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ON GENERALIZED PREOPEN SETS

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Firstly in this paper, we find some conditions under which μ-preopen sets of a GTS or μ-space X may be equivalent to μ-open in X. Finally, we obtain some characterizations of generalized paracompactness of a GTS or μ-space X via μ-preopen sets in X.

1. Introduction. Let X be a nonempty set and $\mathscr{T}$ be a topology on X. For the time being, we simply write X to mean the topological space $(X, \mathscr{T})$. For a subset A of X, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ stand respectively for closure and interior of A in the topological space X.

There are several generalizations of open sets of topological spaces. Two well-discussed generalizations of open sets of topological spaces are semi-open and pre-open sets. A subset A of a topological space X is semi-open [8, Levine] if there exists an open set G such that $G \subset A \subset \mathrm{Cl}(G)$. A subset A of a topological space X is pre-open (see [9, Mashhour et al.]) introduced under the name locally dense by Corson and Michael [2] if there exists an open set U such that $A \subset U \subset \mathrm{Cl}(A)$.

It is observed that the semi-open, pre-open sets of a topological space X possess properties resembling those of open sets of the topological space with some exception e.g. the family of semi-open (or pre-open) sets are not closed even under finite intersections. Starting from this observation, Császár [6] finally introduced and studied the concepts of generalized topology.

Let $\exp(X)$ be the power set of the nonempty set X. A subcollection μ of $\exp(X)$ is called a generalized topology [6] on X if $\varnothing \in \mu$ and the union of arbitrary number of elements of μ is again a member of μ. A generalized topological space [6] is a nonempty set X endowed with a generalized topology μ and it is denoted by (X, μ). We write GT spaces (resp. GTS) to denote generalized topological spaces (X, μ) (resp. generalized topological space (X, μ)). An element of μ is called a μ-open set of (X, μ). The complement of a μ-open set is called a μ-closed set of (X, μ). A generalized topological space (X, μ) is called strong [5] (also called μ-space by Noiri [11]) if $X \in \mu$. For brevity, we retain the term μ-space due to Noiri [11] to mean the strongly generalized topological space (X, μ) as well.

Henceforth unless otherwise mentioned, X stands for a GTS or μ-space to be understood from the context. For a subset A of X, the intersection of all μ-closed sets containing A is the generalized closure [4] of A and is denoted by $c_\mu(A)$. Also for a subset A of X, the union of all μ-open sets contained in A is the generalized interior [4] of A and is denoted by $i_\mu(A)$. It is easy to see that a subset A of X is μ-open (resp. μ-closed) if and only if $A = i_\mu(A)$ (resp. $A = c_\mu(A)$). Also for a subset A of X, we have $c_\mu(A) = X - i_\mu(X - A)$.

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Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$, the set of real numbers.

2. μ-preopen sets. We recall some known definitions and results to use in the sequel.

Definition 1 (Császár [4]). A subset A of X is called *μ-preopen* if $A \subset i_\mu(c_\mu(A))$.

As usual, the complement of a μ-preopen set is called *μ-preclosed*. So a subset A of X is μ-preclosed if $c_\mu(i_\mu(A)) \subset A$.

It is easy to prove that a subset A of X is μ-preopen if and only if there exists a μ-open set G such that $A \subset G \subset c_\mu(A)$.

Definition 2 (Ekici [7]). A subset D of X is said to be *μ-dense* if $c_\mu(D) = X$.

The first author [10] of this paper, studied a covering property, namely μ-precompactness via μ-preopen sets of μ-spaces. In this paper, we obtain some more properties of μ-preopen sets on GT spaces or μ-spaces.

We see that μ-open sets in X are μ-preopen in X but the converse need not be true. It is then quite natural to grow interest in mind when μ-preopen sets in X may be μ-open in X. To find some answers to this point, we first introduce the following.

Definition 3. A GTS or μ-space X is said to be a *μ-door space* if every subset of X is either μ-open or μ-closed.

