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# GROWTH OF PTH MEANS OF THE POISSON-STIELTJES INTEGRALS IN THE POLYDISC

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We prove a sharp bound estimate of the $p$th means of the Poisson-Stieltjes integrals in the unit polydisc for $p > 1$. The estimate is given in terms of the smoothness of a complex-valued Stieltjes measure $\mu$. If the measure $\mu$ is positive, the estimate becomes equivalent to the smoothness condition.

Let $z = (z_1, ..., z_n) \in \mathbb{C}^n$, $|z| = \max\{|z_j|\colon j = 1, ..., n\}$ be the polydisc norm in $\mathbb{C}^n$, $n \in \mathbb{N}$, $U^n = \{z \in \mathbb{C}^n\colon |z| < 1\}$ be the unit polydisc with the skeleton $T^n = \{z \in \mathbb{C}^n\colon |z_j| = 1, 1 \leq j \leq n\}$.

For $z \in U^n, z_j = r_j e^{i\varphi_j}, w_j = e^{i\theta_j}, 1 \leq j \leq n$ we denote by

$$\mathcal{P}(z, w) = \prod_{j=1}^{n} P_0(z_j, w_j)$$

the Poisson kernel for the unit polydisc, where

$$P_0(z_j, w_j) = \mathrm{Re}\frac{w_j + z_j}{w_j - z_j} = \frac{1 - r_j^2}{1 - 2r_j\cos(\varphi_j - \theta_j) + r_j^2}$$

is the Poisson kernel for the unit disc.

The function $P\colon U^n \to \mathbb{R}$ defined by the equality

$$P[d\mu](z) = \int_{T^n} \mathcal{P}(z, w)\, d\mu(w),$$

is called the Poisson-Stieltjes integral of a finite Stieltjes measure with $|\mu|(T^n) < +\infty$, where $|\mu| = |\mu_1 + i\mu_2| = \mu_1^+ + \mu_1^- + \mu_2^+ + \mu_2^-$, $\mu_j^+$, $\mu_j^-$, $j \in \{1, 2\}$ being the positive and the negative variation of the measure $\mu_j$, respectively.

For a function $u$ on $U^n$ and $0 \leq r < 1$, by $u_r$ we denote the function, defined on $T^n$ by the equality

$$u_r(w) = u(rw), \quad w \in T^n.$$

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Let $\psi\colon [-\pi;\pi]\to\mathbb{R}$. The classes of functions of bounded variation and integrated in the $p$th degree on $[-\pi;\pi]$ are denoted by $BV$ and $L^p$, respectively.

We write for $u_r\in L^p(T^n)$, $p>0$,

$$\|u_r\|_p=\left(\frac{1}{(2\pi)^n}\int\limits_{-\pi}^{\pi}\dots\int\limits_{-\pi}^{\pi}|u_r(e^{i\varphi_1};\dots;e^{i\varphi_n})|^p d\varphi_1\dots d\varphi_n\right)^{\frac{1}{p}}.$$

In [6] (Theorem 2.1.3 (c)) the next theorem was proved.

**Theorem A ([6]).** *If $1\le p<\infty, f\in L^p(T^n)$, and $u=P[f]$, then $\|u_r\|_p\le\|f\|_p$ and $\|u_r-f\|_p\to 0$ as $r\to 1^-$.*

The growth of $L_p$-norm in terms of Stieltjes measure was described for analytic and harmonic functions in the unit disc, represented by the generalized Poisson-Stieltjes integral, in the paper [2].

If $\psi\in L^p, p\ge 1$, then by

$$\omega_p(\delta,\psi)=\sup_{0\le h\le\delta}\left(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}|\psi(x+h)-\psi(x)|^p dx\right)^{\frac{1}{p}}$$

we denote the integral modulus of continuity. As in [7] we say that $\psi\in\Lambda^p_\gamma$ if $\psi\in L^p$ and $\omega_p(\delta,\psi)=O(\delta^\gamma)(\delta\downarrow 0)$.

