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ON THE TRANSCENDENTAL MEROMORPHIC SOLUTIONS OF A CERTAIN CLASS OF DIFFERENTIAL EQUATIONS

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We consider the differential equation

$$F^{(k)} - a_2 = e^{\gamma}\{\alpha(F - a_1) + \beta\},$$

where $a_i(z), \alpha(z)(\not\equiv 0, \infty)$, $\beta(z)(\not\equiv \infty)$, $i = 1, 2$ are small functions of F, γ is an entire function and $k \in \mathbb{N}$. Let f be a transcendental meromorphic function such that $N(r, \infty; f) = S(r, f)$ and $n \in \mathbb{N}$ such that $n \geq k + 1$. If $F = f^n$ is a solution of the above differential equation, then

$$F^{(k)} \equiv \frac{a_2\alpha}{a_1\alpha - \beta}F.$$

Also we exhibit an example to fortify the condition of our result.

1. Introduction, definitions and results. In the paper a meromorphic function means it is meromorphic in the open complex plane $\mathbb{C}$. We use the standard notations of Nevanlinna theory e.g., $N(r, f)$, $m(r, f)$, $T(r, f)$, $N(r, a; f)$, $\overline{N}(r, a; f)$, $m(r, a; f)$ etc., see [6]. We denote by $S(r, f)$ a quantity, not necessarily the same at each occurrence, that satisfies the condition $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ except possibly a set of finite linear measure.

A meromorphic function $a = a(z)$ is called a small function of a meromorphic function f, if $T(r, a) = S(r, f)$. Let us denote by $S(f)$ the class of all small functions of f. Clearly if f is a transcendental function, then every polynomial is a member of $S(f)$.

Let f and g be two non-constant meromorphic functions and $a \in S(f) \cap S(g)$. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share the small function a CM (counting multiplicities) and if we do not consider the multiplicities, then we say that f and g share the small function a IM (ignoring multiplicities).

Let k be a positive integer and $a \in S(f)$. We use $N_{k)}(r, a; f)$ to denote the counting function of zeros of $f - a$ with multiplicity not greater than k , $N_{(k+1}(r, a; f)$ to denote the counting function of zeros of $f - a$ with multiplicity greater than k. Similarly we use $\overline{N}_{k)}(r, a; f)$ and $\overline{N}_{(k+1}(r, a; f)$ are their respective reduced functions.

In 1996, Brück [1] studied the relation between f and f' if an entire function f shares only one finite value CM with its derivative f'. In this direction an interesting conjecture was proposed by Brück [1], which is still open in its full generality.

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Conjecture A. *Let f be a non-constant entire function. Suppose*

$$\rho_1(f) := \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r},$$

the hyper-order of f, is not a positive integer or infinity. If f and f' share a finite value a CM, then

$$\frac{f'-a}{f-a} = c \tag{1}$$

for some $c \in \mathbb{C}\setminus\{0\}$.

The conjecture for the special cases (1) $a = 0$ and (2) $N(r,0;f') = S(r,f)$ had been established by Brück [1]. From the differential equations

$$\frac{f'-a}{f-a} = e^{z^n} \text{ and } \frac{f'-a}{f-a} = e^{e^z}, \tag{2}$$

we see that when $\rho_1(f)$ is a positive integer or infinity, the conjecture does not hold.

The conjecture for the case where f is of finite order had been proved by Gundersen and Yang [5], the case that f is of infinite order with $\rho_1(f) < \frac{1}{2}$ had been proved by Chen and Shon [3]. Recently Cao [2] proved that the Brück conjecture is also true when f is of infinite order with $\rho_1(f) = \frac{1}{2}$. But the case $\rho_1(f) > \frac{1}{2}$ is still open. However, the corresponding conjecture for meromorphic functions fails in general (see [5]). For example, if

$$f(z) = \frac{2e^z + z + 1}{e^z + 1},$$

then f and f' share 1 CM, but (1) does not hold.

It is interesting to ask what happens if f is replaced by a power of it, say, f^n in Brück's conjecture. From (3) we see that the conjecture does not hold without any restriction on the hyper-order when $n = 1$. So we only need to focus on the problem when $n \geq 2$.

Perhaps Yang and Zhang [9] were the first to consider the uniqueness of a power of an entire function $F = f^n$ and its derivative F' when they share certain value and that leads to a specific form of the function f.

Yang and Zhang [9] proved that the Brück conjecture holds for the function f^n and the order restriction on f is not needed if n is relatively large. Actually they proved the following result.

