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ON REMOVABLE SINGULARITIES OF MAPPINGS IN UNIFORM SPACES

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The paper is devoted to the study of mappings of two metric spaces that distort the modulus of families of paths by analogy with the Poletskiĭ inequality. We deal with the situation when the mapping acts in a space that admits weak sphericalization, while the corresponding extended metric space is uniform. For such mappings, the possibility of continuous extension to an isolated boundary point is proved and, as a consequence, an analogue of the Sokhotski–Casorati-Weierstrass theorem is obtained.

1. Introduction. This article is devoted to the study of mappings with bounded and finite distortion, which have been actively studied recently, see, e.g., [1]–[9].

In the recent joint publication of the first author, the continuous extension of mappings acting from one metric space to another, to an isolated point of the boundary, was obtained (see [8]). As a consequence, here was established that the image of an arbitrary neighborhood of an essential singular point under the mapping is dense in the appropriate metric space. The last statement is known as Sokhotski–Casorati-Weierstrass theorem, and it was obtained for mappings that distort the modulus of families of paths according to the Poletskiĭ type inequality. Note that the paper [8] dealt with the situation when the mapped space is Ahlfors regular and supports Poincaré inequality. In this article, we will consider a slightly different case, namely, we will assume that the image space under mapping is uniform. Note that the concept of uniform spaces goes back to Näkki and Palka [5].

In what follows, (X, d, μ) and (X', d', μ') are metric spaces with metrics d and d' and locally finite Borel measures μ and μ', correspondingly. Recall, for a given path $\gamma\colon [a, b] \to X$ its length is the supremum of the sums

$$\sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i-1}))$$

over all partitions $a = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_k = b$ of the interval $[a, b]$. The path γ is called *rectifiable* if its length is finite. Given a family of paths Γ in X, a Borel function $\rho\colon X \to [0, \infty]$ is called *admissible* for Γ, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int_{\gamma} \rho \, ds \geqslant 1$$

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for all (locally rectifiable) $\gamma \in \Gamma$.

In what follows, $\operatorname{adm}\Gamma$ is the set of all admissible functions for Γ. Everywhere further, for any sets E, F, and G in X, we denote by $\Gamma(E,F,G)$ the family of all paths $\gamma\colon [0,1] \to X$ such that $\gamma(0) \in E$, $\gamma(1) \in F$, and $\gamma(t) \in G$ for all $t \in (0,1)$. Everywhere further (X,d,μ) and (X',d',μ') are metric spaces with metrics d and d' and locally finite Borel measures μ and μ', correspondingly. We also assume that $\mu(B) > 0$ for all balls B in X. Given $p \geqslant 1$, the p-modulus of the family Γ is the number

$$M_p(\Gamma) = \inf_{\rho \in \operatorname{adm}\Gamma} \int\limits_X \rho^p(x)\, d\mu(x)\,.$$

Should $\operatorname{adm}\Gamma$ be empty, we set $M_p(\Gamma) = \infty$. A family of paths Γ_1 in X is said to be *minorized* by a family of paths Γ_2 in X, abbr. $\Gamma_1 > \Gamma_2$, if, for every path $\gamma_1 \in \Gamma_1$, there is a path $\gamma_2 \in \Gamma_1$ such that γ_2 is a restriction of γ_1. As known, see e.g. [10, Theorem 1(c)],

$$\Gamma_1 > \Gamma_2 \quad \Rightarrow \quad M_p(\Gamma_1) \ \leq \ M_p(\Gamma_2). \tag{1}$$

According to [4, Section 13.1], cf. [7, (1.7)], we say that a space X is *upper Ahlfors α-regular at a point* $x_0 \in X$ if there are $r_0 = r_0(x_0) > 0$ and $C = C(x_0) > 0$ such that

$$\mu(B(x_0,r)) \leqslant C \cdot r^{\alpha} \qquad \forall\, r \in (0,r_0)\,.$$

This definition differs from the generally accepted one, where the corresponding lower bound must also be fulfilled, and the constants r_0 and $C > 0$ should not depend on the point $x_0 \in X$ (see [11, inequalities (8.10)]).

