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LOCALLY UNIVALENCE OF DIRICHLET SERIES SATISFYING A LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER WITH EXPONENTIAL COEFFICIENTS

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Let $h > 0$, $\gamma_0 \neq 0$, $\gamma_1 \neq 0$, $\gamma_2 \leq -1$ and $|\gamma_1|/4 + |\gamma_0|/6 \leq h/5$. It is proved that the differential equation $w'' + (\gamma_0 e^{2hs} + \gamma_1 e^{hs} + \gamma_2)w = 0$ has an entire solution $F(s) = \exp\{s\lambda_1\} + \sum_{k=2}^{\infty} f_k \exp\{s\lambda_k\}$ locally univalent in the half-plane $\{s\colon \operatorname{Re} s < 0\}$ and such that $\ln M(\sigma, F) \sim (\sqrt{|\gamma_0|}/h)e^{h\sigma}$ as $\sigma \to +\infty$, where $M(\sigma, F) = \sup\{|F(\sigma + it)|\colon t \in \mathbb{R}\}$.

1. Introduction. An analytic univalent in $\mathbb{D} = \{z\colon |z| < 1\}$ function

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

is said to be convex if $f(\mathbb{D})$ is a convex domain. It is well known [1, p. 203] that the condition $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0$ $(z \in \mathbb{D})$ is necessary and sufficient for the convexity of f. Due to W. Kaplan ([2]) the function f is said to be close-to-convex in $\mathbb{D}$ (see also [1, p. 583]) if there exists a convex in $\mathbb{D}$ function Φ such that $\operatorname{Re}(f'(z)/\Phi'(z)) > 0\,(z \in \mathbb{D})$. A close-to-convex function f has a characteristic property that the complement G of the domain $f(\mathbb{D})$ can be filled with rays L which go from ∂G and lie in G. Every close-to-convex in $\mathbb{D}$ function f is univalent in $\mathbb{D}$ and, therefore, $f'(0) \neq 0$. Hence it follows that the function f is close-to-convex in $\mathbb{D}$ if and only if the function

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \tag{2}$$

is close-to-convex in $\mathbb{D}$, where $g_n = f_n/f_1$. We remark also, that the function (2) is said to be starlike in $\mathbb{D}$, if $f(\mathbb{D})$ is starlike domain with respect to the origin. It is clear, that every starlike function is close-to-convex.

S. M. Shah [3] indicated conditions on real parameters γ_0, γ_1, γ_2 of the differential equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (\gamma_0 z^2 + \gamma_1 z + \gamma_2)w = 0, \tag{3}$$

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under which there exists an entire transcendental solution (1) such that f and all its derivatives are close-to-convex in $\mathbb{D}$. In particular, he obtained the following result.

Theorem A. *If* $\gamma_1 = \gamma_2 = 0$, $-2 < \gamma_0 < 0$ *then equation* (3) *has an entire solution* (2) *such that all derivatives* $g^{(2n+1)}$ $(n \geq 0)$ *are close-to-convex in* $\mathbb{D}$*, and if* $\gamma_1 = 0$, $0 < |\gamma_0| < 2$, $\gamma_2 = -1$ *then the equation* (3) *has an entire solution* (2) *such that all derivatives* $g^{(2n)}$ $(n \geq 0)$ *are close-to-convex in* $\mathbb{D}$*. For such solutions* $\ln M_g(r) = (1 + o(1))\sqrt{|\gamma_0|}r$ *as* $r \to +\infty$*, where* $M_g(r) = \max\{|g(z)|\colon\ |z| = r\}$.

Substituting $z = e^s$ into (3) we obtain the differential equation

$$\frac{d^2w}{ds^2} + (\gamma_0 e^{2hs} + \gamma_1 e^{hs} + \gamma_2)w = 0, \tag{4}$$

with $h = 1$. We will consider the case, where h is an arbitrary positive number and using a certain result of M.S. Robertson we will investigate the locally univalence in some half-plane of Dirichlet series with positive exponents satisfying the differential equation (4).

