УДК 517.53

S. I. Fedynyak, P. V. Filevych

DISTANCE BETWEEN A MAXIMUM MODULUS POINT AND THE ZERO SET OF AN ANALYTIC FUNCTION

S. I. Fedynyak, P. V. Filevych. *Distance between a maximum modulus point and the zero set of an analytic function*, Mat. Stud. **52** (2019), 10–23.

Let f be an analytic function in the disk $\mathbb{D}_R=\{z\in\mathbb{C}\colon |z|<R\}$, $R\in(0,+\infty]$. A point $w\in\mathbb{D}_R$ is called a maximum modulus point of f if $|f(w)|=M(|w|,f)$, where $M(r,f)=\max\{|f(z)|\colon |z|=r\}$. Denote by $d(w,f)$ the distance between a maximum modulus point w and the zero set of f, i.e., $d(w,f)=\inf\{|w-z|\colon f(z)=0\}$. Let Φ be a continuous function on $[a,\ln R)$ such that $x\sigma-\Phi(\sigma)\to-\infty$, $\sigma\uparrow\ln R$, for every $x\in\mathbb{R}$. Let also $\widetilde{\Phi}$ be the Young-conjugate function of Φ and $\overline{\Phi}(x)=\widetilde{\Phi}(x)/x$ for all sufficiently large x. We prove that if

$$\ln M(r,f)\le(1+o(1))\Phi(\ln r),\quad r\uparrow R,$$

then

$$\varliminf_{|w|\uparrow R} d(w,f)\frac{\overline{\Phi}^{\,-1}(\ln|w|)}{|w|}\ge C_0,$$

where $C_0=0,5416\ldots$. When the Taylor coefficients of f are nonnegative, the constant C_0 can be replaced by π, and the inequality obtained in this case is sharp.

1. Introduction. Let $R\in(0,+\infty]$ and $\mathbb{D}_R=\{z\in\mathbb{C}\colon |z|<R\}$. By $\mathcal{A}_R$ denote the class of analytic functions in $\mathbb{D}_R$ of the form

$$f(z)=\sum_{n=0}^{\infty}a_nz^n \tag{1}$$

such that $f(z)\not\equiv0$. For every function f analytic in $\mathbb{D}_R$ put

$$M(r,f)=\max\{|f(z)|\colon |z|=r\},\quad r\in[0,R).$$

We call a point $w\in\mathbb{D}_R$ a maximum modulus point of f if $|f(w)|=M(|w|,f)$.

Let $A\in(-\infty,+\infty]$ and $\Phi\colon D_\Phi\to\mathbb{R}$ be a real function. We say that $\Phi\in\Omega_A$ if the domain D_Φ of Φ is an infinite interval of the form $[a,A)$, Φ is continuous on D_Φ, and the following condition

$$\forall x\in\mathbb{R}\colon\quad \lim_{\sigma\uparrow A}(x\sigma-\Phi(\sigma))=-\infty \tag{2}$$

holds. It is easy to see that in the case $A<+\infty$ condition (2) is equivalent to the condition $\Phi(\sigma)\to+\infty$, $\sigma\to A-0$, and in the case $A=+\infty$ this condition is equivalent to the

2010 *Mathematics Subject Classification:* 30B10, 30B30, 30C15, 30D20.
Keywords: analytic function; maximum modulus; maximum modulus point; zero set.
doi:10.30970/ms.52.1.10-23

condition $\Phi(\sigma)/\sigma \to +\infty$, $\sigma \to +\infty$. For $\Phi \in \Omega_A$ by $\widetilde{\Phi}$ we denote the Young-conjugate function of Φ, i.e.,

$$\widetilde{\Phi}(x) = \max\{x\sigma - \Phi(\sigma) \colon \sigma \in D_\Phi\}, \quad x \in \mathbb{R}.$$

Note (see Lemma 2 below), that the function $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ is continuous and increasing to A on some interval of the form $(x_0, +\infty)$. Hence the inverse function $\overline{\Phi}^{\,-1}$ is defined on some interval of the form (A_0, A) and $\overline{\Phi}^{\,-1}$ is continuous and increasing to $+\infty$ on (A_0, A).

For $f \in \mathcal{A}_R$ and $\Phi \in \Omega_{\ln R}$ the quantity

$$T_\Phi(f) = \varlimsup_{r\uparrow R} \frac{\ln M(r,f)}{\Phi(\ln r)}$$

is called Φ-*type* of the function f. The notion of Φ-type for a function analytic in $\mathbb{D}_R$ generalizes the classical notion of the type for an entire function of finite positive order.

Let $\mathcal{Z}_R$ be the class of functions $f \in \mathcal{A}_R$ that have infinitely many zeros in $\mathbb{D}_R$, and let $\mathcal{Z}_R^+$ be the subclass of functions $f \in \mathcal{Z}_R$ of the form (1) such that $a_n \geq 0$ for all integers $n \geq 0$. Denote by $d(w,f)$ the distance between a maximum modulus point w and the zero set of a function $f \in \mathcal{Z}_R$, that is

$$d(w,f) = \inf\{|w - z| \colon f(z) = 0\}.$$

The problem of finding asymptotic estimates from below for the quantity $d(w,f)$ as $|w| \to +\infty$ in the case where f is entire was considered in the works of I. Ostrovskii [1], I. Ostrovskii and A.E. Üreyen [2, 3, 4], A.E. Üreyen [5], and S.I. Fedynyak [6]. In particular, by certain conditions on a function $\Phi \in \Omega_{+\infty}$, the authors of these works have specified a function $h\colon [r_0, +\infty) \to (0, +\infty)$ dependent only on Φ such that the following statements are true.

(O_1) *If* $f \in \mathcal{Z}_{+\infty}$ *and* $T_\Phi(f) \leq 1$, *then*

$$\varliminf_{|w|\to+\infty} d(w,f)h(|w|) \geq \frac{1}{e}. \tag{3}$$

(O_2) *If* $f \in \mathcal{Z}_{+\infty}^+$ *and* $T_\Phi(f) \leq 1$, *then*

$$\varliminf_{|w|\to+\infty} d(w,f)h(|w|) \geq 1. \tag{4}$$

(O_3) *There exists a function* $f \in \mathcal{Z}_{+\infty}^+$ *such that* $T_\Phi(f) \leq 1$ *and*

$$\varliminf_{|w|\to+\infty} d(w,f)h(|w|) \leq \pi. \tag{5}$$

Next, we present an overview of relevant results. The first results in this direction, summarized in the following theorem, were obtained in [1].

