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# WEAK SOLUTIONS TO THE COMPLEX MONGE-AMPÈRE EQUATION ON OPEN SUBSETS OF $\mathbb{C}^n$

In the paper, we prove the existence of weak solutions to the complex Monge-Ampère equation in the class $\mathcal{D}(\Omega)$ on an open subset $\Omega$ of $\mathbb{C}^n$.

**1. Introduction.** As is well known, the complex Monge-Ampère operator plays a central role in pluripotential theory and has been extensively studied for many years. For a $\mathcal{C}^2$-smooth plurisubharmonic function $u$ defined on an open subset $\Omega$ of $\mathbb{C}^n$, its complex Monge-Ampère operator is defined by

$$(dd^c u)^n = n! 4^n \det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \right) dV_{2n},$$

where $dV_{2n}$ is the volume form of $\mathbb{C}^n$. However, by an example in [21], Shiffman and Taylor have shown that it is impossible to extend the domain of definition of complex Monge-Ampère operator in a meaningful way to the whole class of plurisubharmonic functions and still have the range contained in the class of non-negative Borel measures (see Appendix 1 in [21]). Bedford and Taylor [4] proved in 1982 that the complex Monge-Ampère operator is defined for locally bounded plurisubharmonic functions. Cegrell in [9] introduced and investigated in 2004 some classes of unbounded plurisubharmonic functions on bounded hyperconvex domains in $\mathbb{C}^n$. He has shown that the complex Monge-Ampère operator is well defined on the class $\mathcal{E}(\Omega)$ as a non-negative Radon measure and it is continuous on decreasing sequences of plurisubharmonic functions in this class. At the same time, in [9] he proved that the class $\mathcal{E}(\Omega)$ is the biggest class of plurisubharmonic functions with this property. To extend the class $\mathcal{E}(\Omega)$, in 2006, Błocki in [6] introduced the class $\mathcal{D}(\Omega)$ of plurisubharmonic functions on an open subset $\Omega$ of $\mathbb{C}^n$ and showed that the complex Monge-Ampère operator can be well defined on this class. In the case $n = 2$ he described this class by the equality: $\mathcal{D}(\Omega) = PSH(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega)$ (see Theorem 1.1 in [5]), where $W^{1,2}_{\text{loc}}(\Omega)$ is the Sobolev space on $\Omega$. He also showed that if $\Omega$ is a bounded hyperconvex domain then $\mathcal{E}(\Omega) = PSH^-(\Omega) \cap \mathcal{D}(\Omega)$. Therefore, by results of Błocki in [6] one knows that the class $\mathcal{D}(\Omega)$ is the biggest class on which the complex Monge-Ampère operator is well-defined when $\Omega$ is an open subset of $\mathbb{C}^n$.

One of the important and central problems of pluripotential theory is to define weak solutions to the complex Monge-Ampère equation. Namely we consider the following problem. Let $\mu$ be a non-negative Radon measure defined in an open subset $\Omega$ of $\mathbb{C}^n$. The problem is to find $w \in PSH(\Omega)$ such that

$$\begin{cases} w \in \mathcal{D}(\Omega), \\ (dd^c w)^n = \mu \text{ in } \Omega. \end{cases} \tag{1}$$

When $\Omega$ is a strictly pseudoconvex domain in $\mathbb{C}^n$, there are some known results concerning to weak solutions of (1). Bedford and Taylor in [3] proved in 1976 that if $\mu = fdV$ with $f \in \mathcal{C}(\overline{\Omega})$ then (1) has a continuous solution on $\overline{\Omega}$. Kołodziej in [20] showed in 1995 that if there exists a bounded subsolution for (1) then we can find $u \in PSH \cap L^\infty(\Omega)$ with $(dd^c u)^n = \mu$ (see Theorem A in [20]).

