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HILBERT TRANSFORM ON W_σ^1

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We obtain a boundedness criterion for the Hilbert transform on the Paley–Wiener space in terms of decomposition. As an application we obtain a simple method of evaluation of the Hilbert transform.

1. Introduction. The Hilbert transform is regularly used in signal processing, mainly, due the property to extend real functions into analytic functions. Signal processing can be considered as a subfield of mathematics, information and electrical engineering that concerns the analysis, synthesis and modification of signals, which are broadly defined as functions conveying 'information about the behavior or attributes of some phenomenon', such as sound, images and biological measurements [1]. The Hilbert transform has many uses, including solving problems in aerodynamics, condensed matter physics, optics, fluids, and engineering. Physicists are familiar with idea of integral transforms primarily through the Fourier transform. It is the most famous of a variety of possibilities including the Hilbert transform. This is useful in mathematics because a problem presented in terms of the transformed variable might be easier to solve than the original. F. W. King presented a method to deal with the numerical evaluation of Kramers–Kroning transforms [2]. Yi-Wen Liu in [3] presented Hilbert transform as an analytic extension problem. He considered several ways to calculate the Hilbert transform.

Definition 1. The *Hilbert transform* of a function $\psi\colon \mathbb{R}\to\mathbb{C}$ is defined for all $z\in\mathbb{R}$ by

$$H(\psi(z))=\frac{1}{\pi}\cdot \text{v.p.}\int\limits_{-\infty}^{+\infty}\frac{\psi(z)}{t-z}dz, \tag{1}$$

when the integral exists.

It is well-known that $H(L^2(\mathbb{R}))=L^2(\mathbb{R})$, but $H(L^1)\neq L^1$. We obtain the criterion of boundedness of the Hilbert transform on the Paley–Wiener space $W_\sigma^1\subset L^1(\mathbb{R})$ and a simple calculation method of Hilbert transform using decomposition.

The Hardy space $H^p(\mathbb{C}^+)$, $1\le p<+\infty$, is the space of analytic function in the half-plane $\mathbb{C}^+=\{z\colon \operatorname{Im} z>0\}$ for which (see [4])

$$\|f\|:=\sup_{y>0}\left\{\int\limits_{-\infty}^{+\infty}|f(x+iy)|^p dx\right\}^{1/p}<+\infty.$$

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Also, by Sedleckii (see [5]) the space $H^p(\mathbb{C}_+)$, $1 < p < +\infty$, can be defined as the space of analytic in $\mathbb{C}^+$ functions f such that

$$\|f\|_* := \sup_{\varphi\in(-\frac{\pi}{2};\frac{\pi}{2})} \left\{ \int\limits_0^{+\infty} |f(re^{i\varphi})|^p dx \right\}^{1/p} < +\infty.$$

We denote by W^p_σ, $\sigma > 0$, the Paley–Wiener space, i.e., the space of entire functions f of exponential type $\le \sigma$ belonging to $L^p(\mathbb{R})$. Space W^p_σ can be defined [6] as the space of entire functions satisfying the condition

$$\sup_{\varphi\in(0;2\pi)} \left\{ \int\limits_0^{+\infty} |f(re^{i\varphi})|^p e^{-p\sigma r|\sin\varphi|} dr \right\}^{1/p} < +\infty. \tag{2}$$

Theorem 1 ([6]). *The space W^2_σ coincides with the space of functions represented as*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\sigma}^{\sigma} \varphi(it)e^{itz}dt, \quad \varphi \in L^2(-i\sigma; i\sigma). \tag{3}$$

Theorem 2 ([12]). *The space W^1_σ coincides with the space of function f represented by (3), where*

$$\varphi(t) = \sum_{k=-\infty}^{+\infty} (-1)^k c_k e^{-\frac{ik\pi t}{\sigma}},$$

and $(c_k) \in l^1$,

$$\sum_{m=-\infty}^{+\infty} \left| \sum_{k=-\infty}^{+\infty} (-1)^{k+m} c_{k+m} \frac{k}{1+k^2} \right| < +\infty.$$

C. Eoff considered the space W^1_π ([14]) as the class of functions representable as cardinal series

$$f(z) = \sqrt{\frac{2}{\pi}} \sum_{n=-\infty}^{+\infty} c_n \frac{\sin \pi z}{z-n}. \tag{4}$$

2. The main result. The decomposition problem for functions in Paley–Wiener space W^1_σ into the sum of two functions each of them characterized by the module which is “big” only in the upper or lower half-planes was investigated by B. V. Vynnytskyi and his followers (see [7], [8]). R. S. Yulmukhametov in [9] solved the problem of decomposition (splitting) into the product of two functions for some subspace of the Paley–Wiener space. Yu. I. Lyubarskii studied the decomposition of functions with a triangle indicator diagram ([10]). I. E. Chyzhykov researched similar issues in terms of subharmonic functions ([11]).

This problem arises as a result of studies of the completeness function ([12]) and is interesting in the theory of integral operators, shift operator ([13]) and convolution type equations.