Theorem A ([1]). *Let* ρ *and* T *be positive numbers,* $\Phi(\ln r) = Tr^\rho$ *and* $h(r) = eT\rho r^{\rho-1}$ *for all* $r \geq r_0$. *Then the statements* (O_1), (O_2), *and* (O_3) *are true.*

A function $\rho(r)$, continuously differentiable on $[r_0, +\infty)$, is called a *proximate order* if $\rho(r) \to \rho$ and $\rho'(r) r \ln r \to 0$ as $r \to +\infty$ for some $\rho \in (0, +\infty)$.

Theorem B ([2]). *Let ρ and T be positive numbers, $\rho(r)$ be a proximate order such that $\rho(r) \to \rho$ as $r \to +\infty$, $\Phi(\ln r) = Tr^{\rho(r)}$ and $h(r) = eT\rho r^{\rho(r)-1}$ for all $r \geq r_0$. Then the statements (O_1), (O_2), and (O_3) are true.*

As in [5], a function $\rho(r)$, continuously differentiable on $[r_0, +\infty)$, is called a *zero proximate order* if $\rho(r) = \psi(\ln r)/\ln r$ for all $r \geq r_0$, where ψ is a function, positive concave on $[\ln r_0, +\infty)$, and also $\rho(r) \to 0$ and $r^{\rho(r)}/\ln r \to +\infty$ as $r \to +\infty$.

Theorem C ([5]). *Let T be a positive number, $\rho(r)$ be a zero proximate order, $\Phi(\ln r) = Tr^{\rho(r)}$ and $h(r) = eTr^{\rho(r)-1}(\rho(r) + \rho'(r)r\ln r)$ for all $r \geq r_0$. Then the statements (O_1) and (O_3) are true.*

As in [5], a function $\rho(r)$, continuously differentiable on $[r_0, +\infty)$, is called an *infinite proximate order* if $\rho(r) = \psi(\ln r)/\ln r$ for all $r \geq r_0$, where ψ is a function, positive increasing convex on $[\ln r_0, +\infty)$, such that $\psi'(x) \to +\infty$ and $\psi''(x)/\psi'^2(x) \to 0$ (along the set of x for which $\psi''(x)$ exists), as $x \to +\infty$.

Theorem D ([5]). *Let T be a positive number, $\rho(r)$ be a infinite proximate order, $\Phi(\ln r) = Tr^{\rho(r)}$ and $h(r) = eTr^{\rho(r)-1}(\rho(r) + \rho'(r)r\ln r)$ for all $r \geq r_0$. Then the statements (O_1) and (O_3) are true.*

Finally, the following theorem is proved in [6].

Theorem E ([6]). *Let $\Phi \in \Omega_{+\infty}$ be a function, continuously differentiable on D_Φ, such that Φ' is increasing on D_Φ, and*

$$h(r) = \frac{1}{r} \cdot \overline{\Phi}^{\,-1}\left(\ln r + \frac{1}{\overline{\Phi}^{\,-1}(\ln r)}\right)$$

for all $r \geq r_0$. Then the statement (O_1) is true.

Remark 1. It follows from Lemma 4 (see below) that for the function h in Theorem E we have $h(r) \sim \overline{\Phi}^{\,-1}(\ln r)/r$, $r \to +\infty$.

Remark 2. For the function h in Theorems A, B, C, and D, we have $h(r) \sim e\Phi'(\ln r)/r$, $r \to +\infty$. Using this relation, we can prove that in each of the cases, considered in these theorems, the following inequality

$$\varliminf_{r\to+\infty} \frac{rh(r)}{\overline{\Phi}^{\,-1}(\ln r)} \geq 1$$

is true (we will not dwell on proving this fact).

Below we prove that the statements (O_1), (O_2), and (O_3) are true for an arbitrary function $\Phi \in \Omega_{+\infty}$ with $h(r) = \overline{\Phi}^{\,-1}(\ln r)/r$, $r \geq 0$. Moreover, the constant $\frac{1}{e}$ in (3) can be replaced by the larger constant

$$C_0 := \max_{x>1} \frac{\ln(2\sqrt{x(x-1)})}{x} = 0,5416\ldots, \tag{6}$$

and the constant 1 in (4) can be replaced by the sharp constant π.

At the same time, we prove similar results for functions analytic in a disk.

2. Main results.

Theorem 1. *Let $R \in (0, +\infty]$ and $\Phi \in \Omega_{\ln R}$. If $f \in \mathcal{Z}_R$ and $T_\Phi(f) \le 1$, then*

$$\varliminf_{|w| \uparrow R} d(w, f) \frac{\overline{\Phi}^{\,-1}(\ln |w|)}{|w|} \ge C_0. \tag{7}$$

Theorem 2. *Let $R \in (0, +\infty]$ and $\Phi \in \Omega_{\ln R}$. If $f \in \mathcal{Z}_R^+$ and $T_\Phi(f) \le 1$, then*

$$\varliminf_{|w| \uparrow R} d(w, f) \frac{\overline{\Phi}^{\,-1}(\ln |w|)}{|w|} \ge \pi. \tag{8}$$

If $f \in \mathcal{A}_R$ is a function of the form (1), $\Phi \in \Omega_{\ln R}$, and $r \in [0, R)$, then let

$$\mu(r, f) = \max\{|a_n| r^n \colon n \ge 0\}, \quad t_\Phi(f) = \varlimsup_{r \uparrow R} \frac{\ln \mu(r, f)}{\Phi(\ln r)}.$$

Theorem 3. *Let $R \in (0, +\infty]$ and $\Phi \in \Omega_{\ln R}$. Then there exists a function $f \in \mathcal{Z}_R^+$ such that $T_\Phi(f) = t_\Phi(f) = 1$ and*

$$\varliminf_{|w| \uparrow R} d(w, f) \frac{\overline{\Phi}^{\,-1}(\ln |w|)}{|w|} = \pi. \tag{9}$$

We need the following theorem to prove Theorem 2.