2. Auxiliary results. Let $0 < \lambda_1 < \lambda_k \uparrow +\infty$ and $f_k \neq 0$ for all $k \geq 2$ and a Dirichlet series

$$F(s) = \exp\{s\lambda_1\} + \sum_{k=2}^{\infty} f_k \exp\{s\lambda_k\}, \quad s = \sigma + it, \tag{5}$$

has the abscissa of absolute convergence $\sigma_a \in (-\infty, +\infty]$. Then there is the greatest real number $\tau \in (-\infty, \sigma_a]$, for which

$$\sum_{k=2}^{\infty} \lambda_k |f_k| \exp\{\tau\lambda_k\} \leq 1. \tag{6}$$

Those functions F given by Dirichlet series (5) which satisfy (6) is said to be of the class τ. Recalling certain concepts of univalence introduced by P. Montel [4], we say that a function f is locally univalent in a region G if f is regular in G and if, for every closed domain $G^* \subset G$ and for every point $z_0 \in G^*$, there exists a positive number ϱ independependent of z_0 such that f is univalent in every disk $\{z\colon\ |z - z_0| < \varrho\}$ lying within G. M.S. Robertson [5] proved the following theorem.

Theorem B. *Let function* (5) *belong to the class* τ*. If* $\tau > -\frac{\ln \lambda_1}{\lambda_1}$ *then* F *is locally univalent in* $\{s\colon\ Re\, s < \tau\}$*, and if* $\tau \leq -\frac{\ln \lambda_1}{\lambda_1}$ *then* F *is locally univalent in*

$$\left\{s\colon Re\, s < \frac{\tau\lambda_2 + \ln \lambda_1}{\lambda_2 - \lambda_1} \leq \tau\right\}.$$

Now we return to differential equation (4), where $h_0 > 0$, $\gamma_0 \neq 0$, $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$. Substituting (5) in (4), we obtain

$$\sum_{k=1}^{\infty} f_k(\lambda_k^2 + \gamma_2) \exp\{s\lambda_k\} = -\sum_{k=1}^{\infty} \gamma_1 f_k \exp\{s(\lambda_k + h)\} - \sum_{k=1}^{\infty} \gamma_0 f_k \exp\{s(\lambda_k + 2h)\}. \tag{7}$$

If $\lambda_1^2 + \gamma_2 \neq 0$ then $(\lambda_1^2 + \gamma_2) \exp\{s\lambda_1\} = -(1 + o(1))\gamma_1 \exp\{s(\lambda_1 + h)\}$, $\sigma \to -\infty$, that is $\lambda_1^2 + \gamma_2 = -(1 + o(1))\gamma_1 \exp\{sh\} \to 0$ $(\sigma \to -\infty)$, which is impossible. Thus, $\lambda_1^2 + \gamma_2 = 0$, $\gamma_2 < 0$ and $\lambda_1 = \sqrt{-\gamma_2}$. Therefore, the equality (7) implies

$$\sum_{k=2}^{\infty} f_k(\lambda_k^2 + \gamma_2) \exp\{s\lambda_k\} = -\sum_{k=1}^{\infty} \gamma_1 f_k \exp\{s(\lambda_k + h)\} - \sum_{k=1}^{\infty} \gamma_0 f_k \exp\{s(\lambda_k + 2h)\}. \tag{8}$$

and since $\lambda_k^2+\gamma_2\neq 0$, we have

$$f_2(\lambda_2^2+\gamma_2)\exp\{s\lambda_2\}=-(1+o(1))\gamma_1\exp\{s(\lambda_1+h)\},\quad \sigma\to-\infty.$$

Hence it follows that $\lambda_2=\lambda_1+h$ and $f_2=\frac{-\gamma_1}{\lambda_2^2+\gamma_2}$. But then from (8) it follows that

$$\sum_{k=3}^{\infty} f_k(\lambda_k^2+\gamma_2)\exp\{s\lambda_k\}=-\sum_{k=2}^{\infty}\gamma_1 f_k\exp\{s(\lambda_k+h)\}-\sum_{k=1}^{\infty}\gamma_0 f_k\exp\{s(\lambda_k+2h)\}. \tag{9}$$