Theorem A ([9]). *Let f be a non-constant entire function, $n \in \mathbb{N}$ such that $n(\geq 7)$ and let $F = f^n$. If F and F' share 1 CM, then $F \equiv F'$ and $f(z)$ assumes the form $f(z) = ce^{\frac{1}{n}z}$, where $c \in \mathbb{C}\setminus\{0\}$.*

Improving all the results obtained in [9], Zhang [11] proved the following theorem.

Theorem B ([11]). *Let f be a non-constant entire function, $k, n \in \mathbb{N}$ and $a(z)(\not\equiv 0,\infty) \in S(f)$. If $f^n - a$ and $(f^n)^{(k)} - a$ share 0 CM and $n \geq k+5$, then $f^n \equiv (f^n)^{(k)}$ and $f(z)$ assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where $c \in \mathbb{C}\setminus\{0\}$ and $\lambda^k = 1$.*

In 2009, Zhang and Yang [12] further improved the above result in the following manner.

Theorem C ([12]). *Let f be a non-constant entire function, $k, n \in \mathbb{N}$ and $a(z)(\not\equiv 0, \infty) \in S(f)$. Suppose $f^n - a$ and $(f^n)^{(k)} - a$ share 0 CM and $n \geq k+2$. Then conclusion of Theorem B holds.*

In 2010, Zhang and Yang [13] further improved Theorem C in the following direction.

Theorem D ([13]). *Let f be a non-constant entire function and $k, n \in \mathbb{N}$. Suppose f^n and $(f^n)^{(k)}$ share 1 CM and $n \geq k+1$. Then conclusion of Theorem B holds.*

In 2011, Lü and Yi [7] proved the following extension of Theorem D.

Theorem E ([7]). *Let f be a transcendental entire function, $k, n \in \mathbb{N}$ such $n \geq k+1$, $F = f^n$ and $Q \not\equiv 0$ be a polynomial. If $F - Q$ and $F^{(k)} - Q$ share 0 CM, then $F \equiv F^{(k)}$ and $f(z) = ce^{wz/n}$, where c and $w \in \mathbb{C} \setminus \{0\}$ such that $w^k = 1$.*

Remark 1. It is easy to see that the condition $n \geq k+1$ in Theorem E is sharp by the following example.

Example 1. Let $f(z) = e^{e^z} \int_0^z e^{-e^t}(1-e^t)t\, dt$ and $n = 1$, $k = 1$. Then

$$\frac{f'(z) - z}{f(z) - z} = e^z$$

and $f'(z) - z$ and $f(z) - z$ share 0 CM, but $f' \not\equiv f$.

Now observing the above theorem, Lü, Li and Yang [8] asked the following question:

Question 1. *What can be said "if $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share the value 0 CM", where Q_1 and Q_2 are polynomials, and $Q_1 Q_2 \not\equiv 0$?*

Lü, Li and Yang [8] answered the above question for $k = 1$ by giving the transcendental entire solutions of the equation

$$F' - Q_1 = Re^{\alpha}(F - Q_2), \tag{3}$$

where $F = f^n$, R is a rational function and α is an entire function and they obtained the following result.

Theorem F ([8]). *Let f be a transcendental entire function and let $F = f^n$ be a solution of equation (3), $n \in \mathbb{N}$ such that $n \geq 2$, then $\frac{Q_1}{Q_2}$ is a polynomial and $f' \equiv \frac{Q_1}{nQ_2} f$.*

We now pose the following questions as open problems.

Question 2. Is Theorem F true for equation (3) with $F^{(k)}$ instead F?

Question 3. Is Theorem F valid for a transcendental meromorphic function?

Question 4. What are properties of the function f or F if Q_1, Q_2 and R in Theorem F are replaced by small functions of f?

Now our objective to write this paper is to solve the above questions. The following theorem is the main result in this paper.

Theorem 1. *Let f be a transcendental meromorphic function such that $N(r,\infty;f)=S(r,f)$ and $a_i(z),\alpha(z)(\not\equiv 0,\infty)\in S(f)$, $\beta(z)(\not\equiv\infty)\in S(f)$, where $i=1,2$. Let $k,n\in\mathbb{N}$ such that $n\geq k+1$. If $F=f^n$ is a solution of the equation*

$$F^{(k)}-a_2=e^{\gamma}\{\alpha(F-a_1)+\beta\},$$

where γ is an entire function, then

$$F^{(k)}\equiv\frac{a_2\alpha}{a_1\alpha-\beta}F.$$

From Theorem 1 we have the following corollary.