Theorem 4. *Let $R \in (0, +\infty]$, $f \in \mathcal{Z}_R^+$ be a function of the form (1), and $(r_j e^{i\varphi_j})$ be the sequence of all zeros of the function f. If $N \colon [0, R) \to (0, +\infty)$ is a function such that*

$$\sum_{n > N(r)} a_n r^n = o(M(r, f)), \quad r \uparrow R, \tag{10}$$

then

$$\varliminf_{j \to \infty} |\varphi_j| N(r_j) \ge \pi. \tag{11}$$

3. Auxiliary results.

Lemma 1. *Let $R \in (0, +\infty]$, $\rho \in (0, R)$, and $f \in \mathcal{A}_R$. Then for every $w, z \in \mathbb{D}_\rho$ we have*

$$|f(w) - f(z)| \le |w - z| \frac{\rho M(\rho, f)}{\sqrt{(\rho^2 - |w|^2)(\rho^2 - |z|^2)}}.$$

Proof. We note that

$$\int_0^{2\pi} \frac{dx}{a - b \cos x} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

whenever $a > b > 0$. Hence, using Cauchy's integral formula and Cauchy–Bunyakovsky inequality, we obtain

$$|f(w) - f(z)| = \left| \frac{1}{2\pi i} \int_{|\tau| = \rho} \left(\frac{f(\tau)}{\tau - w} - \frac{f(\tau)}{\tau - z} \right) d\tau \right| = \frac{|w - z|}{2\pi} \left| \int_{|\tau| = \rho} \frac{f(\tau)}{(\tau - w)(\tau - z)} d\tau \right| =$$

$$= \frac{|w-z|}{2\pi}\left|\int_0^{2\pi}\frac{f(\rho e^{i\theta})}{(\rho e^{i\theta}-w)(\rho e^{i\theta}-z)}\rho e^{i\theta} i d\theta\right| \le$$

$$\le \frac{|w-z|}{2\pi}\rho M(\rho,f)\int_0^{2\pi}\frac{d\theta}{|\rho e^{i\theta}-w||\rho e^{i\theta}-z|} \le$$

$$\le \frac{|w-z|}{2\pi}\rho M(\rho,f)\sqrt{\int_0^{2\pi}\frac{d\theta}{|\rho e^{i\theta}-w|^2}}\sqrt{\int_0^{2\pi}\frac{d\theta}{|\rho e^{i\theta}-z|^2}} =$$

$$= \frac{|w-z|}{2\pi}\rho M(\rho,f)\sqrt{\frac{2\pi}{\rho^2-|w|^2}}\sqrt{\frac{2\pi}{\rho^2-|z|^2}} = |w-z|\frac{\rho M(\rho,f)}{\sqrt{(\rho^2-|w|^2)(\rho^2-|z|^2)}}.$$

□

The following lemma is well known (see, for example, [7, § 3.2], [8]).

Lemma 2. *Let $A \in (-\infty,+\infty]$, $\Phi \in \Omega_A$, and $\varphi(x) = \max\{\sigma \in D_\Phi\colon x\sigma - \Phi(\sigma) = \widetilde{\Phi}(x)\}$, $x \in \mathbb{R}$. Then, the following statements are true:*

- *(i) the function φ is nondecreasing on $\mathbb{R}$;*
- *(ii) the function φ is continuous from the right on $\mathbb{R}$;*
- *(iii) $\varphi(x) \to A$, $x \to +\infty$;*
- *(iv) the right-hand derivative of $\widetilde{\Phi}(x)$ is equal to $\varphi(x)$ at every point $x \in \mathbb{R}$;*
- *(v) if $x_0 = \inf\{x > 0 : \Phi(\varphi(x)) > 0\}$, then the function $\overline{\Phi}(x) = \widetilde{\Phi}(x)/x$ increases to A on $(x_0,+\infty)$;*
- *(vi) the function $\alpha(x) = \Phi(\varphi(x))$ is nondecreasing on $[0,+\infty)$.*

In the following two lemmas, φ and x_0 are defined by Φ in the same way as in Lemma 2.

Lemma 3. *Let $A \in (-\infty,+\infty]$, $\Phi \in \Omega_A$, $\sigma_0 = \overline{\Phi}(x_0+0)$, and $\sigma \in (\sigma_0, A)$. Then the minimum value of the function*

$$h(y) = \frac{\Phi(y)}{y-\sigma}, \quad y \in (\sigma, A),$$

is $\overline{\Phi}^{\,-1}(\sigma)$, which is achieved when $y = \varphi(\overline{\Phi}^{\,-1}(\sigma))$.

Proof. Let $x = \overline{\Phi}^{\,-1}(\sigma)$, $y \in (\sigma, A)$. For $x > x_0 \ge 0$ from the definition of function $\widetilde{\Phi}$ we obtain

$$\sigma = \overline{\Phi}(x) \ge y - \frac{\Phi(y)}{x} = y - \frac{\Phi(y)}{\overline{\Phi}^{\,-1}(\sigma)},$$

so that $\frac{\Phi(y)}{y-\sigma} \ge \overline{\Phi}^{\,-1}(\sigma)$. Thus $h(y) \ge \overline{\Phi}^{\,-1}(\sigma)$ for all $y \in (\sigma, A)$.

We note that by Lemma 2

$$x\varphi(x) - \Phi(\varphi(x)) = \widetilde{\Phi}(x), \quad x \in \mathbb{R}. \tag{12}$$

If $x > x_0$, then $\Phi(\varphi(x)) > 0$, thus from (12) we obtain $\overline{\Phi}(x) < \varphi(x) < A$. For $x = \overline{\Phi}^{\,-1}(\sigma)$ and $y = \varphi(\overline{\Phi}^{\,-1}(\sigma)) = \varphi(x)$ we have $y \in (\sigma, A)$. Then, by (12),

$$h(y) = \frac{\Phi(\varphi(x))}{\varphi(x) - \overline{\Phi}(x)} = x = \overline{\Phi}^{\,-1}(\sigma).$$

□

Lemma 4. *Let $\delta \in (0,1)$, $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, $\sigma_0 = \overline{\Phi}(x_0+0)$, and $y(\sigma) = \varphi(\overline{\Phi}^{-1}(\sigma))$ for all $\sigma \in (\sigma_0, A)$. Then*

$$\overline{\Phi}^{-1}\left(\sigma + \frac{\delta\Phi(y(\sigma))}{\overline{\Phi}^{-1}(\sigma)}\right) \le \frac{\overline{\Phi}^{-1}(\sigma)}{1-\delta}, \quad \sigma \in (\sigma_0, A).$$