In the case $\Omega$ is a bounded hyperconvex domain in $\mathbb{C}^n$, the problem becomes much more complicated. Cegrell in [8] considered in 1998 weak solutions of the above problem for unbounded plurisubharmonic functions. In [9] he proved that if $\Omega$ is a bounded hyperconvex domain and $\mu$ vanishes on all pluripolar sets of $\Omega$ then (1) has a unique weak solution in the class $\mathcal{F}^a(\Omega)$ (see Theorem 5.14 in [9]). Next, Åhag, Cegrell, Czyż and Hiep in [1] studied in 2009 the above problem for non-negative measures carried by a pluripolar set. They claimed that if $\mu \le (dd^c u)^n$ where $u \in \mathcal{E}(\Omega)$ then for every $H \in \mathcal{E}(\Omega) \cap MPSH(\Omega)$ there exists a weak solution $w \in \mathcal{E}(\Omega, H)$ such that $H + u \le w \le H$ satisfying (1), where $MPSH(\Omega)$ denotes the set of maximal plurisubharmonic functions on $\Omega$.

Next, the problem (1) is also studied by some authors under different hypotheses for $\Omega$ and the measure $\mu$. In [12], the authors solved the above problem when $\Omega$ is a bounded domain and $\mu$ vanishes on pluripolar sets. When $\Omega$ is an unbounded pseudoconvex domain in $\mathbb{C}^n$, in [2] Åhag and Czyż treated the problem (1). They showed that Kołodziej's subsolution theorem is also valid. Next, in the case $\Omega$ is an unbounded hyperconvex domain, the problem (1) is studied by L.M.Hai, N.V.Trao and N.X.Hong. In [13] under the assymption that $\Omega$ is an unbounded hyperconvex domain and $\mu$ is a Borel measure on $\Omega$ with $\mu(\Omega) < \infty$ the authors proved that there exists a weak solution in the Cegrell class with boundary values $\mathcal{F}(\Omega, f)$ (see Theorem 5.3). The main aim of this paper is to establish the existence of weak solutions to the equation (1) in the case $\Omega$ is an arbitrary open subset of $\mathbb{C}^n$. Namely, we prove the following.

**Theorem 1.** *Let $\Omega$ be an open subset of $\mathbb{C}^n$ and $u \in \mathcal{D}(\Omega)$, $v \in MPSH(\Omega)$ be such that $u \le v$ in $\Omega$. If $\mu \le (dd^c u)^n$ then there exists a solution $w$ to (1) satisfying $u \le w \le v$ in $\Omega$.*

It should be remarked that to find weak solutions of equation (1) for this case the methods that we use are different from that given by the authors in [12], [13], in particular, [2]. Here we combine results and techniques in the class $\mathcal{D}(\Omega)$ with results obtained in pluripotential theory on Cegrell's classes to claim the existence of desired weak solutions of the above equation on an arbitrary open subset $\Omega$ in $\mathbb{C}^n$.

The organization of the paper is as follows. In Section 2 we recall some notions of pluripotential theory which is necessary for the next results of the paper. In Section 3 we prove Theorem 1.

**2. Preliminairies.** In this section, we recall some elements of pluripotential theory that will be used throughout the paper. All this can be found in [1]–[20]. Let $n$ be a positive integer and

let $\Omega$ be an open set in $\mathbb{C}^n$. We denote by $PSH(\Omega)$ the family of plurisubharmonic functions defined on $\Omega$ and $PSH^-(\Omega)$ denotes the set of negative plurisubharmonic functions on $\Omega$. We first recall the definition of the class $\mathcal{D}(\Omega)$ on an open set $\Omega$ in $\mathbb{C}^n$ (see [6]).

**Definition 1.** A plurisubharmonic function $u$ defined on $\Omega$ belongs to $\mathcal{D}(\Omega)$ if there exists a nonnegative Radon measure $\mu$ on $\Omega$ such that if $\Omega' \Subset \Omega$ is an open subset and $\{u_j\} \subset PSH(\Omega') \cap \mathcal{C}^\infty(\Omega')$ is a sequence which decreases to $u$ in $\Omega'$ then $(dd^c u_j)^n$ tends weakly to $\mu$ in $\Omega'$. The measure $\mu$ is denoted by $(dd^c u)^n$.

Note that by results of Bedford–Taylor in [4] we have $PSH(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega) \subset \mathcal{D}(\Omega)$. Moreover, if $n = 1$ then $SH(\Omega) = \mathcal{D}(\Omega)$.

Next, we recall the following classes of plurisubharmonic functions introduced and investigated by Cegrell in [9] in the case when $\Omega$ is a bounded hyperconvex domain in $\mathbb{C}^n$.