Hence we obtain as $\sigma\to-\infty$

$$f_3(\lambda_3^2+\gamma_2)\exp\{s\lambda_3\}=-(1+o(1))\gamma_1 f_2\exp\{s(\lambda_1+2h)\}-\gamma_0\exp\{s(\lambda_1+2h)\}=$$
$$=-(1+o(1))(\gamma_1 f_2+\gamma_0)\exp\{s(\lambda_1+2h)\},$$

i. e. $\lambda_3=\lambda_1+2h$, $f_3=-\frac{\gamma_1 f_2}{\lambda_3^2+\gamma_2}-\frac{\gamma_0}{\lambda_3^2+\gamma_2}$. Then from (9) it follows that

$$\sum_{k=4}^{\infty} f_k(\lambda_k^2+\gamma_2)\exp\{s\lambda_k\}=-\sum_{k=3}^{\infty}\gamma_1 f_k\exp\{s(\lambda_k+h)\}-\sum_{k=2}^{\infty}\gamma_0 f_k\exp\{s(\lambda_k+2h)\},$$

whence, as above, we obtain

$$\lambda_4=\lambda_3+h,\quad f_4=-\frac{\gamma_1 f_3}{\lambda_4^2+\gamma_2}-\frac{\gamma_0 f_2}{\lambda_4^2+\gamma_2}.$$

Continuing this process, we come to the formulas

$$\lambda_k=\lambda_{k-1}+h,\quad f_k=-\frac{\gamma_1 f_{k-1}}{\lambda_k^2+\gamma_2}-\frac{\gamma_0 f_{k-2}}{\lambda_k^2+\gamma_2}\quad (k\geq 3). \tag{10}$$

Thus, the following statement is true.

Lemma 1. *Let $h>0$, $\gamma_0\neq 0$, $\gamma_1\neq 0$ and $\gamma_2<0$. Then the differential equation* (4) *has a solution*

$$F(s)=\exp\{s\sqrt{|\gamma_2|}\}+\sum_{k=2}^{\infty} f_k\exp\{s\lambda_k\},\quad \lambda_k=\sqrt{|\gamma_2|}+(k-1)h, \tag{11}$$

where $f_2=\frac{-\gamma_1}{\lambda_2^2+\gamma_2}$ and for $k\geq 3$ the coefficients f_k are determined by recurrent formula (10).

We need also the following lemma.

Lemma 2. *Let $h>0$, $\gamma_0\neq 0$, $\gamma_1\neq 0$ and $\gamma_2<0$. If for some $\eta\in\mathbb{R}$*

$$\frac{\lambda_3|\gamma_1|}{\lambda_2(\lambda_3^2+\gamma_2)}e^{h\eta}+\frac{\lambda_4|\gamma_0|}{\lambda_2(\lambda_4^2+\gamma_2)}e^{2h\eta}+\lambda_2|f_2|e^{\eta\lambda_2}+\frac{\lambda_3|\gamma_0|}{\lambda_3^2+\gamma_2}e^{\eta\lambda_3}\leq 1, \tag{12}$$

then for the function (5)

$$\sum_{k=2}^{\infty}\lambda_k|g_k|\exp\{\eta\lambda_k\}\leq 1. \tag{13}$$

Proof. In view of (10)