Proof. By Lemma 3 for all $\sigma \in (\sigma_0, A)$ we have

$$\frac{\Phi(y(\sigma))}{y(\sigma)-\sigma} = \overline{\Phi}^{-1}(\sigma).$$

Hence we get

$$y(\sigma) = \sigma + \frac{\Phi(y(\sigma))}{\overline{\Phi}^{-1}(\sigma)}.$$

Put

$$v(\sigma) = \sigma + \frac{\delta\Phi(y(\sigma))}{\overline{\Phi}^{-1}(\sigma)}.$$

Clearly, if $\sigma \in (\sigma_0, A)$, then $\sigma < v(\sigma) < y(\sigma)$ and also $y(\sigma) - v(\sigma) = (1-\delta)(y(\sigma)-\sigma)$. Therefore, using Lemma 3 again, for every $\sigma \in (\sigma_0, A)$ we obtain

$$\overline{\Phi}^{-1}(v(\sigma)) \le \frac{\Phi(y(\sigma))}{y(\sigma)-v(\sigma)} = \frac{\Phi(y(\sigma))}{(1-\delta)(y(\sigma)-\sigma)} = \frac{\overline{\Phi}^{-1}(\sigma)}{1-\delta}.$$

This proves Lemma 4. □

Lemma 5. *Let $A \in (-\infty, +\infty]$, $\Phi \in \Omega_A$, and Ψ be a convex function on $[\sigma_1, A)$ such that $\Psi(\sigma) \le \Phi(\sigma)$ for all $\sigma \in [\sigma_1, A)$. Then $\Psi'_+(\sigma) \le \overline{\Phi}^{-1}(\sigma)$ for every $\sigma \in [\sigma_2, A)$.*

Proof. Since $\Phi(\sigma) \to +\infty$, $\sigma \uparrow A$, from the conditions imposed on the function Ψ we get $\Psi(y) - \Psi(\sigma) \le \Phi(y)$ for all $\sigma \in [\sigma_2, A)$ and $y \in (\sigma, A)$. We can assume that $\sigma_2 \ge \sigma_1$ and $\sigma_2 \in (\sigma_0, A)$, where σ_0 is the constant from Lemma 3. Set $y = \varphi(\overline{\Phi}^{-1}(\sigma))$. Since Ψ is a convex function on $[\sigma_1, A)$, for every $\sigma \in [\sigma_2, A)$ we get, using Lemma 3,

$$\Psi'_+(\sigma) \le \frac{\Psi(y)-\Psi(\sigma)}{y-\sigma} \le \frac{\Phi(y)}{y-\sigma} = \overline{\Phi}^{-1}(\sigma).$$

Lemma 5 is proved. □

For a function $f \in \mathcal{A}_R$ of the form (1) let $\nu(r,f) = \max\{n \in \mathbb{N}_0 \colon |a_n|r^n = \mu(r,f)\}$ be its central index. It is well known that $\nu(r,f) = r(\ln\mu(r,f))'_+$ for all $r \in (0,R)$. In addition, we have following result ([9]).

Lemma 6. *Let $R \in (0, +\infty]$, (n_k) be an increasing sequence of nonnegative integers, and (c_k) be a sequence, increasing to R. If (a_n) is a complex sequence such that $a_{n_0} \ne 0$, $a_n = 0$ for each $n < n_0$ and*

$$|a_{n_{k+1}}| = |a_{n_0}|\prod_{j=0}^{k} c_j^{n_j-n_{j+1}}, \quad |a_n| \le |a_{n_k}|c_k^{n_k-n}, \quad n \in (n_k, n_{k+1})$$

for all $k \ge 0$, then the power series (1) defines a function f, analytic in the disk $\mathbb{D}_R$ and such that for which:

(i) $\nu(r,f)=n_0$ *for all* $r\in(0,c_0)$*;*

(ii) $\nu(r,f)=n_{k+1}$ *for all* $r\in[c_k,c_{k+1})$ *and* $k\ge 0$*.*

Lemma 7. *Let* $\varepsilon>0$*. Then there exists a number* $\delta=\delta(\varepsilon)>0$ *such that*

$$\Big|1-\Big(1+\frac{z}{n}\Big)^n\Big|\ge(1-\varepsilon)|z|$$

for any $z\in\mathbb{D}_\delta$ *and* $n\in\mathbb{N}$*.*

Proof. Let $z\in\mathbb{C}$, $n\in\mathbb{N}$. Then

$$|1-(1+z)^n|\ge 1+2n|z|-(1+|z|)^n. \tag{13}$$

Indeed, if $n\ge 2$, then, using the Newton binomial formula, we obtain

$$|1-(1+z)^n|=\left|\sum_{k=1}^n C_n^k z^k\right|\ge n|z|-\sum_{k=2}^n C_n^k|z|^k=1+2n|z|-(1+|z|)^n.$$

If $n=1$, then inequality (13) turns to equality.

Further, since $e^r-1\sim r$, $r\to 0$, then there exists $\delta>0$ such that $e^r-1\le(1+\varepsilon)r$, $r\in[0,\delta)$. Thus, if $z\in\mathbb{D}_\delta$ and $n\in\mathbb{N}$, then

$$\left(1+\frac{|z|}{n}\right)^n\le e^{|z|}\le 1+(1+\varepsilon)|z|,$$

so, using inequality (13) with z/n instead of z, we have

$$\Big|1-\Big(1+\frac{z}{n}\Big)^n\Big|\ge 1+2|z|-\left(1+\frac{|z|}{n}\right)^n\ge 1+2|z|-(1+(1+\varepsilon)|z|)=(1-\varepsilon)|z|.$$

Lemma 7 is proved. □

4. Proofs of theorems.

Proof of Theorem 1. Let $f\in\mathcal{Z}_R$ be a function such that $T_\Phi(f)\le 1$, and C_0 be the constant, defined by (6). Let us prove that for the function f relation (7) holds.