Corollary 1. *Let f be a transcendental meromorphic function such that $N(r,\infty;f)=S(r,f)$ and $a_i(z),\alpha(z)(\not\equiv 0,\infty)\in S(f)$, where $i=1,2$. Let $k,n\in\mathbb{N}$ such that $n\geq k+1$. If $F=f^n$ is a solution of the equation*

$$F^{(k)}-a_2=\alpha e^{\gamma}(F-a_1),$$

where γ is an entire function, then

$$F^{(k)}\equiv\frac{a_2}{a_1}F.$$

Remark 2. It is easy to see that the condition $n\geq k+1$ in Theorem 1 is sharp by the following example.

Example 2. Let

$$f(z)=e^{e^{z^2}}+1,$$

where

$$a_1(z)=1-\frac{1}{2(1+e^{z^2})},\quad a_2(z)=-\frac{2z}{1+e^{-z^2}},\quad \alpha(z)=2z,\quad \beta(z)=\frac{z}{1+e^{z^2}}\text{ and }\gamma(z)=z^2.$$

Clearly $a_1,a_2,\alpha,\beta\in S(f)$. Note that

$$f'(z)=2ze^{z^2}e^{e^{z^2}}.$$

Therefore

$$e^{\gamma(z)}\Big\{\alpha(z)(f(z)-a_1(z))+\beta(z)\Big\}=\frac{2z}{e^{-z^2}+1}\Big[\Big(e^{z^2}+1\Big)e^{e^{z^2}}+1\Big],$$

$$f'(z)-a_2(z)=\frac{2z}{1+e^{-z^2}}\Big[\Big(e^{z^2}+1\Big)e^{e^{z^2}}+1\Big].$$

Thus

$$f'(z)-a_2(z)=e^{\gamma(z)}\Big\{\alpha(z)(f(z)-a_1(z))+\beta(z)\Big\},$$

but

$$\frac{a_2(z)\alpha(z)}{a_1(z)\alpha(z)-\beta(z)}f(z)=-2zf(z)\not\equiv f'(z).$$

2. Lemmas. In this section we present the lemmas which will be needed in the sequel.

Lemma 1 ([4]). *Suppose that f is a transcendental meromorphic function and that*

$$f^n(z)P(f(z)) = Q(f(z)),$$

where $P(f(z))$ and $Q(f(z))$ are differential polynomials in f with functions of small proximity related to f as the coefficients and the degree of $Q(f(z))$ is at most n. Then $m(r, P) = S(r, f)$.

Lemma 2 ([6]). *Let f be a non-constant meromorphic function and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $a_i \in S(f)$, $i = 1, 2$. Then*

$$T(r, f) \leq \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f).$$

Lemma 3 ([10]). *Let f be a non-constant meromorphic function and let $a_n(z)(\not\equiv 0), a_{n-1}(z), \ldots, a_0(z) \in S(f)$. Then $T(r, \sum_{i=0}^{n} a_i f^i) = nT(r, f) + S(r, f)$.*

3. Proof of the theorem.

Proof of Theorem 1. Let

$$F = f^n. \tag{4}$$

By the given condition we have

$$e^\gamma = \frac{F^{(k)} - a_2}{\alpha(F - a_1) + \beta}, \text{ i.e., } F^{(k)} - a_2 = e^\gamma\{\alpha(F - a_1) + \beta\}. \tag{5}$$

Note that

$$T(r, e^\gamma) \leq T(r, F^{(k)} - a_2) + T(r, \alpha(F - a_1) + \beta) \leq T(r, F^{(k)}) + T(r, F) + S(r, F)+$$
$$+S(r, f) \leq (k + 1)T(r, f^n) + nT(r, f) + S(r, f) = n(k + 2)T(r, f) + S(r, f),$$

which implies that $S(r, e^\gamma)$ can be replaced by $S(r, f)$. Also we see that

$$T(r, \gamma') = m(r, \gamma') = m\Big(r, \frac{(e^\gamma)'}{e^\gamma}\Big) = S(r, e^\gamma)$$

and so $\gamma' \in S(f)$. Let $\xi = \frac{\alpha'}{\alpha} + \gamma'$. Therefore

$$T(r, \xi) = T\Big(r, \frac{\alpha'}{\alpha} + \gamma'\Big) \leq T\Big(r, \frac{\alpha'}{\alpha}\Big) + T(r, \gamma') = S(r, f)$$

and so $\xi \in S(f)$. Differentiating (5) once we get

$$F^{(k+1)} - a_2' = e^\gamma\gamma' \{\alpha(F - a_1) + \beta\} + e^\gamma \{\alpha'(F - a_1) + \alpha(F' - a_1') + \beta'\}. \tag{6}$$