**Definition 2.** Let $\Omega$ be a bounded hyperconvex domain in $\mathbb{C}^n$. We say that a bounded, negative plurisubharmonic function $\varphi$ on $\Omega$ belongs to $\mathcal{E}_0(\Omega)$ if $\{\varphi < -\varepsilon\} \Subset \Omega$ for all $\varepsilon > 0$ and $\int_\Omega (dd^c \varphi)^n < +\infty$.

Let $\mathcal{F}(\Omega)$ be the family of plurisubharmonic functions $\varphi$ defined on $\Omega$, such that there exists a decreasing sequence $\{\varphi_j\} \subset \mathcal{E}_0(\Omega)$ that converges pointwise to $\varphi$ on $\Omega$ as $j \to \infty$ and

$$\sup_j \int_\Omega (dd^c \varphi_j)^n < \infty.$$

We denote by $\mathcal{E}(\Omega)$ the family of plurisubharmonic functions $\varphi$ defined on $\Omega$ such that for every open set $G \Subset \Omega$ there exists a plurisubharmonic function $\psi \in \mathcal{F}(\Omega)$ satisfying $\psi \geq \varphi$ on $\Omega$ and $\psi = \varphi$ in $G$.

**Definition 3.** Let $\mathcal{K} \in \{\mathcal{F}, \mathcal{E}, \mathcal{D}\}$. We denote by $\mathcal{K}^a(\Omega)$ the subclass of $\mathcal{K}(\Omega)$ such that the complex Monge-Ampère measure $(dd^c.)^n$ vanishes on all pluripolar sets of $\Omega$.

Let $f \in \mathcal{E}(\Omega)$ and $\mathcal{K} \in \{\mathcal{F}^a, \mathcal{E}^a, \mathcal{D}^a, \mathcal{F}, \mathcal{E}, \mathcal{D}\}$. Then we say that a plurisubharmonic function $\varphi$ defined on $\Omega$ belongs to $\mathcal{K}(\Omega, f)$ if there exists a function $\psi \in \mathcal{K}(\Omega)$ such that

$$\psi + f \leq \varphi \leq f \text{ in } \Omega.$$

**Proposition 1.** *If $\Omega$ is a bounded hyperconvex domain in $\mathbb{C}^n$ then $\mathcal{E}(\Omega) = PSH^-(\Omega) \cap \mathcal{D}(\Omega)$.*

*Proof.* See Theorem 2.4 in [6]. $\square$

**Remark 1.** (i) If $\Omega$ is an open subset of $\mathbb{C}^n$ and $u \in \mathcal{D}(\Omega)$ then $u|_{\mathbb{B}} - \sup_{\mathbb{B}} u \in \mathcal{E}(\mathbb{B})$ for all open balls $\mathbb{B} \Subset \Omega$.

(ii) If $\Omega$ is a hyperconvex domain in $\mathbb{C}^n$ and $u, f \in \mathcal{E}(\Omega)$ with $u \leq f$ and $h \in \mathcal{F}^a(\Omega)$ then $\max(u, f + h) \in \mathcal{F}^a(\Omega, f)$.

Next, we recall the following.

**Definition 4.** Let $\Omega \subset \mathbb{C}^n$ and $v \in PSH(\Omega)$. $v$ is said to be a maximal plurisubharmonic function if for all compact subset $K \Subset \Omega$ and for every plurisubharmonic function $w \in PSH(\Omega)$, $w \leq v$ on $\Omega \setminus K$ then $w \leq v$ on $\Omega$.

The family of maximal plurisubharmonic functions on $\Omega$ is denoted by $MPSH(\Omega)$.

For results concerning to maximal plurisubharmonic functions we refer the readers to [19]. In the case $\Omega \subset \mathbb{C}^n$ is a bounded hyperconvex domain and $u \in \mathcal{E}(\Omega)$. Then in [7], Błocki proved that $u \in MPSH(\Omega)$ if and only if it satisfies the homogeneous Monge-Ampère equation $(dd^c u)^n = 0$.

