УДК 517.555

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ENTIRE CURVES HAVING BOUNDED l-INDEX IN ℓ_∞

A. I. Bandura. *Entire curves having bounded l-index in ℓ_∞*, Mat. Stud. **52** (2019), 108–112.

In this paper we propose an approach to introduce a concept of bounded index in an infinite-dimensional space. Our object of investigation is the space ℓ^∞ equipped with the norm $\|x\|_\infty = \sup\{|x_n|\colon n \in \mathbb{N}\}$. We consider entire curves from $\mathbb{C}$ to ℓ_∞ and prove a proposition indicating connection between of the l-index boundedness of every component of the curve and the l-index boundedness of the curve. Moreover, we obtain sufficient conditions of the l-index boundedness of entire curves in the space. They describe local behavior of norm of derivatives of the entire curves on the discs. Also, there is posed a problem on necessary conditions of the l-index boundedness of entire curves in infinite-dimensional spaces.

1. Introduction. In the last years, analytic functions of several variables having bounded index are intensively investigating. Main objects of investigations are such function classes: entire functions of several variables [5, 6, 15, 16], functions analytic in a polydisc [7], in a ball [3, 4] or in the Cartesian product of the complex plane and the unit disc [8].

For entire functions and analytic function in a ball there were proposed two approach to introduce a concept of index boundedness in a multidimensional complex space. They generate so-called functions of bounded L-index in a direction and functions of bounded $\mathbf{L}$-index in joint variables. But a starting point for these all investigations is the definition of entire function of bounded index introduced by B. Lepson [14]: a function $f(z)$, $z \in \mathbb{C}$, is called a *function of bounded index* if there exists a number $N \in \mathbb{Z}_+$ such that for all $p \in \mathbb{Z}_+$ and $z \in \mathbb{C}$ the next inequality

$$\frac{|f^{(p)}(z)|}{p!} \le \max\left\{\frac{|f^{(k)}(z)|}{k!}\colon 0 \le k \le N\right\}$$

holds. Later M. Sheremeta and A. Kuzyk [13, 20] generalized the concept replacing $p!$ in the denominator by $p!l^p(|z|)$ (so-called functions of *bounded l-index*) where $l\colon \mathbb{C} \to \mathbb{R}_+$ is a positive continuous function.

There are papers on analytic curves of bounded l-index. This function class naturally appears if we consider systems of differential equations and investigate properties of their analytic solutions. A concept of bounded index for entire curves was introduced with the sup-norm by L. F. Heath [12] and with the Euclidean norm by R. Roy and S. M. Shah [17]. In these papers the authors replaced the modulus of function by an appropriate norm in the definition. Later there were also proposed definitions of bounded ν-index by R. Roy and

2010 *Mathematics Subject Classification:* 30D20, 30D15.
Keywords: bounded index; bounded l-index; entire curve; ℓ_∞.
doi:10.30970/ms.52.1.108-112

S. M. Shah [18] for entire curves with these norms. In these definitions, R. Roy and S. M. Shah replaced $p!$ by $p!|z|^p$ and so on. Also M. T. Bordulyak and M. M. Sheremeta [9, 21] studied curves of bounded l-index and of bounded l-M-index which are analytic in arbitrary bounded domain on a complex plane and l is a positive continuous functions in the domain. They replaced $p!$ by $p!l^p(|z|)$ in the definition of Lepson and used both mentioned norms. These mathematicians found sufficient conditions providing l-index boundedness of every analytic solutions for some system of linear differential equations. A concept of bounded l-M-index uses idea of replacement the modulus of derivative by maximum modulus of the derivative, i.e. $|f^{(p)}(z)|$ by $\max\{|f^{(p)}(z)|\colon |z|=r\}$ in the definition of Lepson.

Moreover, F. Nuray and R. Patterson [16] investigated vector-valued entire functions $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ of bounded index. They presented an application to study properties of entire solutions of some system of linear partial differential equations. Also there are papers on properties of vector-valued functions which are analytic in the unit ball and have bounded **L**-index in joint variables [1, 2].

2. Problems and main results. Nowadays all considered spaces in theory of bounded index have finite dimension. The infinite-dimensional spaces were not an object of investigation in the theory of function of bounded index. Prof. A. Zagorodnyuk (2011-2016) repeatedly posed the following question in conversations with the author at the conferences "Modern problems of probability theory and mathematical analysis" (Vorokhta, Ukraine):

Problem 1. *Is it possible to construct a theory of bounded index for entire functions F with a countable set of variables, that is, $F\colon \mathbb{C}^\infty \to \mathbb{C}$?*

We are not ready to give full answer to the question. But in this investigation we consider some similar problem:

Problem 2 (O. Skaskiv, 2019)**.** *Is it possible to construct theory of bounded index for entire curves in an infinite-dimensional complex space, i.e. $f\colon \mathbb{C} \to \mathbb{C}^\infty$?*

In these short notices we propose some approach to introduce a concept of bounded index in the space ℓ^∞ that is the sequence space whose elements are the bounded complex sequences. The space is equipped with the norm $\|x\|_\infty = \sup\{|x_n|\colon n \in \mathbb{N}\}$.