If $u$ is a harmonic function in unit disc $U$, $\alpha\ge 0$, then by $u_\alpha(re^{i\varphi})=r^{-\alpha}D^{-\alpha}u(re^{i\varphi})$ we denote the Riemann-Liouville fractional integral [5] in the variable $r$ and

$$M_p(r,u)=\left(\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}|u(re^{it})|^p dt\right)^{\frac{1}{p}},\quad P_\alpha(r,t)=D^\alpha(r^\alpha P_0(r,t)).$$

The fractional integration allows to reduce the case of the power growth to the case of a harmonic function with uniformly bounded $L_1$-norm.

**Theorem B ([2]).** *Let $u$ be a harmonic function in $U$, $p\in[1;+\infty), 0<\gamma\le 1$ and $\alpha>\gamma-1$. In order that $u(re^{i\varphi})=\int_{-\pi}^{\pi}P_\alpha(r,\varphi-\theta)d\psi(\theta)$, where $\psi\in BV\cap\Lambda^p_\gamma$, it is necessary and sufficient for $\gamma<1$, and it is necessary for $\gamma=1$ , that $\sup\limits_{0<r<1}\int_0^{2\pi}|u_\alpha(re^{i\varphi})|d\varphi=M_\alpha<+\infty$ and $M_p(r,u)=O\left((1-r)^{\gamma-\alpha-1}\right), r\uparrow 1$.*

We say that $\mu\in H^{(\beta_1,\dots,\beta_n)}$ if there exists some $C>0$ such that

$$\sup_{\varphi\in[-\pi;\,\pi]^n}|\mu|\ \left(\left\{(e^{i\theta_1},\dots,e^{i\theta_n})\in T^n\colon\ |\theta_j-\varphi_j|\le\delta^{\frac{1}{\beta_j}},1\le j\le n\right\}\right)\le C\delta,\quad 0<\delta<1.$$

The sharp estimate of the growth of the Poisson-Stieltjes integral in the polydisc in terms of the Stieltjes measure $\mu$ from the class $H^{(\beta_1,\dots,\beta_n)}$ is obtained in [4]. Such classes are introduced in [1].

**Theorem C ([4]).** *Let $\mu$ be a finite Borel measure on $T^n$, $n\in\mathbb{N}$, $0<\beta_j<2$, $\frac{1}{\beta_*}=\frac{1}{\beta_1}+\ldots+\frac{1}{\beta_n}$, $1\le j\le n$. In order that*

$$\left|\int\limits_{T^n}\mathcal{P}\left(z,w\right)d\mu\left(w\right)\right|\le C\delta^{1-\frac{1}{\beta_*}},\quad 0<\delta<1,\ |z_j|=1-\delta^{\frac{1}{\beta_j}},$$

*for some positive constant $C$, it is sufficient, and for nonnegative $\mu$ it is necessary that $\mu \in H^{(\beta_1,\dots,\beta_n)}$.*

If $p \geq 1$, then the integral modulus of continuity $\omega_p(\delta;\beta;\mu)$ is defined by

$$\omega_p(\delta;\beta;\mu) =$$

$$= \sup_{|\varphi_1|\leq\delta^{\frac{1}{\beta_1}},\dots,|\varphi_n|\leq\delta^{\frac{1}{\beta_n}}} \left( \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} (\mu((\theta_1;\theta_1+\varphi_1] \times \dots \times (\theta_n;\theta_n+\varphi_n]))^p d\theta_1 \dots d\theta_n \right)^{\frac{1}{p}},$$

where $0 < \delta < 1, \beta = (\beta_1, \dots, \beta_n), \beta_j > 0, 1 \leq j \leq n$.

We say that $\mu \in H_\sigma^{(\beta_1;\dots;\beta_n);p}$ if $\mu \in L^p$ and

$$\omega_p(\delta;\beta;\mu) \leq C \cdot \delta^\sigma, \tag{1}$$

where $C = \text{const}, 0 < \delta < 1, \sigma > 0$.