**Proposition 2.** *Let $\Omega \subset \hat{\Omega}$ be bounded hyperconvex domains in $\mathbb{C}^n$. Assume that $u \in \mathcal{F}(\Omega)$ and define*

$$\hat{u} := \sup\{\varphi \in PSH^-(\hat{\Omega}) : \varphi \leq u \text{ on } \Omega\}.$$

*If $(dd^c u)^n$ is carried by a pluripolar set of $\Omega$ then*

$$(dd^c \hat{u})^n = 1_\Omega (dd^c u)^n \text{ in } \hat{\Omega}.$$

*Proof.* By Lemma 4.5 in [17] we have $\hat{u} \in \mathcal{F}(\hat{\Omega})$ and $(dd^c \hat{u})^n \leq 1_\Omega (dd^c u)^n$ in $\hat{\Omega}$. Since $(dd^c u)^n$ is carried by a pluripolar set of $\Omega$, Theorem 5.11 in [9] implies that $(dd^c u)^n = 1_{\{u=-\infty\}}(dd^c u)^n$ in $\Omega$. Moreover, since $\hat{u} \leq u$ on $\Omega$, by Lemma 4.1 in [1] we get

$$1_\Omega (dd^c u)^n = 1_{\Omega \cap \{u=-\infty\}}(dd^c u)^n \leq 1_{\Omega \cap \{\hat{u}=-\infty\}}(dd^c \hat{u})^n \leq (dd^c \hat{u})^n \leq 1_\Omega (dd^c u)^n.$$

This implies that $(dd^c \hat{u})^n = 1_\Omega (dd^c u)^n$ in $\hat{\Omega}$, and the desired conclusion follows. $\square$

**3. Weak solutions to the complex Monge-Ampère equations** In this section we prove Theorem 1. We need the following auxiliary results.

**Lemma 1.** *Let $\Omega$ be an open set in $\mathbb{C}^n$ and let $\mathbb{B} \Subset \Omega$ be an open ball. Assume that $u \in \mathcal{D}(\Omega)$ and $v \in \mathcal{F}(\mathbb{B})$ such that*

 (i) $\sup_\mathbb{B} u < 0$;

 (ii) $(dd^c v)^n$ *is carried by a pluripolar set $E \subset \mathbb{B}$;*

 (iii) $(dd^c u)^n$ *is carried by a pluripolar set $F \subset \Omega \backslash E$;*

 (iv) $w := \sup\{\varphi \in PSH(\Omega) : \varphi \leq u \text{ on } \Omega \text{ and } \varphi \leq v \text{ on } \mathbb{B}\} \in \mathcal{D}(\Omega)$.

*Then, $(dd^c w)^n = (dd^c u)^n + 1_\mathbb{B}(dd^c v)^n$ in $\Omega$.*

*Proof.* Without loss of generality we can assume that $E \subset \mathbb{B} \cap \{v = -\infty\}$ and $F \subset (\Omega \backslash E) \cap \{u = -\infty\}$. Let $\{\Omega_j\}$ be an increasing sequence of open sets in $\mathbb{C}^n$ such that $\mathbb{B} = \mathbb{B}(a, r) \Subset \Omega_j \Subset \Omega_{j+1} \Subset \Omega$ and $\Omega = \bigcup_{j=1}^\infty \Omega_j$. By the local property of the class $\mathcal{D}(\Omega)$ we may assume that $w \in \mathcal{D}(\Omega_j)$. Set $u_j = u|_{\Omega_j}$. Define

$$w_j := \sup\{\varphi \in PSH(\Omega_j) : \varphi \leq u_j \text{ on } \Omega_j \text{ and } \varphi \leq v \text{ on } \mathbb{B}\}.$$

From $w \leq w_j$ on $\Omega_j$ then $w_j \in \mathcal{D}(\Omega_j)$ and $w_j \searrow w$ in $\Omega$. Hence, Proposition 2.1 in [5] implies that $(dd^c w_j)^n$ converges to $(dd^c w)^n$ in the weak*-topology. Replacing $\Omega$ by $\Omega_j$ if necessary, we can assume that $\Omega$ is bounded and $u$ is a plurisubharmonic function on an open neighborhood of $\overline{\Omega}$. We first prove that

$$(dd^c w)^n = 0 \text{ on } (\{w < u\} \cap \Omega \backslash \overline{\mathbb{B}}) \cup (\{w < \min(u, v)\} \cap \mathbb{B}). \tag{2}$$