Let $\overline{X} := X \cup \{\infty\}$, and let $h\colon \overline{X} \times \overline{X} \to \mathbb{R}$ be a metric. We say that h satisfies the *weak sphericalization condition,* if $(\overline{X},h)$ is a compact metric space while h and d generate the same topology on X (note that this definition is somewhat different from what we have in [8]). A metric space X is called a *space admitting a weak sphericalization*, if there exists a metric $h\colon \overline{X} \times \overline{X} \to \mathbb{R}$ satisfying the weak sphericalization condition. The most important particular case of sphericalization is the introduction of a chordal metric in an extended Euclidean space $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, see [9, Section 12], cf. [8]. Unless otherwise stated, the notion of a neighborhood of a point, as well as the sets ∂E and $\overline{E}$ for $E \subset \overline{X}$ are associated with the topology of the space $\overline{X}$. If we are talking about the mapping f between the spaces (X,d) and (X',d'), while X' admits a weak sphericalization $\left(\overline{X'},h\right)$, the continuity of f should also be understood in the sense of the metrics d and h, respectively.

If the space X admits weak sphericalization, then for the domain $G \subset \overline{X}$ and the sets $E,F \subset G$ we set

$$M_p(\Gamma(E,F,G)) := M_p(\Gamma(E \setminus \{\infty\}, F \setminus \{\infty\}, G \setminus \{\infty\}))\,.$$

Moreover, for a family of paths Γ on $\overline{X}$ we put $M_p(\Gamma) = M_p(\Gamma^*)$, where Γ^* consists of those and only those paths of Γ not passing through the point ∞.

Given $p \geqslant 2$, a space $\overline{X}$ is called *p-uniform* if, for each $r > 0$, there is $\delta = \delta(r) > 0$ such that $M_p(\Gamma(F,F^*,\overline{X})) \geqslant \delta$ whenever F and F^* are continua of $\overline{X}$ with $h(F) \geqslant r$ and $h(F^*) \geqslant r$. The extended Euclidean space is also a very successful and simple example of a uniform space, see [4, (7.29)]. In this case, the exponent p is equal to the dimension of the space n, see ibidem.

A set E is said to be *path connected* if any two points x_1 and x_2 in E can be joined by a path $\gamma\colon [0,1]\to E$, $\gamma(0)=x_1$ and $\gamma(1)=x_2$. Given a metric space (X,d,μ) with a measure μ, a *domain* in X is an open path connected set in X. We say that a domain G is *locally path connected* at a point $x_0\in\partial G$, if, for every neighborhood U of x_0, there is a neighborhood $V\subset U$ such that $V\cap G$ is path connected. Recall that X is *locally (path) connected* if every neighborhood of a point $x\in X$ contains a (path) connected neighborhood.

Given $0<r_1<r_2<\infty$, denote

$$A=A(x_0,r_1,r_2)=\{x\in X\colon r_1<d(x,x_0)<r_2\}\,.$$

Let $p\geqslant 1$ and $q\geqslant 1$, let G be a domain in X, and let $Q\colon G\to[0,\infty]$ be a measurable function. Suppose that the space X' admits weak sphericalization. Similarly to [4, Ch. 7], a mapping $f\colon G\to\overline{X'}$ (or $f\colon\ G\setminus\{x_0\}\to\overline{X'}$) is called a *ring Q-mapping at a point $x_0\in G$ with respect to (p,q)-moduli*, if for some $r_0=r_0(x_0)>0$ and for all $0<r_1<r_2<r_0$ the inequality

$$M_p(f(\Gamma(S(x_0,r_1),S(x_0,r_2),A(x_0,r_1,r_2))))\leqslant\int\limits_{A(x_0,r_1,r_2)\cap G}Q(x)\cdot\eta^q(d(x,x_0))\,d\mu(x)\qquad(2)$$

holds for any measurable function $\eta\colon(r_1,r_2)\to[0,\infty]$ with

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1.\qquad(3)$$

Similarly to [3], we say that a function $\varphi\colon G\to\mathbb{R}$ has *finite mean oscillation at a point* $x_0\in\overline{G}$, abbreviated $\varphi\in FMO(x_0)$, if

$$\overline{\lim\limits_{\varepsilon\to0}}\ \frac{1}{\mu(B(x_0,\varepsilon))}\int\limits_{B(x_0,\varepsilon)}|\varphi(x)-\overline{\varphi}_{\varepsilon}|\ d\mu(x)<\infty\,,\qquad(4)$$

where

$$\overline{\varphi}_{\varepsilon}=\frac{1}{\mu(B(x_0,\varepsilon))}\int\limits_{B(x_0,\varepsilon)}\varphi(x)\ d\mu(x)$$

is the mean value of the function $\varphi(x)$ over the ball $B(x_0,\varepsilon)=\{x\in G\colon d(x,x_0)<\varepsilon\}$ with respect to the measure μ. Here the condition (4) includes the assumption that φ is integrable with respect to the measure μ over the ball $B(x_0,\varepsilon)$ for some $\varepsilon>0$.