**Remark 1.** We note that for $k > 0$ the conditions $\mu \in H_\sigma^{(\beta_1;\dots;\beta_n);p}$ and $\mu \in H_{\sigma k}^{(\frac{\beta_1}{k};\dots;\frac{\beta_n}{k});p}$ are equivalent. It follows directly from the definition of the class $H_\sigma^{(\beta_1;\dots;\beta_n);p}$.

Due to the remark above it is easy to check that the Lebesgue measure $\lambda_n$ on $T^n$ satisfies $\lambda_n \in H_{1/\beta^*}^{(\beta_1,\beta_2,\dots,\beta_n);p}$, where $\frac{1}{\beta^*} = \frac{1}{\beta_1} + \dots + \frac{1}{\beta_n}$, in particular, $\lambda_n \in H_k^{(\frac{n}{k};\dots;\frac{n}{k});p}$, $k > 0$.

In this paper we describe the growth of $p$th means of the Poisson-Stieltjes integrals in the polydisc in terms of belonging of the Stieltjes measure $\mu$ to the class $H_\sigma^{(\beta_1;\dots;\beta_n);p}$.

Denote $w = (e^{i\theta_1}, \dots, e^{i\theta_n}) \in T^n$.

**Theorem 1.** *1.1. Let $\mu$ be a finite complex-valued Stieltjes measure on $T^n$, $n \in \mathbb{N}, p \in (1;+\infty)$, $\sigma > 0, \beta_j > 0, 1 \leq j \leq n$, $\frac{1}{\beta^*} = \frac{1}{\beta_1} + \dots + \frac{1}{\beta_n}$. If $\mu \in H_\sigma^{(\beta_1;\dots;\beta_n);p}$, then*

$$\left\| \int_{T^n} \mathcal{P}(z,w)\, d\mu(w) \right\|_p \leq C\delta^{\sigma - \frac{1}{\beta_*}}, \tag{2}$$

*where $0 < \delta < 1$, $|z_j| = 1 - \delta^{\frac{1}{\beta_j}}$, $C$ is some positive constant.*

*1.2. Let $\mu$ be a nonnegative Stieltjes measure on $T^n$, $n \in \mathbb{N}, p \in (1;+\infty)$, $\sigma > 0$, $\beta_j > 0$, $1 \leq j \leq n$, $\frac{1}{\beta^*} = \frac{1}{\beta_1} + \dots + \frac{1}{\beta_n}$. If for some positive constant $C$*

$$\left\| \int_{T^n} \mathcal{P}(z,w)\, d\mu(w) \right\|_p \leq C\delta^{\sigma - \frac{1}{\beta_*}}, \tag{3}$$

*then $\mu \in H_\sigma^{(\beta_1;\dots;\beta_n);p}$.*

**Corollary 1.** *Let $\mu$ be a nonnegative Stieltjes measure on $T^n$, $n \in \mathbb{N}, p \in (1;+\infty)$ $\sigma > 0$, $\beta_j > 0, 1 \leq j \leq n$, $\frac{1}{\beta^*} = \frac{1}{\beta_1} + \dots + \frac{1}{\beta_n}$. In order that $\mu \in H_\sigma^{(\beta_1;\dots;\beta_n);p}$ it is necessary and sufficient that*

$$\left\| \int_{T^n} \mathcal{P}(z,w)\, d\mu(w) \right\|_p \leq C\delta^{\sigma - \frac{1}{\beta_*}},$$

*where $0 < \delta < 1$, $|z_j| = 1 - \delta^{\frac{1}{\beta_j}}$, $C$ is some positive constant.*

**Remark 2.** The vector parameter $\beta$ shows relative rate of approaching of the coordinates to the skeleton. In particular, if all $\beta_j$ are equal, then the coordinates of $z$ tend to 1 with the same speed.

*Proof of Theorem 1.1.* For fixed $\delta \in (0,1)$ we set $r_j = 1-\delta^{\frac{1}{\beta_j}}$ and let $z_j = r_j e^{i\varphi_j}$, $1 \le j \le n$.