Indeed, let $\{v_j\} \subset \mathcal{E}_0(\mathbb{B}) \cap \mathcal{C}(\overline{\mathbb{B}})$ be such that $v_j \searrow v$ in $\mathbb{B}$ and let $\{u_j\} \subset PSH(\Omega) \cap \mathcal{C}(\Omega)$ be such that $u_j \searrow u$ in $\Omega$ and $\sup_\mathbb{B} u_1 < 0$. We set

$$w_j := \sup\{\varphi \in PSH(\Omega) : \varphi \leq h_j \text{ in } \Omega\},$$

where

$$
h_j = \begin{cases} \min(u_j, v_j) & \text{on } \mathbb{B}, \\ u_j & \text{on } \Omega \backslash \mathbb{B}. \end{cases}
$$

Since $u_j \leq u_1 < 0 = v_j$ on $\partial \mathbb{B}$ so $h_j \in \mathcal{C}(\Omega)$. Then, $w_j \in PSH(\Omega)$ and $w_j \searrow w$ in $\Omega$. Thanks to Corollary 9.2 in [4] we obtain that $(dd^c w_j)^n = 0$ on $\{w_j < h_j\}$. Let $j \to \infty$, by [11] we get the equality (2).

Next, we claim that

$$
(dd^c w)^n = 0 \text{ in } \{u > -\infty\} \cap \{u = w\} \tag{3}
$$

Indeed, let $K \subset \{u > -\infty\} \cap \{u = w\}$ be a compact set. Since $K \subset \{w + \frac{1}{j} > u\}$, by Theorem 4.1 in [18] and using the hypotheses we have

$$
\int_K (dd^c w)^n = \lim_{j \to +\infty} \int_K (dd^c \max(w + \frac{1}{j}, u))^n \leq \int_K (dd^c \max(w, u))^n = \int_K (dd^c u)^n.
$$

Moreover, since $(dd^c u)^n$ is carried by a pluripolar set $F$, we infer

$$
\int_K (dd^c w)^n \leq \int_{F \cap \{u > -\infty\}} (dd^c u)^n = 0.
$$

Hence, $\int_K (dd^c w)^n = 0$. It follows that $(dd^c w)^n = 0$ in $\{u > -\infty\} \cap \{u = w\}$, and the desired conclusion follows. Similarly, we also have

$$
(dd^c w)^n = 0 \text{ in } \mathbb{B} \cap \{v > -\infty\} \cap \{v = w\}.
$$

Combining this with (2) and (3) we infer that

$$
(dd^c w)^n = 0 \text{ on } (\{u > -\infty\} \cap \Omega \backslash \overline{\mathbb{B}}) \cup (\{\min(u, v) > -\infty\} \cap \mathbb{B}). \tag{4}
$$

Now, let $R > 0$ be such that $\Omega \Subset \mathbb{B}(0, R)$ and define

$$
\hat{v} := \sup\{\varphi \in PSH^-(\mathbb{B}(0, R)) : \varphi \leq v \text{ on } \mathbb{B}\}.
$$

Proposition 2 implies that $\hat{v} \in PSH^-(\Omega) \cap \mathcal{D}(\Omega)$ and

$$
(dd^c \hat{v})^n = \mathbf{1}_{\mathbb{B}} (dd^c v)^n = \mathbf{1}_E (dd^c v)^n \text{ in } \Omega.
$$

We claim that

$$
(dd^c w)^n = (dd^c (u + \hat{v}))^n = (dd^c u)^n \text{ on } \{u = -\infty\} \backslash E. \tag{5}
$$

Indeed, let $K \subset \{u = -\infty\} \backslash E$ be a compact set. Since $(dd^c \hat{v})^n = \mathbf{1}_E (dd^c v)^n$, and $K \subset \{u = -\infty\} \backslash E$ then

$$
\int_K (dd^c \hat{v})^n = 0.
$$

Lemma 4.1 and Lemma 4.4 in [1] imply that

$$
\int_K (dd^c u)^n \leq \int_K (dd^c (u + \hat{v}))^n = \sum_{j=0}^n \binom{n}{j} \int_K (dd^c u)^j \wedge (dd^c \hat{v})^{n-j} \leq
$$

$$\leq \sum_{j=0}^{n} \binom{n}{j} \left(\int_{K} (dd^c u)^n\right)^{\frac{j}{n}} \left(\int_{K} (dd^c \hat{v})^n\right)^{\frac{n-j}{n}} = \int_{K} (dd^c u)^n.$$