The terminology proposed below is essentially used by us in our previous article [8] and is somewhat different from that indicated here. Given $2\leqslant\alpha<\infty$ and $1\leqslant q\leqslant\alpha$, the space $X=(X,d,\mu)$ is called *(α,q)-admissible source*, if (X,d,μ) be locally compact and locally path connected upper Ahlfors α-regular metric space, moreover, for each point $x_0\in X$ there is $\gamma>0$ such that

$$\mu(B(x_0,2r))\leqslant\gamma\cdot\log^{\alpha-2}\frac{1}{r}\cdot\mu(B(x_0,r))\qquad(5)$$

for some $r_0>0$ and for all $r\in(0,r_0)$. Concerning relation (5), see also [7, (10.7)]. Similarly, given $p\geqslant 2$, the space $X'=(X',d',\mu')$ is called *p-admissible target*, if (X',d',μ') admits a weak sphericalization, besides that, $\left(\overline{X'},h\right)$ be locally connected p-uniform metric space. Let X and Y be metric spaces. A mapping $f\colon X\to Y$ is *discrete* if $f^{-1}(y)$ is discrete for all $y\in Y$ and f is *open* if f maps open sets onto open sets. The main result of this article is the following statement.

Theorem 1. *Fix $2 \leqslant \alpha < \infty$, $2 \leqslant p < \infty$ and $1 \leqslant q \leqslant \alpha$. Let D be a domain in X, let (X, d, μ) be an (α, q)-admissible source and let (X', d', μ') be an p-admissible target. Suppose that $G := D \setminus \{\zeta_0\}$ is a domain in X, which is locally path connected at $\zeta_0 \in D$, $Q \in FMO(\zeta_0)$ and that balls $B_h(A, r) = \{y \in \overline{X'} \colon h(y, A) < r\}$ do not degenerate into points for each $A \in \overline{X'}$ and every $r > 0$. If $f \colon D \setminus \{\zeta_0\} \to X'$ is an open discrete ring Q-mapping with respect to (p, q)-moduli at ζ_0, and ζ_0 is an essential singularity of f, then $f(U \setminus \{\zeta_0\})$ is dense in X' for an arbitrary neighborhood U of ζ_0.*

In Theorem 1, the "density" and "essential singularity" must be understood in terms of the space $(\overline{X'}, h)$.

2. The main lemma on the removability of singularities. Recall that a pair $E = (A, C)$, where A is an open set in X, and $C \subset A$ is a non-empty compact set, is called *condenser* in X. Given condenser E, we denote Γ_E the family of all paths $\gamma \colon [a, b) \to A$ with $\gamma(a) \in C$ such that $|\gamma| \cap (A \setminus F) \neq \varnothing$ for every compact set $F \subset A$.

Let $D \subset X$, $f \colon D \to X'$ be a discrete open mapping, $\beta \colon [a, b) \to X'$ be a path, and $x \in f^{-1}(\beta(a))$. A path $\alpha \colon [a, c) \to D$ is called a *maximal f-lifting* of β starting at x, if (1) $\alpha(a) = x$; (2) $f \circ \alpha = \beta|_{[a, c)}$; (3) for $c < c' \leqslant b$, there is no a path $\alpha' \colon [a, c') \to D$ such that $\alpha = \alpha'|_{[a, c)}$ and $f \circ \alpha' = \beta|_{[a, c')}$. In the case $X = X' = \mathbb{R}^n$, the openness and discreteness of f imply that every path β with $x \in f^{-1}(\beta(a))$ has a maximal f-lifting starting at x (see [6, Corollary II.3.3]). The following assertion holds.