Theorem 1. *Each μ-preopen set of a μ-door space is μ-open.*

Proof. Let V be a μ-preopen set of a μ-door space X. Then $V \subset i_\mu(c_\mu(V))$. If V is not μ-open, then it is μ-closed. So we have $c_\mu(V) = V$ which implies $i_\mu(c_\mu(V)) = i_\mu(V)$. By the definition, $i_\mu(V) = i_\mu(c_\mu(V)) \subset V$. Since V is not μ-open, $i_\mu(V) = i_\mu(c_\mu(V)) \subsetneq V$ which is a contradiction to the fact that $V \subset i_\mu(V) = i_\mu(c_\mu(V))$. □

Theorem 2. *Each singleton in a μ-space X is either μ-open or μ-closed if each μ-preopen set of X is μ-open.*

Proof. Firstly, suppose that for $x \in X$, $\{x\}$ is not μ-closed. Since each μ-preopen set in X is μ-open, $\{x\}$ is not μ-preclosed and so

$$c_\mu(i_\mu(\{x\})) \nsubseteq \{x\}. \tag{1}$$

Now the following two cases may arise.

Case i. $\{x\}$ is μ-open. Then $c_\mu(i_\mu(\{x\})) = c_\mu(\{x\})$ which satisfies (1).

Case ii. $\{x\}$ is not μ-open. Then $c_\mu(i_\mu(\{x\})) = c_\mu(\varnothing) = \varnothing$ which contradicts (1).

So we conclude that $\{x\}$ is μ-open. □

Theorem 2 may not be true in a GTS since in a GTS, X may not be μ-open and $\varnothing$ may not be μ-closed.

Example 1 (Ekici [7]). Let $X = \{a, b, c\}$ and $\mu = \{\varnothing, \{a\}, \{c, a\}\}$. In the GTS (X, μ), each μ-preopen set is μ-open but $\{c\}$ is neither μ-open nor μ-closed.

Theorem 3. *In a μ-space X, each μ-dense set is μ-open if each μ-preopen set in X is μ-open.*

Proof. Let A be μ-dense in X. Then $i_\mu(c_\mu(A)) = X$, since X is μ-open in X. Then obviously, $A \subset i_\mu(c_\mu(A))$. So A is μ-preopen. By hypothesis, we conclude that A is μ-open. □

In Example 1, each μ-preopen set of the GTS (X, μ) is μ-open and $\{a, b\}$ is μ-dense. But $\{a, b\}$ is not μ-open in X. So we conclude that Theorem 3 need not be true in a GTS.

We call a subset A in a GTS X is purely μ-preopen if A is μ-preopen but not μ-open. So if A is purely μ-preopen in X then $A \neq \varnothing, X$. It also follows that in a GTS X, X is μ-dense.

Theorem 4. *There exists a μ-dense set other than X in a GTS X if X contains a purely μ-preopen set.*

Proof. Let A be a purely μ-preopen set in a GTS X. Then $A \subset i_\mu(c_\mu(A)) \subset c_\mu(A)$. If $c_\mu(A) = X$, then $A \neq X$ is μ-dense in X. If $c_\mu(A) \neq X$, we put $G = i_\mu(c_\mu(A))$. Then we see that $G \neq X$ and $c_\mu(A) = c_\mu(G)$. Now

$$c_\mu((X - G) \cup A) = c_\mu(X - G) \cup c_\mu(A) = (X - G) \cup c_\mu(G) \supset (X - G) \cup G = X.$$

So we get $c_\mu((X - G) \cup A) = X$. Since A is purely μ-preopen and $A \subset G$, $G - A \neq \varnothing$. Then it follows that $(X - G) \cup A \neq X$. Hence $(X - G) \cup A \neq X$ is μ-dense in X. □

Corollary 1. *Let A be μ-preopen in a GTS X and there exist $G \subset X$ such that $A \subset G \subset c_\mu(A)$. Then $(X - G) \cup A$ is μ-dense in X.*

Proof. Similar to that of Theorem 4. □

Theorem 5. *In a μ-space X, each subset of X is μ-preopen if and only if each μ-open set in X is μ-closed.*

Proof. Firstly, we suppose that each subset of X is μ-preopen and U is μ-open in X. So $X - U$ is μ-closed. Since by hypothesis, $c_\mu(X - U)$ is μ-preopen, we have

$$c_\mu(X - U) \subset i_\mu(c_\mu(c_\mu(X - U))) = i_\mu(c_\mu(X - U)) \text{ since } X - U = c_\mu(X - U) = i_\mu(X - U).$$

So we see that $X - U = c_\mu(X - U) \subset i_\mu(X - U)$. Also by definition, $i_\mu(X - U) \subset X - U$. Hence we get $X - U = i_\mu(X - U)$. It means that $X - U$ is μ-open and hence U is μ-closed.