We denote

$$R_{m_1\dots m_n} = \left\{ \begin{array}{r} \left(e^{i\theta_1},...,e^{i\theta_n}\right) \in T^n\colon\ (m_j-1)\,\delta^{\frac{1}{\beta_j}} \le |\theta_j-\varphi_j| \le m_j\delta^{\frac{1}{\beta_j}},\ if\ m_j>1, \\ |\theta_j-\varphi_j| \le \delta^{\frac{1}{\beta_j}}\ ,\ if\ m_j=1,\ 1\le j\le n \end{array} \right\}.$$

Then

$$T^n = \bigcup_{m_1=1}^{N_1} \dots \bigcup_{m_n=1}^{N_n} R_{m_1\dots m_n},$$

where $N_j = [\frac{\pi^{\beta_j}}{\delta}]+1,\ 1\le j\le n.$

As in [6, p.28] we have that

$$\left| \int\limits_{R_{m_1\dots m_n}} \mathcal{P}\left(z,w\right)d\mu\left(w\right) \right| \le |\mu|\left(R_{m_1\dots m_n}\right)\cdot \prod_{j=1}^{n} P_0\left(|z_j|,\tilde{w}_{m_j}\right),$$

where $z\in U^n,\ \tilde{w}_m = \left(\tilde{w}_{m_1},...,\tilde{w}_{m_n}\right),\ \tilde{w}_{m_j} = e^{i\tilde{\theta}_{m_j}},\ \tilde{\theta}_{m_j} = (m_j-1)\,\delta^{\frac{1}{\beta_j}}$, for $1\le j\le n, m_j\ge 1$.

The following estimates are well-known [6, p.31]:

$$P_0\left(re^{i\varphi},e^{i\theta}\right) \le \frac{\pi^2}{\left(\varphi-\theta\right)^2}\left(1-r\right),\quad P_0\left(re^{i\varphi},e^{i\theta}\right) \le \frac{2}{1-r}, \tag{4}$$

where $|\varphi-\theta|\le\pi$, $0<r<1$.

Applying the first estimate, we get

$$P_0\left(|z_j|,\tilde{w}_{m_j}\right) \le \frac{\pi^2}{\tilde{\theta}^2_{m_j}}\left(1-r_j\right) = \frac{\pi^2}{(m_j-1)^2\cdot\delta^{\frac{1}{\beta_j}}} \le \frac{4\pi^2}{m_j^2\cdot\delta^{\frac{1}{\beta_j}}},\quad m_j>1, |z_j| = 1-\delta^{\frac{1}{\beta_j}}. \tag{5}$$

For $m_j = 1$ we obtain

$$P_0\left(|z|,\tilde{w}_0\right) \le \frac{2}{1-r_j} = \frac{2}{m_j^2\delta^{\frac{1}{\beta_j}}}.$$

Therefore, (5) holds for all $m_j\ge 1$.

Now we can estimate the integral over the sets $R_{m_1\dots m_n}$. For $N = \max\limits_j N_j$ we get

$$\left| \sum_{m_1=1}^{N}\dots\sum_{m_n=1}^{N}\ \int\limits_{R_{m_1\dots m_n}} \mathcal{P}\left(z,w\right)d\mu\left(w\right) \right| \le C\sum_{m_1=1}^{N}\dots\sum_{m_n=1}^{N}\ \int\limits_{R_{m_1\dots m_n}} \prod_{j=1}^{n}\frac{1}{m_j^2\delta^{\frac{1}{\beta_j}}}|d\mu(w)| \le$$

$$\le \frac{C}{\prod\limits_{j=1}^{n}\delta^{\frac{1}{\beta_j}}}\sum_{m_1=1}^{N}\dots\sum_{m_n=1}^{N}\frac{|\mu|(R_{m_1\dots m_n})}{\prod\limits_{j=1}^{n}m_j^2} = \frac{C}{\prod\limits_{j=1}^{n}\delta^{\frac{1}{\beta_j}}}\cdot\sum_{m_1=1}^{N}\dots\sum_{m_n=1}^{N}\frac{|\mu|(R_{m_1\dots m_n})}{\prod\limits_{j=1}^{n}m_j}\cdot\frac{1}{\prod\limits_{j=1}^{n}m_j}.$$