Lemma 1. *Let X and X' be locally compact metric spaces, let X be locally connected, let D be a domain in X, and let $f \colon D \to X'$ be a discrete open mapping. If $\beta \colon [a, b) \to X'$ be a path, and $x \in f^{-1}(\beta(a))$, then there exists a maximal f-lifting of β starting at x.*

A complete proof of Lemma 1 can be found in [8, Lemma 2.1].

As in the classical cases of analytic functions and quasiconformal mappings, the result of Theorem 1 may be obtained on the basis of a more general theorem on the removability of isolated singularities. In this regard, we prove the following fundamental statement, which has a relatively abstract form. In what follows, the existing of limit of f at ζ_0 is understood in the sense of the space $(\overline{X'}, h)$.

Lemma 2. *Fix $2 \leqslant \alpha < \infty$, $2 \leqslant p < \infty$ and $1 \leqslant q \leqslant \alpha$. Let D be a domain in X, let (X, d, μ) be an (α, q)-admissible source and let (X', d', μ') be an p-admissible target. Assume that $G := D \setminus \{\zeta_0\}$ is a domain in X, which is locally path connected at $\zeta_0 \in D$.*

Suppose also that there exists $\varepsilon_0 > 0$ and a Lebesgue measurable function $\psi \colon (0, \varepsilon_0) \to [0, \infty]$ with the following property: for every $\varepsilon_2 \in (0, \varepsilon_0]$ there is $\varepsilon_1 \in (0, \varepsilon_2]$ such that the relation

$$0 < I(\varepsilon, \varepsilon_2) := \int\limits_{\varepsilon}^{\varepsilon_2} \psi(t)\, dt < \infty \tag{6}$$

holds for every $\varepsilon \in (0, \varepsilon_1)$. Assume also that the relation

$$\int\limits_{\varepsilon < d(x, \zeta_0) < \varepsilon_0} Q(x) \cdot \psi^q(d(x, \zeta_0))\, d\mu(x) \;=\; o\left(I^q(\varepsilon, \varepsilon_0)\right) \tag{7}$$

holds as $\varepsilon \to 0$.

Let K be some nondegenerate continuum in $\overline{X'}$, and let $f \colon D \setminus \{\zeta_0\} \to X' \setminus K$ be an open, discrete ring Q-mapping at ζ_0 with respect (p, q)-moduli. Now, f has a continuous extension at ζ_0.

Proof. Since X is locally compact, we may consider that $\overline{B(\zeta_0,\varepsilon_0)}$ is a compact set in D. Suppose the contrary, i.e., suppose that f has no limit at ζ_0. Since $(\overline{X'},h)$ is compact, the set

$$C(f,\zeta_0):=\{y\in\overline{X'}:\exists\,\zeta_k\in D\setminus\{\zeta_0\}\colon\zeta_k\overset{d}{\to}\zeta_0, f(\zeta_k)\overset{h}{\to}y, k\to\infty\}$$

is non-empty. Thus, there exist two sequences x_j and $x_j^{\prime}$ in $B(\zeta_0,\varepsilon_0)\setminus\{\zeta_0\}$, $d(x_j,\zeta_0)\to 0$, $d(x_j^{\prime},\zeta_0)\to 0$ such that

$$h\left(f(x_j),\,f(x_j^{\prime})\right)\geqslant a>0$$

for all $j\in\mathbb{N}$. Since G is locally path connected at ζ_0, there exists a sequence $r_k\to 0$, $0<r_k<\varepsilon_0$, $r_1>r_2>r_3>\ldots$, such that $B(\zeta_0,r_k)\subset V_k\subset B(\zeta_0,r_{k-1})$ and $V_k\cap G=V_k\setminus\{\zeta_0\}$ is path connected set. Since $d(x_j,\zeta_0)\to 0$ and $d(x_j^{\prime},\zeta_0)\to 0$ as $j\to\infty$, there is a number $j_1\in\mathbb{N}$ such that x_{j_1} and $x_{j_1}^{\prime}\in B(\zeta_0,r_2)$. Let C_{j_1} be a path joining x_{j_1} and $x_{j_1}^{\prime}$ in $V_2\setminus\{\zeta_0\}\subset B(\zeta_0,r_1)\setminus\{\zeta_0\}$. Similarly, there is a number $j_2\in\mathbb{N}$ such that x_{j_2} and $x_{j_2}^{\prime}\in B(\zeta_0,r_3)$. Let C_{j_2} be a path joining x_{j_2} and $x_{j_2}^{\prime}$ in $V_3\setminus\{\zeta_0\}\subset B(\zeta_0,r_2)\setminus\{\zeta_0\}$. Continuing this process, we obtain some number $j_k\in\mathbb{N}$ such that x_{j_k} and $x_{j_k}^{\prime}\in B(\zeta_0,r_{k+1})$. We join x_{j_k} and $x_{j_k}^{\prime}$ by a path C_{j_k}, which belongs to $V_{k+1}\setminus\{\zeta_0\}\subset B(\zeta_0,r_k)\setminus\{\zeta_0\}$. There is no loss of generality in assuming that x_j and $x_j^{\prime}$ can be joined by the path C_j in $B(\zeta_0,r_j)\setminus\{\zeta_0\}$ (see Figure 1 for this).