The exact formulation of this problem is the following.

Problem (see [12]). *What functions $f \in W^p_\sigma$, $1 \le p \le 2$, admit a decomposition $f = \chi_1 - \chi_2$ with entire functions χ_1, χ_2, where χ_1 belongs to the Hardy space H^p in the half-plane $\{z\colon \operatorname{Im}(z) > 0\}$ and χ_2 belongs to the Hardy space H^p in $\{z\colon \operatorname{Im}(z) < 0\}$.*

For case $p = 2$ a solution of the problem is elementary and is based on the Paley-Wiener theorem

$$\chi_1(z) = \frac{1}{\sqrt{2\pi}} \int_0^\sigma \varphi(it) e^{itz} dt, \quad \chi_2(z) = -\frac{1}{\sqrt{2\pi}} \int_{-\sigma}^0 \varphi(it) e^{itz} dt.$$

But the case $p = 1$ is the most interesting for applications.

Theorem 3 ([12])**.** *For $f \in W^1_\sigma$ the problem has a solution if and only if*

$$\sum_{m=-\infty}^{+\infty} \left| \sum_{k=-\infty}^{+\infty} c_k \frac{(-1)^{m+k} e^{i\pi\sigma m} - 1}{m - \delta m - k} \right| < +\infty, \tag{5}$$

for some $\delta \in (0;1)$ here we suppose that

$$\frac{(-1)^{m+k} e^{i\pi\sigma m} - 1}{m - \delta m - k} = \pi i, \quad \text{if} \quad m - \delta m - k = 0.$$

Note that for an arbitrary $\sigma > 0$ we can rewrite equality (4) as

$$f(t) = \sigma \sqrt{\frac{2}{\pi}} \sum_{k=-\infty}^{+\infty} c_k \frac{\sin \sigma t}{\sigma t - \pi k}. \tag{6}$$

We obtain the following result.

Theorem 4. *Let $f \in W^1_\sigma$, then the following conditions are equivalent:*

i) the representation $f = \chi_1 - \chi_2$ holds, where χ_1, χ_2 is entire functions, $\chi_1 \in H^1(\mathbb{C}^+)$, $\chi_2 \in H^1(\mathbb{C}^-)$;

ii) the Hilbert transform $H(f(z))$ belongs to $L^1(\mathbb{R})$;

iii) $(c_k) \in l^1$ in representation (6) of the function f and inequality (5) holds.

Proof Theorem 4. The proof of the theorem is divided into several stages. Firstly, notice that the equivalency of i) and iii) follows from Theorem 3 and Theorem 2.

Secondly, we showed the implication of i)$\Rightarrow$ ii). If the representation $f = \chi_1 - \chi_2$ true, then

$$H(f(t)) = \text{v.p.} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{t-z} dt = \text{v.p.} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\chi_1(t)}{t-z} dt - \text{v.p.} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\chi_2(t)}{t-z} dt.$$

Since $\chi_1 \in H^1(\mathbb{C}^+)$, we have by Cauchy integral formula

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\chi_1(t)}{t-z} dt = \begin{cases} \chi_1(z), & z \in \mathbb{C}^+, \\ 0, & z \in \mathbb{C}^-, \end{cases}$$

where $\mathbb{C}^- = \{\operatorname{Im} z < 0\}$. Therefore the non-tangential values on $\mathbb{R}$ from $\mathbb{C}^+$ of the integral

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\chi_1(t)}{t-z}dt$$

are equal almost everywhere to $\chi_1(z), z \in \mathbb{R}$. Analogously, we obtain

$$\frac{1}{2\pi i}\int_{+\infty}^{-\infty}\frac{\chi_2(t)}{t-z}dt = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\chi_2(t)}{t-z}dt = \begin{cases}\chi_2(z), & z \in \mathbb{C}^-,\\ 0, & z \in \mathbb{C}^+.\end{cases}$$

Since $\chi_1 \in L^1(-\infty;+\infty)$, $\chi_2 \in L^1(-\infty;+\infty)$, by Sochotsky's formula [16]

$$H(f) \in L^1(-\infty;+\infty).$$

Thirdly, let us prove ii)$\Rightarrow$ i). Consider the Hilbert transform of f.

By (6),

$$H(f(t)) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} c_k\frac{\sqrt{\frac{2}{\pi}}\sigma\frac{\sin\sigma t}{\sigma t-\pi k}}{t-z}dt = \frac{\sigma}{i\sqrt{2\pi^3}}\sum_{k=-\infty}^{+\infty} c_k\int_{-\infty}^{+\infty}\frac{\sin\sigma t}{(\sigma t-\pi k)(t-z)}dt.$$

The series is uniformly convergent and can be integrated term by term.