Using the Hölder inequality ($\frac{1}{p}+\frac{1}{q}=1, p>1, q>1$) and the arguments similar to those in [3], we get

$$\left|\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\int\limits_{R_{m_1...m_n}}\mathcal{P}(z,w)d\mu(w)\right|^p\le$$

$$\le\frac{C}{\prod\limits_{j=1}^{n}\delta^{\frac{p}{\beta_j}}}\cdot\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\frac{|\mu|^p(R_{m_1...m_n})}{\prod\limits_{j=1}^{n}m_j^p}\left(\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\prod_{j=1}^{n}m_j^{-q}\right)^{\frac{p}{q}}\le$$

$$\le\frac{C}{\prod\limits_{j=1}^{n}\delta^{\frac{p}{\beta_j}}}\cdot\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\frac{|\mu|^p(R_{m_1...m_n})}{\prod\limits_{j=1}^{n}m_j^p}.$$

Then for $r_j=1-\delta^{\frac{1}{\beta_j}}$, $1\le j\le n$, $z=re^{i\varphi}=(r_1e^{i\varphi_1},...,r_ne^{i\varphi_n})$

$$\int\limits_{[-\pi;\pi]^n}\left|\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\int\limits_{R_{m_1...m_n}}\mathcal{P}\left(re^{i\varphi},w\right)d\mu(w)\right|^p d\varphi_1...d\varphi_n\le$$

$$\le\frac{C}{\prod\limits_{j=1}^{n}\delta^{\frac{p}{\beta_j}}}\cdot\int\limits_{[-\pi;\pi]^n}\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\frac{|\mu|^p(R_{m_1...m_n})}{\prod\limits_{j=1}^{n}m_j^p}d\varphi_1...d\varphi_n\le$$

$$\le\frac{C}{\prod\limits_{j=1}^{n}\delta^{\frac{p}{\beta_j}}}\cdot\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\frac{1}{\prod\limits_{j=1}^{n}m_j^p}\int\limits_{[-\pi;\pi]^n}|\mu|^p(R_{m_1...m_n})d\varphi_1...d\varphi_n.$$

Using (1) and the latter inequality, we deduce

$$\int\limits_{[-\pi;\pi]^n}\left|\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\int\limits_{R_{m_1...m_n}}\mathcal{P}(z,w)d\mu(w)\right|^p d\varphi_1...d\varphi_n\le$$

$$\le\frac{C}{\prod\limits_{j=1}^{n}\delta^{\frac{p}{\beta_j}}}\cdot\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\frac{1}{\prod\limits_{j=1}^{n}m_j^p}\cdot\omega_p^p(\delta;\beta;\mu)\le\frac{C}{\prod\limits_{j=1}^{n}\delta^{\frac{p}{\beta_j}}}\cdot\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\frac{\delta^{p\sigma}}{\prod\limits_{j=1}^{n}m_j^p}\le$$

$$\le\frac{C\cdot\delta^{p\sigma}}{\prod\limits_{j=1}^{n}\delta^{\frac{p}{\beta_j}}}\cdot\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\prod_{j=1}^{n}\frac{1}{m_j^p}\le C\cdot\delta^{p(\sigma-(\frac{1}{\beta_1}+...+\frac{1}{\beta_n}))}\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\prod_{j=1}^{n}\frac{1}{m_j^p}.$$

Since $p>1$, we have

$$\int\limits_{[-\pi;\pi]^n}\left|\sum_{m_1=1}^{N}...\sum_{m_n=1}^{N}\int\limits_{R_{m_1...m_n}}\mathcal{P}(z,w)d\mu(w)\right|^p d\varphi_1...d\varphi_n\le C\delta^{p(\sigma-\frac{1}{\beta^*})},$$

where $\frac{1}{\beta^*}=\frac{1}{\beta_1}+...+\frac{1}{\beta_n}$. Thus, $\left(\int\limits_{[-\pi;\pi]^n}\left|\int\limits_{T^n}\mathcal{P}(z,w)d\mu(w)\right|^p d\varphi_1...d\varphi_n\right)^{\frac{1}{p}}\le C\delta^{\sigma-\frac{1}{\beta^*}}$.