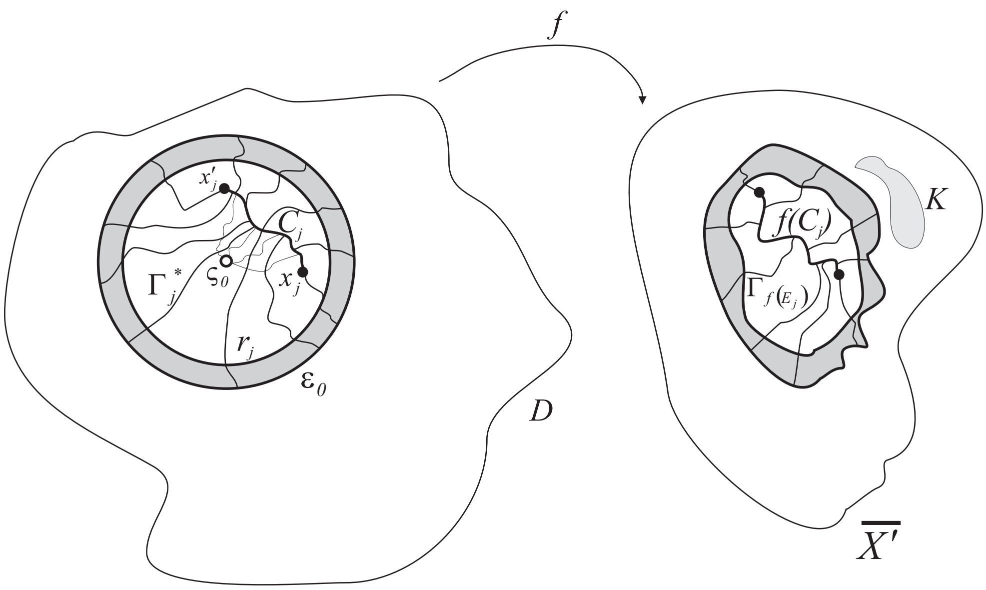

Figure 1: Configuration in the proof of Lemma 2

Let $E_j=(B(\zeta_0,\varepsilon_0)\setminus\{\zeta_0\},C_j)$, and let $\Gamma_{f(E_j)}$ be a family of paths, which corresponds to a condenser $f(E_j)$ in the sense of the notation made before the formulation of the lemma. By the minorizing principle of modulus and by p-uniformity of $\overline{X'}$

$$M_p(\Gamma_{f(E_j)})\geqslant M_p(\Gamma(f(C_j),K,X'\setminus K))\geqslant M_p(\Gamma(f(C_j),K,X'))\geqslant\delta>0\,,\qquad(8)$$

where $\delta>0$ is some positive constant depending only on $\min\{a,h(K)\}$, cf. see [10, Theorem 1(c)]. Now, by the uniformity of the space $\overline{X'}$ and the last relation it follows that the family of paths $\Gamma_{f(E_j)}$ is not empty. Let Γ_j^* be a family of all maximal f-liftings of $\Gamma_{f(E_j)}$ in $B(\zeta_0,\varepsilon_0)\setminus\{\zeta_0\}$ starting at C_j. This family of paths is well defined by Lemma 1.

