in X^1(\mathbb{R})$. Using Lemma 2, we can see that

$$W_{h,k} = \mathsf{N}(h, V_{h,k}) \in X^1(\mathbb{R}).$$

By Theorem 4 we have that $W_{h,n} \to u(t)$ when $n \to \infty$.

As $X^1(\mathbb{R})$ is closed, we obtain the result.

□

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