*Proof of 1.2.* As in [4] for $w \in R_{1...1} = \left\{ \left(e^{i\theta_1}, ..., e^{i\theta_n}\right) \in T^n \colon \ |\theta_j - \varphi_j| \le \delta^{\frac{1}{\beta_j}}, \ m_j = 1, \ 1 \le j \le n \right\}$ one has $P_0\left(r_j e^{i\varphi_j}, w_j\right) \ge C\delta^{-\frac{2}{\beta_j}} \cdot (1 - r_j)$. Since the measure is nonnegative as well as the Poisson kernel, we obtain:

$$\int\limits_{[-\pi;\pi]^n} \Big( \sum_{m_1=1}^{N} ... \sum_{m_n=1}^{N} \int\limits_{R_{m_1...m_n}} \mathcal{P}\left(re^{i\varphi}, w\right) d\mu\left(w\right) \Big)^p d\varphi_1 ... d\varphi_n \ge$$

$$\ge \int\limits_{[-\pi;\pi]^n} \Big( \int\limits_{R_{1...1}} \mathcal{P}\left(re^{i\varphi}, w\right) d\mu\left(w\right) \Big)^p d\varphi_1 ... d\varphi_n \ge$$

$$\ge \int\limits_{[-\pi;\pi]^n} \Big( \int\limits_{R_{1...1}} \prod_{j=1}^{n} \frac{C(1-r_j)}{\delta^{\frac{2}{\beta_j}}} d\mu\left(w\right) \Big)^p d\varphi_1 ... d\varphi_n \ge$$

$$\ge \int\limits_{[-\pi;\pi]^n} \prod_{j=1}^{n} \frac{C\delta^{\frac{p}{\beta_j}}}{\delta^{\frac{2p}{\beta_j}}} \mu^p(R_{1...1}) d\varphi_1 ... d\varphi_n \ge C \prod_{j=1}^{n} \frac{1}{\delta^{\frac{p}{\beta_j}}} \int\limits_{[-\pi;\pi]^n} \mu^p(R_{1...1}) d\varphi_1 ... d\varphi_n.$$

In view of (3) $\delta^{p(\sigma - \frac{1}{\beta^*})} \ge C \prod_{j=1}^{n} \delta^{-\frac{p}{\beta_j}} \cdot \int_{T^n} \mu^p(R_{1...1}) d\varphi_1 ... \varphi_n$. Therefore,

$\int_{[-\pi;\pi]^n} \mu^p(R_{1...1}) d\varphi_1 ... d\varphi_n \le C \cdot \delta^{p\sigma}$ and $\omega_p(\delta; \beta; \mu) \le C \cdot \delta^{\sigma}$.

□

**Example.** Let $\mu$ be the unitary mass, concentrated at the point $\overline{w}(1, ..., 1) \in T^n$. Then

$$\int\limits_{T^n} \mathcal{P}(z, \overline{w}) d\mu(\overline{w}) = \frac{1 - |z_1|^2}{|1 - z_1|^2} \cdot ... \cdot \frac{1 - |z_n|^2}{|1 - z_n|^2}.$$

For a set $E \subset [-\pi, \pi]^n$ we define $\mu^*(E) = \mu(\{(e^{i\theta_1}; ...; e^{i\theta_n}) \in T^n \colon (\theta_1, ..., \theta_n) \in E\})$. We write $E_{(\varphi_1, ..., \varphi_n)} = \{(\theta_1, ..., \theta_n) \in [-\pi; \pi]^n \colon \ -\varphi_1 \le \theta_1 \le 0; ...; -\varphi_n \le \theta_n \le 0\}$. Note that

$$\mu^*\big(\mathop{\times}_{j=1}^{n} (\theta_j, \theta_j + \varphi_j]\big) = \begin{cases} 1, & (\theta_1, ..., \theta_n) \in E_{(\varphi_1, ..., \varphi_n)} \\ 0, & (\theta_1, ..., \theta_n) \notin E_{(\varphi_1, ..., \varphi_n)} \end{cases}, \quad |\varphi_j| < \pi, \quad j \in \{1, 2, \dots, n\}.$$