Now dividing (6) by (5), we have

$$\frac{F^{(k+1)} - a_2'}{F^{(k)} - a_2} = \gamma' + \frac{\alpha'(F - a_1) + \alpha(F' - a_1') + \beta'}{\alpha(F - a_1) + \beta},$$
$$\alpha F^{(k+1)}F - (a_1\alpha - \beta)F^{(k+1)} - a_2'\alpha F + a_1a_2'\alpha - a_2'\beta =$$

$$= \gamma'(F^{(k)} - a_2)\{\alpha(F - a_1) + \beta\} + \alpha'(F - a_1)(F^{(k)} - a_2)$$
$$+\alpha(F^{(k)} - a_2)(F' - a_1') + \beta'(F^{(k)} - a_2),$$
$$\alpha F^{(k+1)}F - (\alpha' + \alpha\gamma')F^{(k)}F - \alpha F^{(k)}F' =$$
$$= \big(a_1\alpha - \beta\big)F^{(k+1)} - \big(a_1\alpha' + a_1'\alpha - \beta' + a_1\alpha\gamma' - \beta\gamma'\big)F^{(k)} - a_2\alpha F' +$$
$$+\big(a_2'\alpha - a_2\alpha\gamma' - a_2\alpha'\big)F + a_1a_2\alpha\gamma' + a_1a_2\alpha' - a_1a_2'\alpha + a_1'a_2\alpha - a_2\beta\gamma' - a_2\beta' + a_2'\beta$$

and so

$$F^{(k+1)}F - \xi F^{(k)}F - F^{(k)}F' = \tag{7}$$
$$= \frac{1}{\alpha}\Big\{\big(a_1\alpha - \beta\big)F^{(k+1)} - \big(a_1\alpha' + a_1'\alpha - \beta' + a_1\alpha\gamma' - \beta\gamma'\big)F^{(k)} - a_2\alpha F' +$$
$$+\big(a_2'\alpha - a_2\alpha\gamma' - a_2\alpha'\big)F + a_1a_2\alpha\gamma' + a_1a_2\alpha' - a_1a_2'\alpha + a_1'a_2\alpha - a_2\beta\gamma' - a_2\beta' + a_2'\beta\Big\}.$$

Now we consider following two cases.
Case 1. Suppose $F^{(k+1)}F - \xi F^{(k)}F - F^{(k)}F' \not\equiv 0$.
Immediately we have following two subcases.
Subcase 1.1. Suppose $n > k+1$. By induction we deduce from (4) that

$$F^{(k)} = \sum_{\lambda} A_{1\lambda} f^{l_0^\lambda}(f')^{l_1^\lambda}\dots(f^{(k)})^{l_k^\lambda},$$

where $l_0^\lambda, l_1^\lambda, \dots, l_k^\lambda$ are non-negative integers satisfying

$$\sum_{j=0}^{k} l_j^\lambda = n,\ n-k \le l_0^\lambda \le n-1$$

and $A_{1\lambda}$ are constants and

$$F^{(k+1)} = \sum_{\lambda} B_{1\lambda} f^{p_0^\lambda}(f')^{p_1^\lambda}\dots(f^{(k+1)})^{p_{k+1}^\lambda},$$

where $p_0^\lambda, p_1^\lambda, \dots, p_{k+1}^\lambda$ are non-negative integers satisfying $\sum_{j=0}^{k+1} p_j^\lambda = n$, $n-k-1 \le p_0^\lambda \le n-1$, i.e., $0 \le p_0^\lambda \le n-1$ and $B_{1\lambda}$ are constants. Note that

$$F^{(k)} = f^{n-k-1}\sum_{\lambda} A_{1\lambda} f^{l_0^\lambda-(n-k-1)}(f')^{l_1^\lambda}\dots(f^{(k)})^{l_k^\lambda}, \tag{8}$$

where $1 \le l_0^\lambda - (n-k-1) \le k$ and

$$F^{(k+1)} = f^{n-k-1}\sum_{\lambda} B_{1\lambda} f^{p_0^\lambda-(n-k-1)}(f')^{p_1^\lambda}\dots(f^{(k+1)})^{p_{k+1}^\lambda}, \tag{9}$$