It follows that

$$\int_{K} (dd^c (u + \hat{v}))^n = \int_{K} (dd^c u)^n.$$

Moreover, since $u + \hat{v} \leq w \leq u$ in $\Omega$, then Theorem 4.1 in [1] implies that

$$\int_{K} (dd^c u)^n \leq \int_{K} (dd^c w)^n \leq \int_{K} (dd^c (u + \hat{v}))^n = \int_{K} (dd^c u)^n,$$

and the required conclusion follows. Similarly, we also have

$$(dd^c w)^n = (dd^c (u + \hat{v}))^n = (dd^c \hat{v})^n = (dd^c v)^n \text{ on } (\mathbb{B} \cap \{v = -\infty\}) \backslash F.$$

Combining this with (4), (5) and using the hypotheses we infer

$$(dd^c w)^n = (dd^c u)^n + 1_{\mathbb{B}} (dd^c v)^n \text{ in } \Omega \backslash \partial \mathbb{B}. \tag{6}$$

Let $\mathbb{B}'$ be an open ball such that $\mathbb{B} \Subset \mathbb{B}' \Subset \Omega$ and $\sup_{\mathbb{B}'} u < 0$. Put

$$v' := \sup\{\varphi \in PSH^-(\mathbb{B}') : \varphi \leq v \text{ on } \mathbb{B}\}.$$

By Proposition 2 we have $v' \in \mathcal{F}(\mathbb{B}')$ and $(dd^c v')^n = 1_{\mathbb{B}} (dd^c v)^n$ in $\mathbb{B}'$. Since $w \leq u < 0$ in $\mathbb{B}'$ and $w \leq v$ in $\mathbb{B}$ so $w \leq v'$ on $\mathbb{B}'$. This implies that

$$w = \sup\{\varphi \in PSH(\Omega) : \varphi \leq u \text{ on } \Omega \text{ and } \varphi \leq v' \text{ on } \mathbb{B}'\}.$$

Applying (6) we get that

$$(dd^c w)^n = (dd^c u)^n + 1_{\mathbb{B}'} (dd^c v')^n \text{ in } \Omega \backslash \partial \mathbb{B}'.$$

Therefore, $(dd^c w)^n = (dd^c u)^n + 1_{\mathbb{B}} (dd^c v)^n$ in $\Omega$.                    $\square$

**Lemma 2.** *Let $\Omega$ be an open set in $\mathbb{C}^n$ and let $u, v \in \mathcal{D}(\Omega)$ be such that $u \leq v$ in $\Omega$. Assume that $(dd^c v)^n$ is carried by a pluripolar set of $\Omega$ and $\mu$ is a non-negative Radon measure defined in $\Omega$ with $\mu \leq 1_{\{u > -\infty\}} (dd^c u)^n$. Then, for every open ball $\mathbb{B} \Subset \Omega$, there exists $w \in \mathcal{D}(\Omega)$ satisfying*

(i) $u \leq w \leq v$ on $\Omega$;

(ii) $(dd^c w)^n \geq \mu$ in $\Omega$;

(iii) $(dd^c w)^n = \mu + (dd^c v)^n$ in $\mathbb{B}$.

*Proof.* Fix a ball $\mathbb{B} \Subset \Omega$. Let $\mathbb{B}'$ be an open ball such that $\mathbb{B} \Subset \mathbb{B}' \Subset \Omega$ and $u \in \mathcal{E}(\mathbb{B}')$. Without loss of generality we may assume that $(dd^c v)^n$ is carried by $\{v = -\infty\}$. Define

$$g := (\sup\{\varphi \in PSH^-(\mathbb{B}') : \varphi \leq u \text{ on } \mathbb{B}' \backslash \mathbb{B}\})^*,$$
$$f := \sup\{\varphi \in PSH^-(\mathbb{B}') : \varphi \leq \min(g, v) \text{ on } \mathbb{B}'\}.$$

Then, $g \in \mathcal{E}(\mathbb{B}') \cap MPSH(\mathbb{B})$. Since $u \leq v$ in $\Omega$ so $u \leq f \leq v$ in $\mathbb{B}'$ and $f = g = u$ on $\mathbb{B}' \backslash (\mathbb{B} \cup E)$ for some pluripolar set $E \subset \partial \mathbb{B}$. By Lemma 4.1 in [1], Corollary 3.1 in [15] and using the hypotheses we have

$$(dd^c v)^n = 1_{\{v=-\infty\}}(dd^c v)^n \leq 1_{\{f=-\infty\}}(dd^c f)^n \leq (dd^c f)^n \leq (dd^c v)^n + (dd^c g)^n = (dd^c v)^n$$

on $\mathbb{B}$. It follows that $(dd^c f)^n = (dd^c v)^n$ on $\mathbb{B}$.