Let $\Gamma^1_{E_j}$ be a family of all paths $\alpha\colon [a,\, c) \to B(\zeta_0, \varepsilon_0) \setminus \{\zeta_0\}$ starting at C_j, for which $\alpha(t_k) \to \zeta_0$ for some sequence $t_k \to c-0$, $t_k \in [a,\, c)$, as $k \to \infty$. Similarly, let $\Gamma^2_{E_j}$ be a family of all paths $\alpha\colon [a,\, c) \to B(\zeta_0, \varepsilon_0) \setminus \{\zeta_0\}$ starting at C_j, for which $d\,(\alpha(t_k), S(\zeta_0, \varepsilon_0)) \to 0$ for some sequence $t_k \in [a,\, c)$ such that $t_k \to c-0$ as $k \to \infty$. By [8, Lemma 4.2]

$$\Gamma^*_j \subset \Gamma^1_{E_j} \cup \Gamma^2_{E_j}\,. \tag{9}$$

Observe that $f(\Gamma^*_j) < \Gamma_{f(E_j)}$ and $f(\Gamma^*_j) \subset f(\Gamma^1_{E_j}) \cup f(\Gamma^2_{E_j})$. Now, by (1) and (9), we obtain that

$$M_p\left(\Gamma_{f(E_j)}\right) \leqslant M_p(f(\Gamma^1_{E_j}))\, +\, M_p(f(\Gamma^2_{E_j}))\,. \tag{10}$$

By [8, Lemma 4.1] $M_p(f(\Gamma^1_{E_j})) = 0$.

Note that an arbitrary path $\gamma \in \Gamma^2_{E_j}$ is not contained entirely both in $B(\zeta_0, \varepsilon_0 - \frac{1}{m})$ and $X \setminus B(\zeta_0, \varepsilon_0 - \frac{1}{m})$ for sufficiently large m. Thus, there exists $y_1 \in |\gamma| \cap S(\zeta_0, \varepsilon_0 - \frac{1}{m})$ (see [12, Theorem 1, § 46, item I]). Let $\gamma\colon [0,1] \to X$ and let $t_1 \in (0,1)$ be such that $\gamma(t_1) = y_1$. There is no loss of generality in assuming that $|\gamma|_{[0,t_1)}| \subset B(\zeta_0, \varepsilon_0 - 1/m)$. We put $\gamma_1 := \gamma|_{[0,t_1)}$. Observe that $|\gamma_1| \subset B(\zeta_0, \varepsilon_0 - 1/m)$, moreover, γ_1 is not included entirely either in $\overline{B(\zeta_0, r_j)}$ or in $X \setminus \overline{B(\zeta_0, r_j)}$. Consequently, there exists $t_2 \in (0, t_1)$ with $\gamma_1(t_2) \in S(\zeta_0, r_j)$ (see [12, Theorem 1, § 46, item I]). There is no loss of generality in assuming that $|\gamma|_{[t_2,t_1]}| \subset X \setminus \overline{B(\zeta_0, r_j)}$. Put $\gamma_2 = \gamma_1|_{[t_2,t_1]}$. Observe that γ_2 is a subcurve of γ. By the said above, $\Gamma^2_{E_j} > \Gamma(S(\zeta_0, r_j), S(\zeta_0, \varepsilon_0 - \frac{1}{m}), A(\zeta_0, r_j, \varepsilon_0 - \frac{1}{m}))$ for sufficiently large $m \in \mathbb{N}$. Set $A_j = \{x \in X\colon r_j < d(x, \zeta_0) < \varepsilon_0 - \frac{1}{m}\}$ and

$$\eta_j(t) = \begin{cases} \psi(t)/I(r_j, \varepsilon_0 - \frac{1}{m}), & t \in (r_j,\, \varepsilon_0 - \frac{1}{m}),\\ 0, & t \in \mathbb{R} \setminus (r_j,\, \varepsilon_0 - \frac{1}{m}). \end{cases}$$

Here $I(a,b)$ is defined in (6). Observe that

$$\int\limits_{r_j}^{\varepsilon_0 - \frac{1}{m}} \eta_j(t)\, dt = \frac{1}{I\left(r_j, \varepsilon_0 - \frac{1}{m}\right)} \int\limits_{r_j}^{\varepsilon_0 - \frac{1}{m}} \psi(t)\, dt = 1\,.$$