We set $\Psi(\sigma)=\ln M(e^\sigma,f)$ for each $\sigma<\ln R$. Then the condition $T_\Phi(f)\le 1$ can be rewritten as $\Psi(\sigma)\le(1+o(1))\Phi(\sigma)$, $\sigma\uparrow\ln R$. Since the function Ψ is convex on the interval $(-\infty,\ln R)$, by Lemma 5 we obtain

$$\Psi'_+(\ln r)\le(1+o(1))\overline{\Phi}^{\,-1}(\ln r),\quad r\uparrow R. \tag{14}$$

Suppose, on the contrary, that inequality (7) does not hold, that is, there exists a positive number $\lambda<C_0$ such that for the set E of maximum modulus points w, satisfying the inequality

$$d(w,f)\frac{\overline{\Phi}^{\,-1}(\ln|w|)}{|w|}\le\lambda,$$

we have $\sup E=R$.

For each maximum modulus point w of the function f we fix the nearest zero of f and denote this zero by $z(w)$. Note that by Lemma 1 for arbitrary $\rho \in (\max\{|w|, |z(w)|\}, R)$ the inequality

$$M(|w|, f) \le d(w, f) \frac{\rho M(\rho, f)}{\sqrt{(\rho^2 - |w|^2)(\rho^2 - |z(w)|^2)}} \tag{15}$$

holds.

Since $\lambda < C_0$, for some $\mu > 1$, according to (6), we have

$$\lambda < \frac{\ln(2\sqrt{\mu(\mu-1)})}{\mu}. \tag{16}$$

Let $w \in E$. Put

$$\rho(w) = |w| + \frac{\mu\lambda|w|}{\overline{\Phi}^{-1}(\ln|w|)}.$$

Then $|w| < \rho(w)$. In addition, since

$$|w - z(w)| = d(|w|, f) \le \frac{\lambda|w|}{\overline{\Phi}^{-1}(\ln|w|)},$$

we have also $|z(w)| < \rho(w)$. It is also clear that $\rho(w) \sim |z(w)| \sim |w|$ as $|w| \uparrow R$.

Since

$$\ln\rho(w) = \ln|w| + \ln\left(1 + \frac{\mu\lambda}{\overline{\Phi}^{-1}(\ln|w|)}\right) \le \ln|w| + \frac{\mu\lambda}{\overline{\Phi}^{-1}(\ln|w|)}, \tag{17}$$

by Lemma 4 we obtain $\overline{\Phi}^{-1}(\ln\rho(w)) \sim \overline{\Phi}^{-1}(\ln|w|)$ as $|w| \uparrow R$. Then, according to (14),

$$\Psi'_+(\ln\rho(w)) \le (1 + o(1))\overline{\Phi}^{-1}(\ln|w|), \quad |w| \uparrow R. \tag{18}$$

Using (17) and (18), we get

$$\ln\frac{M(\rho(w), f)}{M(|w|, f)} = \int_{\ln|w|}^{\ln\rho(w)} \Psi'_+(x)dx \le \Psi'_+(\ln\rho(w))(\ln\rho(w) - \ln|w|) \le (1 + o(1))\mu\lambda$$

as $|w| \uparrow R$. Therefore,

$$\varliminf_{|w|\uparrow R} \frac{M(|w|, f)}{M(\rho(w), f)} \ge \frac{1}{e^{\mu\lambda}}. \tag{19}$$

Noting that the relations

$$\rho^2(w) - |w|^2 \sim \frac{2\mu\lambda|w|^2}{\overline{\Phi}^{-1}(\ln|w|)}, \quad \rho^2(w) - |z(w)|^2 \ge (1 + o(1))\frac{2(\mu-1)\lambda|w|^2}{\overline{\Phi}^{-1}(\ln|w|)},$$

hold as $|w| \uparrow R$, and using inequality (15) with $\rho = \rho(w)$ and also (19), for $w \in E$ we obtain

$$\lambda \ge \varlimsup_{|w|\uparrow R} d(w, f)\frac{\overline{\Phi}^{-1}(\ln|w|)}{|w|} \ge$$

$$\ge \varliminf_{|w|\uparrow R} \frac{M(|w|, f)}{M(\rho(w), f)} \frac{\sqrt{(\rho^2(w) - |w|^2)(\rho^2(w) - |z(w)|^2)}}{\rho(w)} \frac{\overline{\Phi}^{-1}(\ln|w|)}{|w|} \ge \frac{2\lambda\sqrt{\mu(\mu-1)}}{e^{\mu\lambda}},$$

which contradicts (16). □

Proof of Theorem 4. Let $f \in \mathcal{Z}_R^+$ be a function of the form (1), $(r_j e^{i\varphi_j})$ be the sequence of all zeros of the function f, and $N\colon [0,R) \to (0,+\infty)$ be a function such that (10) holds. We prove that then the inequality (11) holds.

First of all, note that each of the numbers $r_j e^{-i\varphi_j}$ is also a zero of the function f in view of the fact that its Taylor coefficients are nonnegative.

Suppose, on the contrary, that inequality (11) does not hold. Then, as stated above, there exists a number $\varepsilon \in (0,\pi)$ such that the set

$$E = \left\{ j \ge 0\colon \ 0 \le \varphi_j \le \frac{\pi-\varepsilon}{N(r_j)} \right\}$$

is infinite.

Let $j \in E$. Then for every integer $k \in [0, N(r_j)]$ we have $0 \le k\varphi_j \le \pi - \varepsilon$, which implies that

$$-\frac{\pi}{2} + \frac{\varepsilon}{2} \le k\varphi_j - \left(\frac{\pi}{2} - \frac{\varepsilon}{2}\right) \le \frac{\pi}{2} - \frac{\varepsilon}{2}.$$

Taking

$$\eta = \frac{\pi}{2} - \frac{\varepsilon}{2}, \quad \delta = \cos\left(\frac{\pi}{2} - \frac{\varepsilon}{2}\right),$$

we see that $\cos(k\varphi_j - \eta) \ge \delta > 0$ for all integers $k \in [0, N(r_j)]$. Therefore, in view of (10),

$$\left|\mathrm{Re}(e^{i\eta} f(r_j e^{i\varphi_j}))\right| = \left|\sum_{k=0}^{\infty} a_k r_j^k \cos(k\varphi_j - \eta)\right| \ge \sum_{k \le N(r_j)} a_k r_j^k \delta - \sum_{k > N(r_j)} a_k r_j^k =$$
$$= \delta M(r_j, f) - (1+\delta) \sum_{k > N(r_j)} a_k r_j^k = (\delta - o(1)) M(r_j, f), \quad j \to \infty.$$

So, for all sufficiently large $j \in E$ we have $f(r_j e^{i\varphi_j}) \ne 0$. This contradicts the fact that the numbers $r_j e^{i\varphi_j}$ are zeros of the function f. □

Proof of Theorem 2. Let $f \in \mathcal{Z}_R^+$ be a function of the form (1) such that $T_\Phi(f) \le 1$. We prove that then inequality (8) holds.