Now, since the measure $1_{\{u>-\infty\}}(dd^c u)^n$ vanishes on all pluripolar sets of $\Omega$ and $\mu \leq 1_{\{u>-\infty\}}(dd^c u)^n$ in $\Omega$ so by Lemma 5.14 in [9] there exists $h \in \mathcal{F}^a(\mathbb{B})$ such that $(dd^c h)^n = \mu$ in $\mathbb{B}$. It is easy to see that $\max(u, h + f) \in \mathcal{F}^a(\mathbb{B}, f)$ and

$$(dd^c f)^n \leq \mu + (dd^c v)^n \leq (dd^c \max(u, h + f))^n \text{ in } \mathbb{B}.$$

Thanks to Lemma 4.1 in [16] there is $\psi \in \mathcal{F}^a(\mathbb{B}, f)$ such that $u \leq \psi \leq f$ and

$$(dd^c \psi)^n = \mu + (dd^c v)^n \text{ in } \mathbb{B}.$$

Let $w$ be the smallest plurisubharmonic majorant of the function

$$\eta = \begin{cases} \psi & \text{in } \mathbb{B}, \\ u & \text{in } \Omega \backslash \mathbb{B}. \end{cases}$$

Since $u \leq \psi \leq f \leq g$ on $\mathbb{B}$, we have $w \in \mathcal{D}(\Omega)$ and $u \leq w \leq v$ in $\Omega$. It is easy to see that

$$(dd^c w)^n \geq \mu \text{ in } \Omega \backslash \mathbb{B} \tag{7}$$

and

$$(dd^c w)^n = \mu + (dd^c v)^n \text{ on } \mathbb{B}. \tag{8}$$

Indeed, by the definition of $w$ we note that $w = u$ in the interior of $\Omega \setminus \mathbb{B}$ and $w = \psi$ in $\mathbb{B}$. Hence, we have $(dd^c w)^n \geq \mu$ on the interior of $\Omega \setminus \mathbb{B}$ and (8) holds. In order to prove $(dd^c w)^n \geq \mu$ on $\Omega \setminus \mathbb{B}$ it suffices to prove $(dd^c w)^n \geq \mu$ on $\partial \mathbb{B}$. By the definition of $w$ it follows that $w = u$ on $\partial \mathbb{B} \setminus E$, where $E$ is a pluripolar subset of $\partial \mathbb{B}$ containing $\{u = -\infty\}$. Let $K \subset \partial \mathbb{B} \setminus E$ be a compact set. Since $K \subset \{u + \frac{1}{j} > w\}$, by Theorem 4.1 in [18] we have

$$\mu(K) \leq \int_K (dd^c u)^n = \lim_{j \to +\infty} \int_K (dd^c \max(u + \frac{1}{j}, w))^n \leq \int_K (dd^c \max(u, w))^n = \int_K (dd^c w)^n.$$

It follows that $(dd^c w)^n \geq \mu$ on $\partial \mathbb{B} \setminus E$. Hence, $(dd^c w)^n \geq \mu$ on $\partial \mathbb{B}$. Combining this with (7) and (8) we obtain $(dd^c w)^n \geq \mu$ in $\Omega$. The proof is complete. $\qquad \square$

*Proof of Theorem 1.* Let $\{\mathbb{B}_j\}$ be a sequence of open balls such that $\mathbb{B}_j \Subset \Omega$ and $\Omega = \bigcup_{j=1}^{\infty} \mathbb{B}_j$. Put $\mathbb{B}_0 := \varnothing$. We first claim that there exists a decreasing sequence $\{f_j\} \subset \mathcal{D}(\Omega)$ such that $u \leq f_j \leq v$ and

$$(dd^c f_j)^n = 1_{(\bigcup_{k=0}^{j-1} \mathbb{B}_k) \cap \{u=-\infty\}} \mu \text{ on } \Omega.$$