$$\sum_{k=2}^{\infty}\lambda_k|f_k|\exp\{\eta\lambda_k\}\le\lambda_2|f_2|\exp\{\eta\lambda_2\}+$$
$$+\sum_{k=3}^{\infty}\left(\frac{\lambda_k|\gamma_1||f_{k-1}|}{\lambda_k^2+\gamma_2}\exp\{\eta\lambda_k\}+\frac{\lambda_k|\gamma_0||f_{k-2}|}{\lambda_k^2+\gamma_2}\exp\{\eta\lambda_k\}\right)=$$
$$=\lambda_2|f_2|\exp\{\eta\lambda_2\}+\sum_{k=2}^{\infty}\frac{\lambda_{k+1}|\gamma_1||f_k|}{\lambda_{k+1}^2+\gamma_2}\exp\{\eta\lambda_{k+1}\}+\sum_{k=1}^{\infty}\frac{\lambda_{k+2}|\gamma_0||f_k|}{\lambda_{k+2}^2+\gamma_2}\exp\{\eta\lambda_{k+2}\}=$$
$$=\lambda_2|f_2|\exp\{\eta\lambda_2\}+\frac{\lambda_3|\gamma_0||f_1|}{\lambda_3^2+\gamma_2}\exp\{\eta\lambda_3\}+$$
$$+\sum_{k=2}^{\infty}\frac{\lambda_{k+1}|\gamma_1||f_k|}{\lambda_{k+1}^2+\gamma_2}\exp\{\eta\lambda_{k+1}\}+\sum_{k=2}^{\infty}\frac{\lambda_{k+2}|\gamma_0||f_k|}{\lambda_{k+2}^2+\gamma_2}\exp\{\eta\lambda_{k+2}\},$$

whence

$$\sum_{k=2}^{\infty}\left(1-\frac{\lambda_{k+1}|\gamma_1|\exp\{\eta(\lambda_{k+1}-\lambda_k)\}}{\lambda_k(\lambda_{k+1}^2+\gamma_2)}-\frac{\lambda_{k+2}|\gamma_0|\exp\{\eta(\lambda_{k+2}-\lambda_k)\}}{\lambda_k(\lambda_{k+2}^2+\gamma_2)}\right)\times$$
$$\times\lambda_k|f_k|\exp\{\eta\lambda_k\}\le\lambda_2|f_2|\exp\{\eta\lambda_2\}+\frac{\lambda_3|\gamma_0|}{\lambda_3^2+\gamma_2}\exp\{\eta\lambda_3\}.$$

Since the sequences $\left(\frac{\lambda_{k+1}|\gamma_1|}{\lambda_k(\lambda_{k+1}^2+\gamma_2)}\right)$ and $\left(\frac{\lambda_{k+2}|\gamma_1|}{\lambda_k(\lambda_{k+2}^2+\gamma_2)}\right)$ are decreasing and $\lambda_{k+1}-\lambda_k=h$, we have

$$\sum_{k=2}^{\infty}\left(1-\frac{\lambda_3|\gamma_1|\exp\{h\eta\}}{\lambda_2(\lambda_3^2+\gamma_2)}-\frac{\lambda_4|\gamma_0|\exp\{2h\eta\}}{\lambda_2(\lambda_4^2+\gamma_2)}\right)\lambda_k|f_k|\exp\{\eta\lambda_k\}\le$$
$$\le\lambda_2|f_2|\exp\{\eta\lambda_2\}+\frac{\lambda_3|\gamma_0|}{\lambda_3^2+\gamma_2}\exp\{\eta\lambda_3\}.$$

From (12) it follows that

$$\frac{\lambda_3|\gamma_1|\exp\{h\eta\}}{\lambda_2(\lambda_3^2+\gamma_2)}+\frac{\lambda_4|\gamma_0|\exp\{2h\eta\}}{\lambda_2(\lambda_4^2+\gamma_2)}<1.$$