We calculate this integral using the residue theorem and Sochotsky's theorem [16]. For $\operatorname{Im} z > 0$ we get

$$\int_{-\infty}^{+\infty}\frac{\sin\sigma t}{(\sigma t-\pi k)(t-z)}dt = \frac{1}{2i}\int_{-\infty}^{+\infty}\frac{e^{i\sigma t}}{(\sigma t-\pi k)(t-z)}dt - \frac{1}{2i}\int_{-\infty}^{+\infty}\frac{e^{-i\sigma t}}{(\sigma t-\pi k)(t-z)}dt =$$
$$= \frac{1}{2i}\int_{-\infty}^{+\infty}f_1(t)dt - \frac{1}{2i}\int_{-\infty}^{+\infty}f_2(t)dt.$$

We can see that the poles of the function $f_1(t)$ are at z, $\frac{\pi k}{\sigma}$ and the pole of the function $f_2(t)$ is at $\frac{\pi k}{\sigma}$. Note that all these poles are simple. So,

$$\operatorname*{res}_{t=z} f_1(t) = \lim_{t\to z}(t-z)f_1(t) = \lim_{t\to z}\frac{(t-z)e^{i\sigma t}}{(\sigma t-\pi k)(t-z)} = \frac{e^{i\sigma z}}{\sigma z-\pi k},$$
$$\operatorname*{res}_{t=\frac{\pi k}{\sigma}} f_1(t) = \lim_{t\to\frac{\pi k}{\sigma}}\left(t-\frac{\pi k}{\sigma}\right)f_1(t) = \lim_{t\to z}\frac{(t-\frac{\pi k}{\sigma})e^{i\sigma t}}{(\sigma t-\pi k)(t-z)} = \frac{e^{i\pi k}}{\pi k-\sigma z},$$
$$\operatorname*{res}_{t=\frac{\pi k}{\sigma}} f_2(t) = \lim_{t\to\frac{\pi k}{\sigma}}\left(t-\frac{\pi k}{\sigma}\right)f_2(t) = \lim_{t\to z}\frac{(t-\frac{\pi k}{\sigma})e^{-i\sigma t}}{(\sigma t-\pi k)(t-z)} = \frac{e^{-i\pi k}}{\pi k-\sigma z}.$$

Therefore

$$\int_{-\infty}^{+\infty}\frac{\sin\sigma t}{(\sigma t-\pi k)(t-z)}dt = \pi\left(\operatorname*{res}_{t=z} f_1(t) + \frac{1}{2}\operatorname*{res}_{t=\frac{\pi k}{\sigma}} f_1(t) + \frac{1}{2}\operatorname*{res}_{t=\frac{\pi k}{\sigma}} f_2(t)\right) = \frac{\pi((-1)^k - e^{i\sigma z})}{\pi k-\sigma z}.$$

Analogously for $\operatorname{Im} z < 0$ we get

$$\int\limits_{-\infty}^{+\infty} \frac{\sin \sigma t}{(\sigma t - \pi k)(t - z)} dt = \frac{\pi((-1)^k - e^{-i\sigma z})}{\pi k - \sigma z}.$$

Consequently, by Sochotsky's formula ([16])

$$H(f(z)) = \frac{\sigma}{i\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k \left(\frac{(-1)^k - e^{i\sigma z}}{\pi k - \sigma z} + \frac{(-1)^k - e^{-i\sigma z}}{\pi k - \sigma z} \right) = \frac{\sigma}{i\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k \frac{(-1)^k - \cos \sigma z}{\pi k - \sigma z}.$$

We define

$$\chi_1(t) = \frac{f - iH(f)}{2}, \quad \chi_2(t) = \frac{-f - iH(f)}{2}.$$

Since

$$\chi_1(t) = \sum_{m=-\infty}^{+\infty} c_k \frac{e^{i\sigma z} - (-1)^k}{2i}$$

and the sum converges absolutely and uniformly on each compact of $\mathbb{C}^+$, χ_1 is an entire function of exponential type $\leq \sigma$. The function $f(z)$ is an entire function of exponential type $\leq \sigma$ and belongs to $L^1(\mathbb{R})$. Finally, $\chi_1 \in L^1(\mathbb{R})$ and $\chi_2 \in L^1(\mathbb{R})$. □

3. Example and corollary.

Example. The Hilbert transform of the function

$$f(t) = \frac{1 - \cos \sigma t}{t^2}$$

does not belong to $L^1(\mathbb{R})$.

We have,

$$\chi_1(z) = \frac{\pi(i - \sigma z + \sin \sigma z - i \cos \sigma z)}{2iz^2}, \quad \chi_2(z) = \frac{\pi(-i + \sigma z + \sin \sigma z + i \cos \sigma z)}{2iz^2}.$$

Therefore,

$$H(f(z)) = i(\chi_1(z) + \chi_2(z)) = \frac{\sin \sigma z - \sigma z}{z^2} \notin L^1(\mathbb{R}).$$

The following corollary gives a simple way for evaluation of the Hilbert transform.

Corollary. *If one of the conditions of Theorem 4 is fulfilled, then*

$$H(f(z)) = \frac{\sigma}{i\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} c_k \frac{(-1)^k - \cos \sigma z}{\pi k - \sigma z}.$$

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