By the definition of ring Q-mapping at ζ_0 with respect to (p,q)-moduli and by (10), we obtain that

$$M_p(f(\Gamma_{E_j})) \leqslant \frac{1}{I^q(r_j, \varepsilon_0 - \frac{1}{m})} \int\limits_{r_j < d(x,\zeta_0) < \varepsilon_0} Q(x)\, \psi^{\,q}(d(x,\zeta_0))\, d\mu(x)\,.$$

Passing to the limit as $m \to \infty$, we obtain that

$$M_p(f(\Gamma_{E_j})) \leqslant \mathcal{S}(r_j) := \frac{1}{I^q(r_j, \varepsilon_0)} \int\limits_{r_j < d(x,\zeta_0) < \varepsilon_0} Q(x)\, \psi^{\,q}(d(x,\zeta_0))\, d\mu(x).$$

Formula (7) shows that $\mathcal{S}(r_j) \to 0$ as $j \to \infty$ and, consequently, from (10) it follows that

$$M_p\left(\Gamma_{f(E_j)}\right) \to 0\,, \qquad j \to \infty\,. \tag{11}$$

On the other hand, (11) contradicts (8). Thus, f has a limit at ζ_0, as required. □

3. On the main result and the proof of Theorem 1. The following statement can be found in [4, Lemma 13.2].

Proposition 1. *Let G be a domain in upper α-regular metric space (X,d,μ) at $\alpha\geqslant 2$. Assume that $x_0\in\overline{G}$ and $Q\colon G\to[0,\infty]$ belongs to $FMO(x_0)$. If*

$$\mu(G\cap B(x_0,2r))\leqslant\gamma\cdot\log^{\alpha-2}\frac{1}{r}\cdot\mu(G\cap B(x_0,r))$$

for some $r_0>0$ and every $r\in(0,r_0)$, then

$$\int\limits_{\varepsilon<d(x,x_0)<\varepsilon_0} Q(x)\cdot\psi^{\alpha}(d(x,x_0))\,d\mu(x)=o(I^{\alpha}(\varepsilon,\varepsilon_0))\,,\qquad \varepsilon\to 0\,,$$

$I(\varepsilon,\varepsilon_0):=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt$ and $\psi(t):=\frac{1}{t\log\frac{1}{t}}$.

Let us prove the following important assertion.

Theorem 2. *Fix $2\leqslant\alpha<\infty$, $2\leqslant p<\infty$ and $1\leqslant q\leqslant\alpha$. Let D be a domain in X, let (X,d,μ) be an (α,q)-admissible source and let (X',d',μ') be an p-admissible target. Suppose that $G:=D\setminus\{\zeta_0\}$ is a domain in X, which is locally path connected at $\zeta_0\in D$.*

Let K be some nondegenerate continuum in $\overline{X'}$ and let $f\colon D\setminus\{\zeta_0\}\to X'\setminus K$ be an open, discrete ring Q-mapping at ζ_0 with respect (p,q)-moduli. If $Q\in FMO(\zeta_0)$, then f has a continuous extension at ζ_0.

Proof. We show that the condition $Q\in FMO(\zeta_0)$ implies the conditions (6)–(7) at ζ_0. In fact, putting $\psi(t)=\log^{-\alpha/q}\frac{1}{t}$, we obtain the relations (6)–(7) from Proposition 1. Now we obtain the desired conclusion by Lemma 2. □

Proof of Theorem 1. By virtue of the fact that the point ζ_0 is an essential singular point of the mapping f, the ball $B(\zeta_0,r)$ contains an infinite number of points for each $r>0$. We carry out the proof by contradiction. Assume that there exist a neighborhood U of ζ_0 and $A\in\overline{X'}$ such that

$$h(f(x),A)\geqslant\delta_0 \tag{12}$$

for every $x\in U\setminus\{\zeta_0\}$. Since $D\setminus\{\zeta_0\}$ is locally path connected at ζ_0, there exists a neighborhood $V\subset U$ of ζ_0 such that $V\setminus\{\zeta_0\}$ is path connected.