Conversely, let each μ-open set in X is μ-closed and A be any subset of X. We see that $X - c_\mu(A)$ is μ-open and so also μ-closed. Therefore $X - c_\mu(A) = c_\mu(X - c_\mu(A)) = X - i_\mu(c_\mu(A))$ which implies $c_\mu(A) = i_\mu(c_\mu(A))$. Hence $A \subset c_\mu(A) = i_\mu(c_\mu(A))$. Thus A is μ-preopen. □

Theorem 6. *In a GTS X, each subset $A(\neq \varnothing, X)$ of X is μ-preopen if and only if each μ-open set $U(\neq \varnothing, X)$ in X is μ-closed.*

Proof. Similar to that of Theorem 5. □

3. The Generalized paracompactness via μ-preopen sets. In the study of covering properties of GT spaces or μ-spaces (X, μ), 'μ-open cover' of X is defined to be the collection $\mathscr{C}$ of μ-open sets of X such that $\bigcup_{U \in \mathscr{C}} U = X$. Now in GT spaces, X may not be μ-open and so even the union of all μ-open sets of X may not be equal to X. Hence such definition of μ-open covers of GT spaces become void. To avoid such nullified cases, we change the existing ideas of covering properties in GT spaces slightly.

In this section, we write 'a space' to mean a GTS or μ-space and a space (X,μ) is to be denoted by X.

Suppose that $X_\mu = \bigcup_{G\in\mu} G$. Obviously, X_μ is μ-open. If X is a GTS with $X \notin \mu$, then X_μ is not μ-closed. Also note that in the case of μ-spaces, $X_\mu = X$. A collection $\mathscr{C}$ of subsets of X is called a cover of X if $\bigcup_{G\in\mathscr{C}} G = X_\mu$. We write '$\mu$-open collection of X' and 'μ-preopen collection of X' to mean a collection consisting μ-open sets and μ-preopen sets respectively of X. A μ-open collection (resp. μ-preopen collection) $\mathscr{C}$ of X is said to be a μ-open (resp. μ-preopen) cover of X if $\bigcup_{G\in\mathscr{C}} G = X_\mu$.

Let $\mathscr{U}$ and $\mathscr{V}$ be two covers of X. The cover $\mathscr{V}$ is called a refinement [12, p. 144] of the cover $\mathscr{U}$ if for each $V \in \mathscr{V}$, there exists a $U \in \mathscr{U}$ such that $V \subset U$. If the covers $\mathscr{U}$ and $\mathscr{V}$ both are μ-open covers of X, then $\mathscr{V}$ is called a μ-open refinement of $\mathscr{U}$ [1]. However, if $\mathscr{V}$ is a μ-preopen cover and $\mathscr{U}$ is a μ-open cover of X, then $\mathscr{V}$ is called a μ-preopen refinement of $\mathscr{U}$.

Definition 4 (cf. Arar [1], Deb Ray and Bhowmick [3])**.** A collection $\mathscr{U}$ of subsets of X is called *μ-locally finite* if for each $x \in X_\mu$, there exists a μ-open set U with $x \in U$ meeting only finitely many members of $\mathscr{U}$.

Definition 5 (cf. Arar [1], Deb Ray and Bhowmick [3])**.** A space X is called a *μ-paracompact* space if each μ-open cover of X has a μ-locally finite μ-open refinement.

Definition 6. A μ-preopen set G in a space X is said to be *capped* by a μ-open set if $G \subset U$ and $G \subset V$ for μ-open sets U, V in X, then there exists a μ-open set W such that $G \subset W \subset U \cap V$.

Example 2. Let $a, b \in \mathbb{R}$, $X = [-1, 1]$ and $\mu = \{\varnothing\} \cup \{[-1, a) \mid 0 \leq a \leq 1\}$. Then (X, μ) is a GTS. We note that μ-preopen sets are always contained in a μ-open set. If A is a μ-preopen set contained in the μ-open set $[-1, 0)$, then A is capped by the μ-open set $[-1, 0)$ irrespective of all μ-open sets containing A. If a μ-preopen set A is contained in the μ-open sets $[-1, a), [-1, b), 0 \leq a, b \leq 1$, then A is capped by the μ-open set $[-1, c), c = \min\{a, b\}$. So we see that all μ-preopen sets in X are capped by a μ-open set in X.