Fix an arbitrary $\varepsilon > 0$ and select $\delta > 0$ so that the inequality

$$\delta < 1 - \frac{1}{\sqrt{1+\varepsilon}}$$

holds. Taking

$$\eta = (1+\varepsilon)(1-\delta)^2 - 1,$$

we see that $\eta > 0$. Put $\Psi(\sigma) = \ln M(e^\sigma, f)$ for each $\sigma < \ln R$. From the condition $T_\Phi(f) \le 1$ we have $\Psi(\sigma) \le (1+\eta)\Phi(\sigma)$ for all $\sigma \in [\sigma_1, \ln R)$. Since the function Ψ is convex on the interval $(-\infty, \ln R)$, by Lemma 5 we obtain

$$\Psi'_+(\ln r) \le (1+\eta)\overline{\Phi}^{\,-1}(\ln r), \quad r \in [R_1, R). \tag{20}$$

Further, if $N > 0$, $r \in (0,R)$, and $\rho \in (r,R)$, then

$$\sum_{k>N} a_n r^n < \frac{M(r,f)}{e^{(\ln\rho - \ln r)(N - \Psi'_+(\ln\rho))}}. \tag{21}$$

Indeed, since the coefficients a_n are nonnegative, $M(r,f) = f(r)$ for all $r \in [0,R)$, and therefore

$$\sum_{k>N} a_n r^n < \left(\frac{r}{\rho}\right)^N \sum_{k>N} a_n \rho^n \le \left(\frac{r}{\rho}\right)^N M(\rho,f) = M(r,f)\left(\frac{r}{\rho}\right)^N e^{\Psi(\ln\rho)-\Psi(\ln r)}.$$

It remains to take into account that

$$\Psi(\ln\rho) - \Psi(\ln r) = \int_{\ln r}^{\ln\rho} \Psi'_+(x)dx \le (\ln\rho - \ln r)\Psi'_+(\ln\rho).$$

As in Lemma 4, let $\sigma_0 = \overline{\Phi}(x_0+0)$ and $y(\sigma) = \varphi(\overline{\Phi}^{\,-1}(\sigma))$ for all $\sigma \in (\sigma_0, \ln R)$. Applying inequality (21) with $N = (1+\varepsilon)\overline{\Phi}^{\,-1}(\ln r)$ and $\rho > r$ such that

$$\ln\rho = \ln r + \frac{\delta\Phi(y(\ln r))}{\overline{\Phi}^{\,-1}(\ln r)},$$

and using inequality (20) and Lemma 4, we obtain

$$\sum_{k>(1+\varepsilon)\overline{\Phi}^{\,-1}(\ln r)} a_n r^n < M(r,f)\exp\left(-\frac{\delta\Phi(y(\ln r))}{\overline{\Phi}^{\,-1}(\ln r)}\left((1+\varepsilon)\overline{\Phi}^{\,-1}(\ln r) - \frac{1+\eta}{1-\delta}\Psi'_+(\ln r)\right)\right) =$$

$$= \frac{M(r,f)}{e^{\delta^2(1+\varepsilon)\Phi(y(\ln r))}} = o(M(r,f)), \quad r \uparrow R.$$

Since $\varepsilon > 0$ is arbitrary, hence by Theorem 4 we have

$$\varliminf_{j\to\infty} |\varphi_j|\overline{\Phi}^{\,-1}(\ln r_j) \ge \pi, \tag{22}$$

where $(r_j e^{i\varphi_j})$ is the sequence of all zeros of the function f.

Further, since the coefficients a_n of the function f are nonnegative, it is easy to prove (see, for example, [1]), that $d(w,f) = d(|w|,f)$ for an arbitrary maximum modulus point w of this function. Therefore, if inequality (8) is not satisfied, then there exists a number $\lambda < \pi$ such that for the set

$$E = \left\{ r \in (e^{\sigma_0}, R) \colon d(r,f)\frac{\overline{\Phi}^{\,-1}(\ln r)}{r} \le \lambda \right\}$$

we have $\sup E = R$.

For each point $r \in [0,R)$ we fix the number of the nearest zero of f and denote this number by $j(r)$.

Let $r \in E$ and $r \uparrow R$. Then $d(r,f) = o(r)$, so it is clear from geometric considerations that $\varphi_{j(r)} \to 0$. Further, since $|r_{j(r)} - r| \le d(r,f)$, we obtain $r_{j(r)} \sim r$. In addition,

$$\ln r_{j(r)} \le \ln(r + d(r,f)) \le \ln\left(r + \frac{\lambda r}{\overline{\Phi}^{\,-1}(\ln r)}\right) \le \ln r + \frac{\lambda}{\overline{\Phi}^{\,-1}(\ln r)},$$

and therefore, by Lemma 4, $\overline{\Phi}^{\,-1}(\ln r_{j(r)}) \le (1+o(1))\overline{\Phi}^{\,-1}(\ln r)$.

Using the above relations and the obvious inequality $r_{j(r)} \sin|\varphi_{j(r)}| \le d(r,f)$ and assuming that $r \in E$, we get

$$\varlimsup_{r\uparrow R} |\varphi_{j(r)}|\overline{\Phi}^{\,-1}(\ln r_{j(r)}) = \varlimsup_{r\uparrow R} \sin|\varphi_{j(r)}|\overline{\Phi}^{\,-1}(\ln r_{j(r)}) \le \varlimsup_{r\uparrow R} d(r,f)\frac{\overline{\Phi}^{\,-1}(\ln r)}{r} \le \lambda,$$

which contradicts the inequality (22). $\square$

Proof of Theorem 3. Let $\Phi\in\Omega_{\ln R}$. Suppose, as above, $x_0=\inf\{x>0\colon \Phi(\varphi(x))>0\}$. From condition (2) and the properties of the function $\widetilde{\Phi}$ given in Lemma 2 it follows that there exists an increasing sequence (n_k) of nonnegative integers such that for it and for the sequences (c_k) and (r_k), where $c_k=\exp\{\overline{\Phi}(n_{k+1})\}$ and $r_k=\exp\{\varphi(n_{k+1})\}$ for all integers $k\geq 0$, we have $n_0=0$, $n_1>x_0$, and also

$$kn_k\ln r-\Phi(\ln r)\leq kn_k\ln c_0,\quad r\in[c_k,R),\ k\geq 0; \tag{23}$$

$$n_k=o(n_{k+1}),\quad k\to\infty; \tag{24}$$

$$\min\{(n_{k+1}-n_k)(\ln c_{k+1}-\ln c_k),(n_{k+2}-n_{k+1})(\ln c_{k+1}-\ln r_k)\}\geq 2\ln(k+1),\ k\geq 0. \tag{25}$$