We may consider that V is open. Otherwise, let D_* be a connected component of $\mathrm{Int}\,(U)\setminus\{\zeta_0\}$, containing $V\setminus\{\zeta_0\}$. Since X is locally path connected and $\mathrm{Int}\,(U)\setminus\{\zeta_0\}$ is open, D_* is open; see [12, Theorem 4.II.49.6]. Let $\bigcup\limits_{\alpha\in\mathfrak{A}}D_{\alpha}$ be a union of all components of $\mathrm{Int}\,(U)\setminus\{\zeta_0\}$ excluding D_*, where $\mathfrak{A}$ is some set of indexes α. Note that $\zeta_0\in\overline{D_*}$, because $B(\zeta_0,r)$ contains an infinite number of points for each $r>0$. Now, $\zeta_0\notin\overline{D_{\alpha}}$ for $\alpha\in\mathfrak{A}$. Since D_{α} are components of $\mathrm{Int}\,(U)\setminus\zeta_0$ and $\zeta_0\notin\overline{D_{\alpha}}$, D_{α} are closed in $\mathrm{Int}\,(U)$. Now, $V_*:=\mathrm{Int}\,(U)\setminus\bigcup\limits_{\alpha\in\mathfrak{A}}D_{\alpha}$ is the open neighborhood of ζ_0, where $V_*\setminus\{\zeta_0\}=D_*$ is connected. Moreover, since D_* is open and X is locally path connected, D_* is path connected, see e.g. [4, Proposition 13.1]. Thus, we may consider that V is open subset of U, V is neighborhood of ζ_0 and $V\setminus\{\zeta_0\}$ is path connected.

By (12), $f(x)\in X'\setminus B_h(A,\delta_0)$ for $x\in V\setminus\{\zeta_0\}$, where

$$B_h(A,\delta_0)=\{y\in\overline{X'}\colon h(y,A)<\delta_0\}\,.$$

Since $\overline{X'}$ is locally connected and compact space and, in addition, any balls $B_h(A,r)=\{y\in \overline{X'}\colon h(y,A)<r\}$ do not degenerate into points, there exist a non-degenerate continuum K such that $K\subset B_h(A,\delta_0)$. Now, it follows from (12) that f does not take values in K. By Theorem 2, f has isolated singularity as $x\to\zeta_0$, that contradicts to assumption of the theorem. □

REFERENCES

1. E. Afanas'eva, A. Golberg, R. Salimov, *Finite mean oscillation in upper ular metric spaces*, Lobachevskii J. Math., **38** (2017), №2, 206–212.
2. E. Afanas'eva, R. Salimov, *Boundary behavior of mappings in $\lambda(\varepsilon)$-regular metric spaces*, J. Math. Sci., **211** (2015), №5, 617–623.
3. A. Ignat'ev, V. Ryazanov, *Finite mean oscillation in the mapping theory*, Ukr. Mat. Visn., **2** (2005), №3, 395–417 (in Russian); translation *Finite mean oscillation in the mapping theory*, Ukr. Math. Bull., **2** (2005), №3, 403–424.
4. O. Martio, V. Ryazanov, U. Srebro, E. Yakubov, Moduli in Modern Mapping Theory, Springer Monographs in Mathematics, Springer, New York etc., 2009.
5. R. Näkki and B. Palka, *Uniform equicontinuity of quasiconformal mappings*, Proc. Amer. Math. Soc., **37** (1973), №2, 427–433.
6. S. Rickman, Quasiregular mappings, Berlin etc., Springer-Verlag, 1993.
7. V. Ryazanov, R. Salimov, *Weakly flat spaces and boundaries in the mapping theory*, Ukr. Math. Visnyk, **4** (2007), №2, 199–233 (in Russian); translation *Weakly flat spaces and boundaries in the mapping theory*, Ukr. Math. Bull., **4** (2007), №2, 199–233.
8. E.A. Sevost'yanov, A.A. Markysh, *On Sokhotski-Casorati-Weierstrass theorem on metric spaces*, Complex Variables and Elliptic Equations, published online https://www.tandfonline.com/doi/full/10.1080/17476933.2018.1557155.
9. J. Väisälä, Lectures on n-dimensional quasiconformal mappings, Lecture Notes in Math., V.229, Springer–Verlag, Berlin etc., 1971.
10. B. Fuglede, *Extremal length and functional completion*, Acta Math., **98** (1957), 171–219.
11. J. Heinonen, Lectures on Analysis on metric spaces, Springer Science+Business Media, New York, 2001.
12. K. Kuratowski, Topology, V.2, Academic Press, New York–London, 1968.

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