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ON ONE VARIATIOANAL PROBLEM REDUCING TO DIFFERENTIAL-BOUNDARY OPERATOR

In memoriam of V. E. Lyantse

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One quadratic functional in the real space $L_2(\Omega)$ $\left(\Omega\subset\mathbb{R}^2\right)$ is considered. The conditions are being necessary for the finding of its minimum are indicated and the problem of finding of corresponding sufficient conditions is formulated.

Let $\Omega\subset\mathbb{R}^2$ be a bounded domain with the smooth boundary line $\partial\Omega\overset{def}{=}\omega$. Here and below we suppose that all mentioned functions belong to the (real) Sobolev space $H^2(\Omega)=$ $=\{u\in L_2(\Omega)\colon u',u''\in L_2(\Omega)\}$(if the opposite is not stated). Let us consider the following (bilinear) forms:

$$\pi_0(u,v)=\int\limits_{\Omega}\left[\frac{\partial u}{\partial x_1}\cdot\frac{\partial v}{\partial x_1}+\frac{\partial u}{\partial x_2}\cdot\frac{\partial v}{\partial x_2}+p(x)u(x)v(x)\right]dx,$$

where $p\in C(\overline{\Omega})$ and $\overline{\Omega}$ is the closure of Ω(for the simplicity), $x=(x_1,x_2)\in\Omega$;

$$\pi_1(u,v)=\int\limits_{\omega}\alpha(\xi)\,u(\xi)\,v(\xi)\,d\xi,$$

where $\alpha\in C(\omega),\ \xi=(\xi_1,\xi_2)\in\omega$;

$$\pi_2(u,v)=\int\limits_{\Omega}\int\limits_{\Omega}\psi(x,y)u(x)v(y)dxdy,$$

where $\psi\in L_2(\Omega\times\Omega)$ and $\psi(x,y)=\psi(y,x)$;

$$\pi_3(u,v)=\frac{1}{2}\int\limits_{\Omega}\int\limits_{\omega}\phi(x,\xi)\,[u(x)v(\xi)+u(\xi)v(x)]dxd\xi,$$

where $\phi\in L_2(\Omega\times\omega)$ ("interaction between the interior and boundary points");

$$\pi_4(u,v)=\int\limits_{\omega}\int\limits_{\omega}c(\xi,\eta)\,u(\xi)\,v(\eta)\,d\xi d\eta,$$

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where $c \in C(\omega \times \omega)$ and $c(\varsigma, \eta) = c(\eta, \varsigma)$.

Put

$$\pi(u,v) = \sum_{i=0}^{4} \pi_i(u,v).$$

Applying the first Green formula we obtain

$$\pi(u,v) = \int_\Omega (-\Delta u + p(x)u)\, v dx + \int_\omega \frac{\partial u}{\partial n} v(\xi)\, d\xi + \int_\omega \alpha(\xi)\, u(\xi)\, v(\xi) d\xi +$$

$$+ \int_\Omega \int_\Omega \psi(x,y) u(x) v(y) dx dy + \frac{1}{2} \int_\Omega \int_\omega \phi(x,\xi) [u(\xi)\, v(x) + u(x) v(\xi)] d\xi dx +$$

$$+ \int_\omega \int_\omega c(\xi,\eta)\, u(\xi) v(\eta)\, d\xi d\eta, \tag{1}$$

where n is the exterior normal to ω.

Further, suppose that $f \in L_2(\Omega)$ and for each $v \in L_2(\Omega)$ put

$$\pi[v] = \pi(v,v), \qquad Mv = \int_\Omega f(x) v(x) dx.$$

Let us consider the variational problem

$$J[v] \equiv \pi[v] - 2Mv \to \min, \quad v \in H^2(\Omega). \tag{2}$$

It is known [1] that if $u \in H^2(\Omega)$ is a local minimum of the functional J then

$$\forall v \in H^2(\Omega) \quad \pi(u,v) = Mu. \tag{3}$$

Further, for each

$$v \in H_0^2(\Omega) \equiv \left\{ v \in H^2(\Omega) : v \downarrow \omega = \frac{\partial v}{\partial n} \downarrow \omega = 0 \right\}$$

we have

$$\pi(u,v) = \int_\Omega (-\Delta u + p(x)u) v dx + \int_\Omega \int_\Omega \psi(x,y) u(x) v(y) dx dy +$$

$$+ \frac{1}{2} \int_\Omega \int_\omega \phi(x,\xi) u(\xi) v(x) d\xi dx =$$

$$= \int_\Omega \Big(-\Delta u + p(x) u + \int_\Omega \psi(x,y) u(x) dy + \frac{1}{2} \int_\omega \phi(x,\xi)\, u(\xi) d\xi \Big) v(x) dx.$$

Since $H_0^2(\Omega)$ is dense in $L_2(\Omega)$ we obtain

$$-\Delta u + p(x) u + \int_\Omega \psi(x,y) u(y) dy + \frac{1}{2} \int_\omega \phi(x,\xi) u(\xi) d\xi = f. \tag{4}$$

Whence using (1)–(4) we conclude that

$$(\forall v \in H^2(\Omega)): \quad \int\limits_{\omega} \frac{\partial u}{\partial n} v(\xi) d\xi + \int\limits_{\omega} \alpha(\xi) u(\xi) v(\xi) d\xi +$$

$$+\frac{1}{2} \int\limits_{\Omega} \int\limits_{\omega} \phi(x,\xi) u(\xi) v(x) d\xi dx + \int\limits_{\omega} \int\limits_{\omega} c(\xi,\eta) u(\xi) v(\eta) d\xi d\eta = 0,$$

i.e., $(\forall v \in H^2(\Omega)$:

$$\int\limits_{\omega} \Big(\frac{\partial u}{\partial n} + \alpha(\xi) u(\xi) + \frac{1}{2} \int\limits_{\Omega} \phi(x,\xi) u(x) dx + \int\limits_{\omega} c(\xi,\eta) u(\eta) u(\eta) \Big) v(\eta) d\eta = 0. \tag{5}$$

But (5) implies

$$\frac{\partial u}{\partial n} + \alpha\left(\xi\right) u\left(\xi\right) + \frac{1}{2} \int\limits_{\Omega} \phi\left(x,\xi\right) u\left(x\right) dx + \int\limits_{\omega} c\left(\xi,\eta\right) u\left(\eta\right) d\eta = 0. \tag{6}$$

Theorem. *If $u \in H^2(\Omega)$ is a solution of the variational problem (2) then u satisfices the relations (4) and (6).*

Problem 1. *Under what additional conditions the relations (4) and (6) are sufficient for being the solutions of problem (2)?*

Problem 2. *Is it correct to consider*

$$\pi_3\left[u\right] \equiv \pi_3\left(u,u\right) = \int\limits_{\Omega} \int\limits_{\omega} \phi(x,\xi) \left[u(x)u(\xi)\right] dx d\xi$$

as a potential energy of some mechanical system ("interaction between the interior and boundary points") ?

REFERENCES

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