Therefore,

$$\sum_{k=2}^{\infty}\lambda_k|f_k|\exp\{\eta\lambda_k\}\le\frac{\lambda_2|f_2|\exp\{\eta\lambda_2\}+\frac{\lambda_3|\gamma_0|}{\lambda_3^2+\gamma_2}\exp\{\eta\lambda_3\}}{1-\frac{\lambda_3|\gamma_1|\exp\{h\eta\}}{\lambda_2(\lambda_3^2+\gamma_2)}-\frac{\lambda_4|\gamma_0|\exp\{2h\eta\}}{\lambda_2(\lambda_4^2+\gamma_2)}},$$

i. e. in view of (12) the inequality (13) holds. □

3. Main result. The following theorem is true.

Theorem. *Let $h>0$, $\gamma_0\ne0$, $\gamma_1\ne0$, $\gamma_2<0$ and for some $\eta\in\mathbb{R}$*

$$\frac{(\sqrt{|\gamma_2|}+2h)|\gamma_1|}{4h(\sqrt{|\gamma_2|}+h)^2}e^{h\eta}+\frac{(\sqrt{|\gamma_2|}+3h)|\gamma_0|}{3h(\sqrt{|\gamma_2|}+h)(3h+2\sqrt{|\gamma_2|})}e^{2h\eta}+$$

$$+\frac{|\gamma_1|(\sqrt{|\gamma_2|}+h)\exp\{\eta\sqrt{|\gamma_2|}\}}{h(h+\sqrt{|\gamma_2|})}e^{h\eta}+\frac{(\sqrt{|\gamma_2|}+2h)|\gamma_0|\exp\{\eta\sqrt{|\gamma_2|}\}}{4h(\sqrt{|\gamma_2|}+h)}e^{2h\eta}\le 1. \qquad (14)$$

Then the differential equation (4) *has the entire solution* (11) *locally univalent in the half-plan*

$$\left\{s\colon\ Re\, s<\min\left\{\eta,\ \frac{\eta(\sqrt{|\gamma_2|}+h)+\ln\sqrt{|\gamma_2|}}{h}\right\}\right\} \qquad (15)$$

and such that $\ln M(\sigma,F)=(1+o(1))\frac{\sqrt{|\gamma_0|}}{h}e^{h\sigma}$ *as* $\sigma\to+\infty$.

Proof. The conditions (14) and (12) are equivalent. Therefore, by Lemma 2 the condition (13) holds and, thus, F belongs to the class $\tau\ge\eta$. By Theorem B the function F is locally univalent in half-plane $\left\{s\colon\ \mathrm{Re}\, s<\min\left\{\tau,\ \frac{\tau\lambda_1+\ln\lambda_1|}{\lambda_2-\lambda_1}\right\}\right\}$ and, thus, in half-plane (15).

Now we show that series (11) is entire. Indeed, since for every $\sigma\in\mathbb{R}$ there exists $k_0=k_0(\sigma)\ge 3$ such that

$$\frac{|\gamma_1|}{\lambda_{k+1}^2+\gamma_2}e^{h\sigma}+\frac{|\gamma_0|}{l_{k+2}^2+\gamma_2}e^{2h\sigma}\le\frac{1}{2}\quad(k\ge k_0),$$

we have, as above,

$$\sum_{k=k_0}^{\infty}|f_k|\exp\{\sigma\lambda_k\}\le\sum_{k=k_0-1}^{\infty}\frac{|\gamma_1||f_k|}{\lambda_{k+1}^2+\gamma_2}\exp\{\sigma\lambda_k\}\exp\{\sigma(\lambda_{k+1}-\lambda_k)\}+$$
$$+\sum_{k=k_0-2}^{\infty}\frac{|\gamma_0||f_k|}{\lambda_{k+2}^2+\gamma_2}\exp\{\sigma\lambda_k\}\exp\{\sigma(\lambda_{k+2}-\lambda_k)\}=$$
$$=\frac{|\gamma_1||f_{k_0-1}|\exp\{\sigma\lambda_{k_0}\}}{\lambda_{k_0}^2+\gamma_2}+\frac{|\gamma_0||f_{k_0-2}|\exp\{\sigma\lambda_{k_0}\}}{\lambda_{k_0}^2+\gamma_2}+\frac{|\gamma_0||f_{k_0-1}|\exp\{\sigma\lambda_{k_0-1}\}}{\lambda_{k_0-1}^2+\gamma_2}+$$
$$+\sum_{k=k_0}^{\infty}\left(\frac{|\gamma_1|}{\lambda_{k+1}^2+\gamma_2}e^{h\sigma}+\frac{|\gamma_0|}{\lambda_{k+2}^2+\gamma_2}e^{2h\sigma}\right)|f_k|\exp\{\sigma\lambda_k\}\le$$
$$\le\frac{|\gamma_1||f_{k_0-1}|\exp\{\sigma\lambda_{k_0}\}}{\lambda_{k_0}^2+\gamma_2}+\frac{|\gamma_0||f_{k_0-2}|\exp\{\sigma\lambda_{k_0}\}}{\lambda_{k_0}^2+\gamma_2}+\frac{|\gamma_0||f_{k_0-1}|\exp\{\sigma\lambda_{k_0-1}\}}{\lambda_{k_0-1}^2+\gamma_2}+$$
$$+\frac{1}{2}\sum_{k=k_0}^{\infty}|f_k|\exp\{\sigma\lambda_k\},$$