Put $a_0=1$,

$$a_{n_{k+1}}=\prod_{j=0}^{k}c_j^{n_j-n_{j+1}},\quad k\geq 0,$$

and $a_n=0$ if $n\in(n_k,n_{k+1})$ for some $k\geq 0$. By Lemma 6 the power series

$$f(z)=\sum_{k=0}^{\infty}a_{n_k}z^{n_k}$$

defines a function f, analytic in the disk $\mathbb{D}_R$, such that $\nu(r,f)=0$ for all $r\in(0,c_0)$ and $\nu(r,f)=n_{k+1}$ for all $r\in[c_k,c_{k+1})$ and $k\geq 0$.

Let $r\in[c_k,c_{k+1})$ and $k\geq 0$. Then

$$\ln c_k=\overline{\Phi}(n_{k+1})=\sup\left\{\sigma-\frac{\Phi(\sigma)}{n_{k+1}}\colon \sigma\in D_\Phi\right\}\geq \ln r-\frac{\Phi(\ln r)}{n_{k+1}}. \tag{26}$$

From inequalities (26) and (23) we obtain, respectively,

$$n_{k+1}\ln\frac{r}{c_k}\leq\Phi(\ln r),\quad n_k\ln\frac{c_k}{c_0}\leq n_k\ln\frac{r}{c_0}\leq\frac{\Phi(\ln r)}{k},$$

so for every $k\geq 1$ we have

$$\ln\mu(r,f)=\ln\mu(c_k,f)+\int_{c_k}^{r}\frac{\nu(t,f)}{t}dt\leq n_k\ln\frac{c_k}{c_0}+n_{k+1}\ln\frac{r}{c_k}\leq$$
$$\leq\frac{\Phi(\ln r)}{k}+\Phi(\ln r)=\frac{k+1}{k}\Phi(\ln r).$$

This implies that $t_\Phi(f)\leq 1$.

We further note that equality (12), which follows from Lemma 2, can be rewritten as

$$\overline{\Phi}(x)=\varphi(x)-\frac{\Phi(\varphi(x))}{x},\quad x\in\mathbb{R}.$$

Putting $x=n_{k+1}$, we get

$$\ln c_k=\ln r_k-\frac{\Phi(\ln r_k)}{n_{k+1}}. \tag{27}$$

Since $\mu(r,f)\geq\mu(0,f)=1$, $r\in(0,R)$, we obtain, using (27),

$$\ln\mu(r_k,f)\geq\int_{c_k}^{r_k}\frac{\nu(t,f)}{t}dt\geq n_{k+1}\ln\frac{r_k}{c_k}=\Phi(\ln r_k).$$

This and the inequality $t_\Phi(f) \le 1$ imply that $t_\Phi(f) = 1$.

Put

$$\delta_k = \frac{1}{\sqrt{k+1}},\ m_k = n_{k+1} - n_k,\ h_k = \frac{c_k \delta_k}{m_k}$$

for all $k \ge 0$. Then, according to (24) and (27), for $k \ge k_0$ we obtain

$$c_k + h_k = c_k \left(1 + \frac{\delta_k}{m_k}\right) \le c_k \left(1 + \frac{\Phi(\ln r_k)}{n_{k+1}}\right) \le c_k \exp\left\{\frac{\Phi(\ln r_k)}{n_{k+1}}\right\} = r_k. \tag{28}$$

Note that

$$(c_k - h_k)^{n_{k+1}} \sim c_k^{n_{k+1}} \sim (c_k + h_k)^{n_{k+1}}, \quad k \to \infty.$$

Therefore, as $k \to \infty$, we have, uniformly for $r \in [c_k - h_k, c_k + h_k]$,

$$a_{n_k} r^{n_k} \sim a_{n_k} c_k^{n_k} = \mu(c_k, f), \quad a_{n_{k+1}} r^{n_{k+1}} \sim a_{n_{k+1}} c_k^{n_{k+1}} = \mu(c_k, f). \tag{29}$$

For each integer $p \ge 0$ we set

$$b_p = c_p e^{i\pi/m_p}, \quad C_p = \{z \in \mathbb{C} \colon |z - b_p| = h_p\}$$

and consider the functions

$$P_p(z) = a_{n_p} z^{n_p} + a_{n_{p+1}} z^{n_{p+1}}, \quad g_p(z) = f(z) - P_p(z).$$

Let δ be a number whose existence follows from Lemma 7 for $\varepsilon = 1/2$, and let $z \in C_p$. Then $z = b_p + h_p e^{i\theta}$ for some $\theta \in \mathbb{R}$. For all $p \ge p_0$ we have $\delta_p < \delta$. Putting $|z| = r$ and using Lemmas 7 and (29), we get

$$|P_p(z)| = |a_{n_p} z^{n_p} + a_{n_{p+1}} z^{n_{p+1}}| = a_{n_p} r^{n_p} \left|1 + \left(\frac{z}{c_p}\right)^{m_p}\right| = a_{n_p} r^{n_p} \left|1 + \left(e^{i\pi/m_p} + \frac{\delta_p}{m_p} e^{i\theta}\right)^{m_p}\right| =$$
$$= a_{n_p} r^{n_p} \left|1 - \left(1 + \frac{\delta_p}{m_p} e^{i(-\pi/m_p + \theta)}\right)^{m_p}\right| \ge a_{n_p} r^{n_p} \frac{\delta_p}{2} \ge \frac{\delta_p}{3} \mu(c_p, f), \quad p \ge p_1.$$