So,

$$\sup_{|\varphi_1| \le \delta^{\frac{1}{\beta_1}}, ..., |\varphi_n| \le \delta^{\frac{1}{\beta_n}}} \left( \frac{1}{(2\pi)^n} \int\limits_{-\pi}^{\pi} ... \int\limits_{-\pi}^{\pi} (\mu^*((\theta_1; \theta_1 + \varphi_1] \times ... \times (\theta_n; \theta_n + \varphi_n]))^p d\theta_1 ... d\theta_n \right)^{\frac{1}{p}} =$$

$$= \sup_{|\varphi_1| \le \delta^{\frac{1}{\beta_1}}, ..., |\varphi_n| \le \delta^{\frac{1}{\beta_n}}} \left( \frac{1}{(2\pi)^n} \int\limits_{E_{(\varphi_1, ..., \varphi_n)}} d\theta_1 ... d\theta_n \right)^{\frac{1}{p}} = \sup_{|\varphi_1| \le \delta^{\frac{1}{\beta_1}}, ..., |\varphi_n| \le \delta^{\frac{1}{\beta_n}}} \left( \frac{|\varphi_1| \cdot ... \cdot |\varphi_n|}{(2\pi)^n} \right)^{\frac{1}{p}} \asymp$$

$$\asymp \left( \delta^{\frac{1}{\beta_1}} \cdot ... \cdot \delta^{\frac{1}{\beta_n}} \right)^{\frac{1}{p}} = \left( \delta^{\frac{1}{\beta^*}} \right)^{\frac{1}{p}}. \tag{6}$$

Comparing (1) to (6), we obtain that $\frac{1}{\beta^*} \cdot \frac{1}{p} = \sigma, \beta^* = \frac{1}{p\sigma}$. Thus $\mu \in H^{(\beta_1, ..., \beta_n); p}_{\frac{1}{p\beta^*}}$.

Let us now estimate the Poisson-Stieltjes integral directly:

$$\int\limits_{[-\pi,\pi]^n} \left(\frac{1-|z_1|^2}{|1-z_1|^2}\cdot\ldots\cdot\frac{1-|z_n|^2}{|1-z_n|^2}\right)^p d\varphi_1 \ldots d\varphi_n =$$

$$=\int\limits_{-\pi}^{\pi}\left(\frac{1-{r_1}^2}{|1-r_1e^{i\varphi_1}|^2}\right)^p d\varphi_1 \ldots \int\limits_{-\pi}^{\pi}\left(\frac{1-{r_n}^2}{|1-r_ne^{i\varphi_n}|^2}\right)^p d\varphi_n \asymp$$

$$\asymp (1-{r_1}^2)^p\cdot\ldots\cdot(1-{r_n}^2)^p\cdot\frac{1}{(1-r_1)^{2p-1}}\cdot\ldots\cdot\frac{1}{(1-r_n)^{2p-1}}\asymp\frac{1}{\delta^{\frac{p-1}{\beta_1}}}\cdot\ldots\cdot\frac{1}{\delta^{\frac{p-1}{\beta_n}}}\asymp\frac{1}{\delta^{\frac{p-1}{\beta^*}}},$$

where $1-r_j=\delta^{\frac{1}{\beta_j}}, 1\le j\le n$. Thus,

$$\left(\int\limits_{T^n}\left(\frac{1-|z_1|^2}{|1-z_1|^2}\cdot\ldots\cdot\frac{1-|z_n|^2}{|1-z_n|^2}\right)^p d\varphi_1 \ldots d\varphi_n\right)^{\frac{1}{p}}\asymp\frac{1}{\delta^{(p-1)\sigma}}.$$

It coincides with the conclusion of Theorem 1.1:

$$\left\|\int\limits_{T^n}\mathcal{P}\left(z,w\right)d\mu\left(w\right)\right\|_p\asymp C\delta^{\sigma-\frac{1}{\beta^*}}.$$

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