whence it follows that $\sum_{k=k_0}^{\infty}|f_k|\exp\{\sigma\lambda_k\}<+\infty$, i. e. Dirichlet series (11) is absolutely convergent in $\mathbb{C}$.

For the study of the growth of (11) we use the Wiman-Valiron method. Let

$$\mu(\sigma,F)=\max\{|f_k|\exp\{\sigma\lambda_k\}\colon\ k\ge 1\}$$

be the maximal term of the Dirichlet series of form (5) and

$$\nu(\sigma,F)=\max\{k\colon\ |f_k|\exp\{\sigma\lambda_k\}=\mu(\sigma,F)\}$$

be its central index.

We suppose that the exponents of the Dirichlet series (5) satisfy the condition

$$\int_0^{\infty}t^{-2}\ln n(t)dt<+\infty,$$

where $n(t)=\sum\limits_{0<\lambda_n\le t} 1$, and put

$$\eta(x)=\int_x^{\infty} t^{-2}\ln\, n(t)dt,\quad l(x)=\frac{1}{\eta(x)}\ln^{-2}\frac{1}{\eta(x)},\quad k(x)=x\sqrt{1/l(x)}.$$

Then [6] for every $m\in\mathbb{N}$ and all s, $\operatorname{Re}s=\tau$, $|\tau-\sigma|<1/(30k(\lambda_\nu))$,

$$F^{(m)}(s)=\lambda_\nu^m(F(s)+o(M(\tau,F))),\quad \nu=\nu(\sigma,F),\tag{16}$$

as $0\le\sigma\to+\infty$ outside of some set $E\subset[0+\infty)$ of finite measure and E is contained in the union of intervals $[R_\nu+\tau_{\nu-1},\ R_\nu+\tau_\nu)$ and $\tau_\nu-\tau_{\nu-1}\to 0$ as $\nu\to\infty$.

Let $\delta(\sigma)$ be an arbitrary positive function on $[0,+\infty)$, which tends to zero as $\sigma\to+\infty$, and $\Delta(\sigma)=\{s\colon\ \operatorname{Re}s=\sigma,\ |F(s)|\ge(1-\delta(\sigma))M(\sigma,F)\}$. Then choosing $\tau=\sigma$ we obtain from (16)

$$F^{(m)}(s)=\lambda_\nu^m F(s)(1+\varepsilon(\sigma)),\quad s\in\Delta(\sigma),\tag{17}$$

where $\varepsilon(\sigma)\to 0$ as $\sigma\to+\infty$, $\sigma\notin E$.