where $0 \le p_0^\lambda - (n-k-1) \le k$. Now from (4), (8) and (9), we get

$$F^{(k+1)}F - \xi F^{(k)}F - F^{(k)}F' = f^n f^{n-k-1}P(f), \tag{10}$$

where

$$P(f) = \sum_{\lambda} B_{1\lambda} f^{p_0^\lambda-(n-k-1)}(f')^{p_1^\lambda}\dots(f^{(k+1)})^{p_{k+1}^\lambda} -$$

$$-\xi\sum_{\lambda}A_{1\lambda}f^{l_0^\lambda-(n-k-1)}(f')^{l_1^\lambda}\dots(f^{(k)})^{l_k^\lambda}-n\sum_{\lambda}A_{1\lambda}f^{l_0^\lambda-(n-k)}(f')^{l_1^\lambda+1}\dots(f^{(k)})^{l_k^\lambda}$$

is a differential polynomial in f of degree $k+1$. In particular every monomial of $P(f)$ is of the form

$$R_\lambda(\alpha,\alpha',\gamma')f^{q_0^\lambda}(f')^{q_1^\lambda}\dots(f^{(k+1)})^{q_{k+1}^\lambda},$$

where $q_0^\lambda,\dots,q_{k+1}^\lambda$ are non-negative integers satisfying $\sum_{j=0}^{k+1}q_j^\lambda=k+1$ and $q_0^\lambda\le k$, $R_\lambda(\alpha,\alpha',\gamma')$ is a polynomial in α,α' and γ' with constant coefficients. Now from (7) and (10) we have

$$f^nf^{n-k-1}P(f)=Q(f), \tag{11}$$

where

$$Q(f)=\frac{1}{\alpha}\Big\{\big(a_1\alpha-\beta\big)F^{(k+1)}-\big(a_1\alpha'+a_1'\alpha-\beta'+a_1\alpha\gamma'-\beta\gamma'\big)F^{(k)}-a_2\alpha F'+$$
$$+\big(a_2'\alpha-a_2\alpha\gamma'-a_2\alpha'\big)F+a_1a_2\alpha\gamma'+a_1a_2\alpha'-a_1a_2'\alpha+a_1'a_2\alpha-a_2\beta\gamma'-a_2\beta'+a_2'\beta\Big\}$$

is a differential polynomial in f of degree n. Since $f^nf^{n-k-1}P(f)\equiv F^{(k+1)}F-\xi F^{(k)}F-F^{(k)}F'\not\equiv 0$, it follows that $P(f)\not\equiv 0$. Now from Lemma 1 and (11) we have

$$m(r,\infty;P(f))=S(r,f)\text{ and }m(r,\infty;f^{n-k-1}P(f))=S(r,f).$$

Since $N(r,\infty;f)=S(r,f)$, we have

$$P(f)\in S(f)\text{ and }f^{n-k-1}P(f)\in S(f).$$

Note that

$$m(r,\infty;f^{n-k-1})=m\Big(r,\infty;f^{n-k-1}P(f)\frac{1}{P(f)}\Big)\le m(r,\infty;f^{n-k-1}P(f))+m\Big(r,\infty;\frac{1}{P(f)}\Big)\le$$
$$\le S(r,f)+T(r,P(f))=S(r,f),$$

i.e., $m(r,\infty;f)=S(r,f)$. Therefore $T(r,f)=N(r,\infty;f)+m(r,\infty;f)=S(r,f)$, which is impossible.

Subcase 1.2. Suppose $n=k+1$. Now from (4) we deduce that

$$F^{(k)}=\frac{d^k}{dz^k}\Big\{f^{k+1}\Big\}=\frac{d^{k-1}}{dz^{k-1}}\Big\{(k+1)f^kf'\Big\}=(k+1)\frac{d^{k-2}}{dz^{k-2}}\Big\{kf^{k-1}(f')^2+f^kf''\Big\}= \tag{12}$$
$$=(k+1)k\,\frac{d^{k-2}}{dz^{k-2}}\Big\{f^{k-1}(f')^2\Big\}+(k+1)\frac{d^{k-2}}{dz^{k-2}}\Big\{f^kf''\Big\}=$$
$$=(k+1)k\,\frac{d^{k-3}}{dz^{k-3}}\Big\{(k-1)f^{k-2}(f')^3\Big\}+(k+1)k\,\frac{d^{k-3}}{dz^{k-3}}\Big\{2f^{k-1}f'f''\Big\}+$$
$$+(k+1)\frac{d^{k-3}}{dz^{k-3}}\Big\{kf^{k-1}f'f''\Big\}+(k+1)\frac{d^{k-3}}{dz^{k-3}}\Big\{f^kf'''\Big\}=$$
$$=(k+1)k(k-1)\frac{d^{k-3}}{dz^{k-3}}\Big\{f^{k-2}(f')^3\Big\}+2(k+1)k\frac{d^{k-3}}{dz^{k-3}}\Big\{f^{k-1}f'f''\Big\}+$$
$$+(k+1)k\frac{d^{k-3}}{dz^{k-3}}\Big\{f^{k-1}f'f''\Big\}+(k+1)\frac{d^{k-3}}{dz^{k-3}}\Big\{f^kf'''\Big\}=\dots\dots=$$