Let $F\colon \mathbb{C} \to \ell^\infty$ be an entire curve, i.e. $F = (f_1, f_2, \ldots, f_n, \ldots)$, where every f_j is an entire function, and $\|F(z)\|_\infty < +\infty$ for any $z \in \mathbb{C}$.

The notation $F^{(k)}(z)$ stands for $(f_1^{(k)}(z), \ldots, f_n^{(k)}(z), \ldots)$.

By B^∞ we denote the space consisting of entire curves $F\colon \mathbb{C} \to \ell^\infty$ which are bounded on every compactum, i.e. $F \in B^\infty$ if for every compactum set $G \subset \mathbb{C}$ there exists $C > 0$ such that for any $z \in G$ one has $\|F(z)\|_\infty \le C$. The condition provides that any derivative of every function from B^∞ also belongs to B^∞.

Let $l\colon \mathbb{C} \to \mathbb{R}_+$ be a positive continuous function.

A function $F \in B^\infty$ is called a *function of bounded l-index*, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and for all $z \in \mathbb{C}$ one has

$$\frac{\|F^{(m)}(z)\|_\infty}{m!l^m(z)} \le \max_{0\le k\le m_0} \frac{\|F^{(k)}(z)\|_\infty}{k!l^k(z)}. \tag{1}$$

The least such integer m_0 is called the l-index of the curve F and is denoted by $N(F;l)$. If $l \equiv 1$ then F is said to be of bounded index. A simple example of such a curve of bounded index is $(e^z, e^{z/2}, \ldots, e^{z/n}, \ldots)$.

Let us denote

$$\lambda_{\mathbf{b}}(\eta)=\sup_{t_1,t_2\in\mathbb{C}}\left\{\frac{l(t_1)}{l(t_2)}\colon |t_1-t_2|\le\frac{\eta}{\min\{l(t_1),l(t_2)\}}\right\}.$$

By Q we denote the class of positive continuous function $l\colon \mathbb{C}\to\mathbb{R}_+$, satisfying the condition

$$(\forall\eta\ge 0)\colon\;\; \lambda_{\mathbf{b}}(\eta)<+\infty, \tag{2}$$

It is known that entire curve of bounded index can has every component of unbounded index. For a finite-dimensional curve such examples are contructed in [12]. Also it is known that if every component of entire curve $F\colon \mathbb{C}\to\mathbb{C}^n$ has bounded index then the curve is of bounded index (see [9]).

We prove first the following assertion.

Proposition 1. *Let $l\colon \mathbb{C}\to\mathbb{R}_+$ be a positive continuous function, each component f_j of an entire curve $F\colon \mathbb{C}\to\ell^\infty$ be of bounded l-index, and* $\sup\{N(f_s;l)\colon s\in\mathbb{N}\}$ *be finite. Then F has bounded l-index by the sup-norm with $N(F;l)\le\sup\{N(f_s;l)\colon s\in\mathbb{N}\}$.*

Proof. In the proof we implicitly use the following elementary fact: "Let $A_{k,s}(z)\ge 0$ be continuous function in $\mathbb{C}$, $0\le s\le N$, $k\in\mathbb{N}$. Denote $A(z)=\sup\{A_{k,s}(z)\colon 0\le s\le N, k\in\mathbb{N}\}$, $A_1(z)=\sup_{0\le s\le N}\sup_{k\in\mathbb{N}}A_{k,s}(z)$, $A_2(z)=\sup_{k\in\mathbb{N}}\sup_{0\le s\le N}A_{k,s}(z)$. Then $A_1(z)=A_2(z)=A(z)$ for each $z\in\mathbb{C}$."

In view of this fact for all $j\ge N=\sup\{N(f_s,l)\colon s\in\mathbb{N}\}$ we have

$$\frac{\|F^{(j)}(z)\|_\infty}{j!l^j(z)}=\frac{\sup\{|f_s^{(j)}(z)|\colon s\in\mathbb{N}\}}{j!l^j(z)}\le$$
$$\le\sup\left\{\max\left\{\frac{|f_s^{(k)}(z)|}{k!l^k(z)}\colon 0\le k\le N(f_s;l)\right\}\colon s\in\mathbb{N}\right\}\le$$
$$\le\sup\left\{\max\left\{\frac{|f_s^{(k)}(z)|}{k!l^k(z)}\colon 0\le k\le N\right\}\colon s\in\mathbb{N}\right\}=$$
$$=\sup\left\{\frac{|f_s^{(k)}(z)|}{k!l^k(z)}\colon 0\le k\le N, s\in\mathbb{N}\right\}=\max\left\{\frac{\|F^{(k)}(z)\|_\infty}{k!l^k(z)}\colon 0\le k\le N\right\},$$

that is $N(F;l)\le\sup\{N(f_s;l)\colon s\in\mathbb{N}\}$. Proposition 1 is proved. □

Positivity and continuity are weak restrictions on the function l. So, we will assume validity of some additional conditions which the function l satisfies.