Suppose that the Dirichlet series (11) satisfies (4). Substituting (17) in (4) we obtain $\lambda_\nu^2=|\gamma_0|e^{2h\sigma}(1+\varepsilon_1(\sigma))$, where $\varepsilon_1(\sigma)\to 0$ as $\sigma\to+\infty$, $\sigma\notin E$. Therefore,

$$\lambda_{\nu(\sigma,F)}=(1+o(1))\sqrt{|\gamma_0|}e^{h\sigma},\quad \sigma\to+\infty,\ \sigma\notin E.\tag{18}$$

If $\sigma\in E$, that is $\sigma_{\nu-1}=R_\nu+\tau_{\nu-1}\le\sigma<\sigma_\nu=R_\nu+\tau_\nu$ for some ν, then $\sigma_\nu-\sigma_{\nu-1}\to 0$ and, thus,

$$e^{h\sigma_{\nu-1}}=(1+o(1))e^{h\sigma}=(1+o(1))e^{h\sigma_\nu}$$

as $\nu\to\infty$. Therefore,

$$\begin{aligned}(1+o(1))e^{h\sigma}=(1+o(1))e^{h\sigma_{\nu-1}}=\lambda_{\nu(\sigma_{\nu-1},F)}\le\lambda_{\nu(\sigma,F)}\le\lambda_{\nu(\sigma_\nu,F)}=\\ =(1+o(1))e^{h\sigma_\nu}=(1+o(1))e^{h\sigma},\quad \sigma\to+\infty,\end{aligned}$$

i. e. (20) is true as $\sigma\to+\infty$. But [7, p. 182] $\ln\,\mu(\sigma,F)=\ln\,\mu(0,F)+\int_0^\sigma\lambda_{\nu(t,F)}dt$. Therefore,

$$\ln\,\mu(\sigma,F)=(1+o(1))\frac{\sqrt{|\gamma_0|}}{h}e^{h\sigma}$$

as $\sigma\to+\infty$ and, since $\ln\lambda_n\sim\ln\, n$ as $n\to\infty$, we have [8,9]

$$\ln\, M(\sigma,F)=(1+o(1))\ln\,\mu(\sigma,F),\quad \sigma\to+\infty.$$

□

Suppose that $\gamma_2\le-1$ and $\eta=0$. Then instead of (15) we have the half-plane

$$\{s\colon\ \operatorname{Re}s<\min\{0,(\ln\sqrt{|\gamma_2|})/h\}\}=\{s\colon\ \operatorname{Re}s<0\},$$

and condition (14) has the form

$$\frac{(\sqrt{|\gamma_2|}+2h)|\gamma_1|}{4(\sqrt{|\gamma_2|}+h)^2}+\frac{(\sqrt{|\gamma_2|}+3h)|\gamma_0|}{3(\sqrt{|\gamma_2|}+h)(3h+2\sqrt{|\gamma_2|})}+\frac{|\gamma_1|(\sqrt{|\gamma_2|}+h)}{(h+\sqrt{|\gamma_2|})}+\frac{(\sqrt{|\gamma_2|}+2h)|\gamma_0|}{4(\sqrt{|\gamma_2|}+h)}\le h.\tag{19}$$

Since

$$\frac{\sqrt{|\gamma_2|}+2h}{4(\sqrt{|\gamma_2|}+h)^2}\le\frac14,\quad \frac{\sqrt{|\gamma_2|}+3h}{3(\sqrt{|\gamma_2|}+h)(3h+2\sqrt{|\gamma_2|})}\le\frac13\ \text{ and }\ \frac{\sqrt{|\gamma_2|}+2h}{4(\sqrt{|\gamma_2|}+h)}\le\frac12,$$

condition (19) holds if $\frac{5|\gamma_1|}{4}+\frac{5|\gamma_0|}{6}\le h$, and the theorem implies the following corollary.

Corollary. *Let $h > 0$, $\gamma_0 \neq 0$, $\gamma_1 \neq 0$, $\gamma_2 \leq -1$ and $5|\gamma_1|/4 + 5|\gamma_0|/6 \leq h$. Then differential equation* (4) *has entire solution* (11) *locally univalent in the half-plane $\{s\colon \operatorname{Re} s < 0\}$ and such that*

$$\ln M(\sigma, F) = (1 + o(1))\frac{\sqrt{|\gamma_0|}}{h}e^{h\sigma}$$

as $\sigma \to +\infty$.

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