Indeed, let $f_1 := v$. Then, $f_1 \in \mathcal{D}(\Omega)$ and $(dd^c f_1)^n = 0$ in $\Omega$. Let $j \geq 1$ be an integer number. Assume by induction that we have determined $f_j$. We find $f_{j+1}$ as follows. Let $\mathbb{B}'_j$ be an

open ball such that $\mathbb{B}_j \Subset \mathbb{B}_j' \Subset \Omega$. By Theorem 4.14 in [1], there exists $v_j \in \mathcal{F}(\mathbb{B}_j')$ such that $v_j \geq u$ and

$$(dd^c v_j)^n = 1_{(\mathbb{B}_j \setminus \bigcup_{k=0}^{j-1} \mathbb{B}_k) \cap \{u=-\infty\}} \mu \text{ on } \mathbb{B}_j'.$$

Set $c_j := \sup_{\mathbb{B}_j'} v + \lambda_j$ where $\lambda_j$ is chosen such that $c_j \geq 0$. Next, we define

$$f_{j+1} = \sup\{\varphi \in PSH(\Omega) : \varphi \leq f_j - c_j \text{ in } \Omega \text{ and } \varphi \leq v_j \text{ in } \mathbb{B}_j'\} + c_j.$$

Then we have $u \leq f_{j+1} \leq f_j \leq v$ on $\Omega$ and by Theorem 1.2 in [6] we have $f_{j+1} \in \mathcal{D}(\Omega)$. Lemma 1 implies that

$$(dd^c f_{j+1})^n = (dd^c f_j)^n + 1_{\mathbb{B}_j'}(dd^c v_j)^n = 1_{(\bigcup_{k=0}^{j} \mathbb{B}_k) \cap \{u=-\infty\}} \mu \text{ in } \Omega.$$

This proves the claim. Put $f := \lim_{j \to \infty} f_j$ in $\Omega$. Then, $f \in PSH(\Omega)$ and $u \leq f \leq v$ in $\Omega$. By Theorem 1.2 in [6] and using the main result in [11] we infer that $f \in \mathcal{D}(\Omega)$ and

$$(dd^c f)^n = 1_{\{u=-\infty\}} \mu \text{ in } \Omega.$$

We now set

$$w := \sup\{\varphi \in \mathcal{D}(\Omega) : \varphi \leq f \text{ and } (dd^c \varphi)^n \geq 1_{\{u>-\infty\}} \mu\}.$$

Because $u \leq w \leq f \leq v$ in $\Omega$, by Theorem 1.2 in [6] we have $w \in \mathcal{D}(\Omega)$. Since the measure $1_{\{u>-\infty\}} \mu$ vanishes on pluripolar sets of $\Omega$, by Proposition 4.3 in [18] and using the Choquet lemma we can choose a increasing sequence $\{\varphi_j\} \subset \mathcal{D}(\Omega)$ such that $\varphi_j \nearrow w$ a.e. in $\Omega$ and

$$(dd^c \varphi_j)^n \geq 1_{\{u>-\infty\}} \mu \text{ in } \Omega.$$

By the main result in [11] we note that $(dd^c \varphi_j)^n \to (dd^c w)^n$ weakly in $\Omega$, and hence,

$$(dd^c w)^n \geq 1_{\{u>-\infty\}} \mu \text{ in } \Omega.$$

Let $\mathbb{B} \Subset \Omega$ be an open ball. By Lemma 2 there exists $\psi \in \mathcal{D}(\Omega)$ such that

 (i) $w \leq \psi \leq f$ on $\Omega$;

 (ii) $(dd^c \psi)^n \geq 1_{\{u>-\infty\}} \mu$ in $\Omega$;

 (iii) $(dd^c \psi)^n = 1_{\{u>-\infty\}} \mu + (dd^c f)^n$ in $\mathbb{B}$.

From the definition of $w$ it is easy to see that $w = \psi$ in $\Omega$, and therefore, by (iii) we have

$$(dd^c w)^n = 1_{\{u>-\infty\}} \mu + (dd^c f)^n = \mu \text{ in } \mathbb{B},$$

and the desired conclusion follows.                                      $\square$

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