In addition, using (28) and (25), for every $p \ge p_2$ we have

$$\left|\sum_{k \le p-1} a_{n_k} z^{n_k}\right| \le \sum_{k \le p-1} a_{n_k} (c_p + h_p)^{n_k} = a_{n_p} (c_p + h_p)^{n_p} \sum_{k \le p-1} \frac{a_{n_k}}{a_{n_p}} (c_p + h_p)^{n_k - n_p} \le$$
$$\le 2\mu(c_p, f) \sum_{k \le p-1} \frac{a_{n_k}}{a_{n_p}} c_p^{n_k - n_p} = 2\mu(c_p, f) \sum_{k \le p-1} \prod_{j=k}^{p-1} \left(\frac{c_j}{c_p}\right)^{n_{j+1} - n_j} \le$$
$$\le 2\mu(c_p, f) \sum_{k \le p-1} \left(\frac{c_{p-1}}{c_p}\right)^{n_p - n_{p-1}} = 2\mu(c_p, f) p \left(\frac{c_{p-1}}{c_p}\right)^{n_p - n_{p-1}} \le \frac{2}{p} \mu(c_p, f);$$

$$\left|\sum_{k \ge p+2} a_{n_k} z^{n_k}\right| \le \sum_{k \ge p+2} a_{n_k} (c_p + h_p)^{n_k} = a_{n_{p+1}} (c_p + h_p)^{n_{p+1}} \sum_{k \ge p+2} \frac{a_{n_k}}{a_{n_{p+1}}} (c_p + h_p)^{n_k - n_{p+1}} \le$$
$$\le 2\mu(c_p, f) \sum_{k \ge p+2} \frac{a_{n_k}}{a_{n_{p+1}}} r_p^{n_k - n_{p+1}} = 2\mu(c_p, f) \sum_{k \ge p+2} \prod_{j=p+1}^{k-1} \left(\frac{r_p}{c_j}\right)^{n_{j+1} - n_j} \le$$

$$\leq 2\mu(c_p, f) \sum_{k\geq p+2} \left(\frac{r_p}{c_{k-1}}\right)^{n_k-n_{k-1}} \leq 2\mu(c_p, f) \sum_{k\geq p+2} \left(\frac{r_{k-2}}{c_{k-1}}\right)^{n_k-n_{k-1}} \leq$$

$$\leq 2\mu(c_p, f) \sum_{k\geq p+2} \frac{1}{(k-1)^2} \leq \frac{2}{p}\mu(c_p, f).$$

So for all $p \geq p_3$ and $z \in C_p$ we get

$$|g_p(z)| \leq \frac{4}{p}\mu(c_p, f) < \frac{\delta_p}{3}\mu(c_p, f) \leq |P_p(z)|.$$

Thus, according to Rouché's theorem, the function f has in the disk $\{z \in \mathbb{C}\colon |z - b_p| < h_p\}$ the same number of zeros as the binomial P_p. Since b_p is a zero of P_p, in this disk the function f has a zero, which we denote by z_p. If $p \to \infty$, then

$$|b_p - z_p| < h_p = \delta_p c_p/m_p = o(c_p/n_{p+1}), \quad |c_p - b_p| = |c_p - c_p e^{i\pi/m_p}| \sim c_p\pi/n_{p+1}.$$

This implies, by the triangle inequality, that

$$|c_p - z_p| \sim c_p\pi/n_{p+1} = c_p\pi/\overline{\Phi}^{\,-1}(\ln c_p), \quad p \to \infty.$$

Therefore,

$$\varliminf_{r\uparrow R} d(r, f)\frac{\overline{\Phi}^{\,-1}(\ln r)}{r} \leq \varliminf_{p\to\infty} d(c_p, f)\frac{\overline{\Phi}^{\,-1}(\ln c_p)}{c_p} \leq \varliminf_{p\to\infty} |c_p - z_p|\frac{\overline{\Phi}^{\,-1}(\ln c_p)}{c_p} = \pi.$$

This and Theorem 2 imply equality (9).

Next, we note that the above estimates imply the relation $M(c_k, f) \sim 2\mu(c_k, f)$, $k \to \infty$. Since the functions $\ln M(r, f)$ and $\ln \mu(r, f)$ are convex and linear with respect to $\ln r$ on $[c_k, c_{k+1}]$, respectively, and $\mu(r, f) \leq M(r, f)$ by the Cauchy inequality, we have

$$0 \leq \ln M(r, f) - \ln \mu(r, f) \leq \ln 3$$

for all $r < R$ sufficiently close to R. This and the equality $t_\Phi(f) = 1$ imply that $T_\Phi(f) = 1$. □

REFERENCES

1. I.V. Ostrovskii, *Distance between a maximum modulus point of an entire function and its zero set*, Operator theory, subharmonic functions, Kyiv: Naukova Dumka, 1991, 67–75. (in Russian)
2. I. Ostrovskii, A.E. Üreyen, *Distance between a maximum modulus point and zero set of an entire function*, Complex Variables, Theory Appl., **48** (2003), №7, 583–598.
3. I. Ostrovskii, A.E. Üreyen, *Maximum modulus points and zero sets of entire functions of regular growth*, C. R. Acad. Sci. Paris. Ser. I., **341** (2005), №8, 481–484.
4. I. Ostrovskii, A.E. Üreyen, *On maximum modulus points and zero sets of entire functions of regular growth*, Rocky Mt. J. Math., **38** (2008), №2, 583–618.
5. A.E. Üreyen, *On maximum modulus points and the zero set for an entire function of either zero or infinite order*, Comput. Methods Funct. Theory, **4** (2005), №2, 341–354.
6. S.I. Fedynyak, *On maximum modulus points and zero set for an entire function*, Mat. Stud., **30** (2008), №2, 169–172.

7. M.A. Evgrafov, Asymptotic estimates and entire functions, Moscow: Nauka, 1979. (in Russian)
8. T.Ya. Hlova, P.V. Filevych, *Generalized types of the growth of Dirichlet series*, Carpathian Math. Publ., **7** (2015), №2, 172–187.
9. P.V. Filevych, *On the slow growth of power series convergent in the unit disk*, Mat. Stud., **16** (2001), №2, 217–221.

Department of Applied Mathematics and Statistics
Ukrainian Catholic University, Lviv, Ukraine
napets.fed@gmail.com
Department of Computational Mathematics and Programming
Lviv Polytechnic National University, Lviv, Ukraine
p.v.filevych@gmail.com

Received 21.03.2019