Theorem 7. *If a μ-preopen collection $\mathscr{C} = \{G_\alpha \mid \alpha \in \Delta\}$ of X is μ-locally finite, then there exists a μ-open collection $\mathscr{U} = \{U_\alpha \mid G_\alpha \subset U_\alpha, \alpha \in \Delta\}$ such that $\mathscr{U}$ and $\mathscr{V} = \{c_\mu(U) \mid U \in \mathscr{U}\}$ both are μ-locally finite.*

Further, if $\mathscr{C}$ is a cover of X, then $\mathscr{U}$ and $\mathscr{V}$ both are also covers of X.

Proof. Given the collection $\mathscr{C}$ of μ-preopen sets is μ-locally finite. So for each $x \in X_\mu$, there exists a μ-open set N_x such that $G_\alpha \cap N_x \neq \varnothing$ for finitely many $\alpha \in \Delta$. Let $G_{\alpha_k} \cap N_x \neq \varnothing$ for $\alpha_k \in \Delta$, $k \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. Then $G_\alpha \cap N_x = \varnothing$ for $\alpha \in \Delta - \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ which implies $c_\mu(G_\alpha) \cap N_x = \varnothing$ for $\alpha \in \Delta - \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Now for each $\alpha \in \Delta$, there exists a μ-open set U_α such that $G_\alpha \subset U_\alpha \subset c_\mu(G_\alpha)$. It gives $c_\mu(G_\alpha) = c_\mu(U_\alpha)$. So $c_\mu(U_\alpha) \cap N_x = \varnothing$ for $\alpha \in \Delta - \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Hence we see that N_x may intersect only finitely many members of $\mathscr{V} = \{c_\mu(U_\alpha) \mid \alpha \in \Delta\}$. Thus $\mathscr{V}$ is μ-locally finite.

Since for each $\alpha \in \Delta$, $U_\alpha \subset c_\mu(U_\alpha)$ and the collection $\mathscr{V} = \{c_\mu(U_\alpha) \mid \alpha \in \Delta\}$ is μ-locally finite, the collection $\{U_\alpha \mid \alpha \in \Delta\}$ is also μ-locally finite.

Now we suppose that $\mathscr{C}$ is a cover of X. Then for each $x \in X_\mu$, there exists a $G_\alpha \in \mathscr{C}$ such that $x \in G_\alpha$. So we get $x \in G_\alpha \subset U_\alpha \subset c_\mu(G_\alpha) = c_\mu(U_\alpha)$. Hence it follows that $\mathscr{U}$ and $\mathscr{V}$ both are covers of X. $\square$

Theorem 8. *Let μ-preopen sets in a space X are capped by μ-open sets of X. Then the space X is μ-paracompact if and only if each μ-open covers of X has a μ-locally finite μ-preopen refinement.*

Proof. The necessity follows easily since μ-open sets on a space X are also μ-preopen in X.

To prove the sufficiency, let $\mathscr{C} = \{G_\alpha \mid \alpha \in \Delta\}$ be a μ-open cover of X and $\mathscr{V} = \{V_\beta \mid \beta \in \Gamma\}$ be a μ-locally finite μ-preopen refinement of $\mathscr{C}$. By Theorem 7, there exists a μ-locally finite μ-open cover $\mathscr{U} = \{U_\beta \mid U_\beta \subset V_\beta, \beta \in \Gamma\}$ and for each $x \in X_\mu$, there exists a $\beta(x) \in \Gamma$ such that $x \in V_{\beta(x)} \subset U_{\beta(x)}$. But for $\beta(x) \in \Gamma$, there is an $\alpha(x) \in \Delta$ such that $x \in V_{\beta(x)} \subset G_{\alpha(x)}$. Since μ-preopen sets in X are capped by μ-open sets, there exists a μ-open set H_x such that $x \in V_{\beta(x)} \subset H_x \subset U_{\beta(x)} \cap G_{\alpha(x)} \subset G_{\alpha(x)}$. So $\mathscr{H} = \{H_x \mid x \in X_\mu\}$ is an μ-open refinement of $\mathscr{C}$. Since $\mathscr{U}$ is μ-locally finite and we see that $H_x \subset U_{\beta(x)}$, $\mathscr{H}$ is also μ-locally finite. Hence we get a μ-locally finite μ-open refinement $\mathscr{H} = \{H_x \mid x \in X_\mu\}$ of $\mathscr{C}$. It implies that X is μ-paracompact. □

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