Let us denote

$$\lambda_{\mathbf{b}}(\eta)=\sup_{t_1,t_2\in\mathbb{C}}\left\{\frac{l(t_1)}{l(t_2)}\colon |t_1-t_2|\le\frac{\eta}{\min\{l(t_1),l(t_2)\}}\right\}.$$

By Q we denote the class of positive continuous function $l\colon \mathbb{C}\to\mathbb{R}_+$, satisfying the condition

$$(\forall\eta\ge 0)\colon\;\; \lambda_{\mathbf{b}}(\eta)<+\infty, \tag{3}$$

Theorem 1. *Let $l \in Q$, $F \in B^\infty$. If for each $\eta > 0$ there exist $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for every $z_0 \in \mathbb{C}$ there exists $k_0 = k_0(z_0) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, and*

$$\max\left\{\left\|F^{(k_0)}(z)\right\|_\infty : |z - z_0| = \frac{\eta}{l(z_0)}\right\} \leq P_1 \left\|F^{(k_0)}(z_0)\right\|_\infty. \tag{4}$$

then the function F is of bounded l-index.

Proof. Our proof is based on the proof of an appropriate theorem for entire functions of one variable [10, 11, 20, 22]. Suppose that for each $\eta > 0$ there exist $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for every $z_0 \in \mathbb{C}$ there exists $k_0 = k_0(z) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, for which inequality (4) holds. We choose $\eta > 1$ and $j_0 \in \mathbb{N}$ such that $P_1 \leq \eta^{j_0}$. For given $z_0 \in \mathbb{C}$, $k_0 = k_0(z_0)$ and $j \geq j_0$ by Cauchy's formula for every component $f_s(z)$ as a function of one variable z one has

$$f_s^{(k_0+j)}(z_0) = \frac{j!}{2\pi i}\int_{|z-z_0|=\eta/l(z_0)} \frac{f_s^{(k_0)}(z)}{(z-z_0)^{j+1}}dz \quad \forall s \in \mathbb{N}. \tag{5}$$

Therefore, in view of (4) and (5) we have

$$\frac{\|F^{(k_0+j)}(z_0)\|_\infty}{j!} = \sup_{s\in\mathbb{N}} \frac{|f_s^{(k_0+j)}(z_0)|}{j!} \leq \sup_{s\in\mathbb{N}} \frac{1}{2\pi}\int_{|z-z_0|=\eta/l(z_0)} \frac{|f_s^{(k_0)}(z)|}{|z-z_0|^{j+1}}|dz| \leq$$
$$\leq \frac{1}{2\pi}\frac{l^{j+1}(z_0)}{\eta^{j+1}}\int_{|z-z_0|=\eta/l(z_0)} \sup_{s\in\mathbb{N}}|f_s^{(k_0)}(z)||dz| = \frac{1}{2\pi}\frac{l^{j+1}(z_0)}{\eta^{j+1}}\int_{|z-z_0|=\eta/l(z_0)} \|F^{(k_0)}(z)\|_\infty|dz| \leq$$
$$\leq \frac{l^j(z_0)}{\eta^j}\max\left\{\left\|F^{(k_0)}(z)\right\|_\infty : |z-z_0| = \frac{\eta}{l(z_0)}\right\} \leq P_1\frac{l^j(z_0)}{\eta^j}\|F^{(k_0)}(z_0)\|_\infty,$$

Obviously that $\frac{j!k_0!}{(j+k_0)!} \leq 1$. Therefore, for all $j \geq j_0$, $z_0 \in \mathbb{C}$

$$\frac{\|F^{(k_0+j)}(z_0)\|_\infty}{(k_0+j)!l^{k_0+j}(z_0)} \leq \frac{j!k_0!}{(j+k_0)!}\frac{P_1}{\eta^j}\frac{\|F^{(k_0)}(z_0)\|_\infty}{k_0!l^{k_0}(z_0)} \leq \eta^{j_0-j}\frac{\|F^{(k_0)}(z_0)\|_\infty}{k_0!l^{k_0}(z_0)} \leq \frac{\|F(z_0)\|_\infty}{k_0!l^{k_0}(z_0)}.$$

Since $k_0 \leq n_0$, the numbers $n_0 = n_0(\eta)$ and $j_0 = j_0(\eta)$ are independent of z_0, this inequality means that the entire curve F has bounded l-index and $N(F; l) \leq n_0 + j_0$. The proof of Theorem 1 is complete. □

Theorem 1 presents one of the possible sufficient conditions of l-index boundedness. Proposition 1 and Theorem 1 lead to the following question:

Problem 3. *Is it possible to deduce other known facts of theory of bounded index for space B^∞, particularly, necessary conditions of l-index boundedness (for example see* [10, 11, 19, 20]*)?*

In this communication, we considered only the space ℓ_∞, because we did not know how to effectively introduce this concept in the case of other infinite-dimensional spaces, in particular in the space ℓ_p $(p \in \mathbb{N}_0)$.

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Received 24.04.2019
Revised 24.08.2019