$$= (k+1)!f(f')^k + \frac{k(k-1)}{4}(k+1)!f^2(f')^{k-2}f'' + \ldots + (k+1)f^k f^{(k)}.$$

Therefore

$$\frac{f'}{f}F^{(k)} = (k+1)!(f')^{k+1} + \frac{k(k-1)}{4}(k+1)!f(f')^{k-1}f'' + \ldots + (k+1)f^{k-1}f'f^{(k)} \tag{13}$$

and

$$F^{(k+1)} = (k+1)!(f')^{k+1} + \frac{k(k+1)}{2}(k+1)!f(f')^{k-1}f'' + \ldots + (k+1)f^k f^{(k+1)}. \tag{14}$$

Putting (4), (12), (13) and (14) into (7), we have

$$f^n P(f) = Q(f), \tag{15}$$

where $Q(f)$ is a differential polynomial in f of degree n and

$$P(f) = F^{(k+1)} - \xi F^{(k)} - n\frac{f'}{f}F^{(k)} = -k(k+1)!(f')^{k+1} - (k+1)!\xi f(f')^k + \tag{16}$$
$$+\frac{k(k+1)(3-k)(k+1)!}{4}f(f')^{k-1}f'' + \ldots + (k+1)f^k f^{(k+1)} -$$
$$-(k+1)\xi f^k f^{(k)} - (k+1)^2 f^{k-1}f'f^{(k)} = -k(k+1)!(f')^{k+1} + R_1(f),$$

is a differential polynomial in f of degree $k+1$ and $R_1(f)$ is a differential polynomial in f such that each term of $R_1(f)$ contains f^m for some $m(1 \le m \le n-1)$ as a factor.

Since $f^n P(f) \equiv F^{(k+1)}F - \xi F^{(k)}F - F^{(k)}F' \not\equiv 0$, it follows that $P(f) \not\equiv 0$. Then by Lemma 1 we get $m(r,\infty;P) = S(r,f)$. Since $N(r,\infty;f) = S(r,f)$, we have

$$P(f) \in S(f) \text{ and } P'(f) \in S(f). \tag{17}$$

Let z_1 be a zero of f with multiplicity $p_1(\ge 2)$. Then from (16) we see that z_1 is a zero of $P(f)$ with multiplicity $(p_1-1)(k+1) \ge 2(p_1-1) \ge p_1$. Consequently we have

$$N_{(2}(r,0;f) \le N(r,0;P(f)) \le T(r,P(f)) = S(r,f).$$

Also from (16) we have

$$m\Big(r,\frac{P(f)}{f^{k+1}}\Big) = S(r,f).$$

Consequently

$$m\Big(r,\frac{1}{f^{k+1}}\Big) \le m\Big(r,\frac{P(f)}{f^{k+1}}\Big) + m\Big(r,\frac{1}{P(f)}\Big) \le T(r,P(f)) + S(r,f) = S(r,f),$$

i.e., $m\big(r,\frac{1}{f}\big) = S(r,f)$. Therefore

$$T(r,f) = N(r,0;f) + m\big(r,\frac{1}{f}\big) = N_{1)}(r,0;f) + S(r,f). \tag{18}$$

Note that from (16) we get

$$P'(f) = A_1(f')^k f'' + B_1(f')^{k+1} + S_1(f), \tag{19}$$

is a differential polynomial in f, where $A_1 = -\frac{1}{4}k(k+1)^2(k+1)!$, $B_1 = -(k+1)!\xi$ and $S_1(f)$ is a differential polynomial in f such that each term of $S_1(f)$ contains f^m for some m $(1 \le m \le n-1)$ as a factor.

Let z_2 be a simple zero of f. Then from (16) and (19) we have

$$P(f(z_2)) = -k(k+1)!(f'(z_2))^{k+1} \text{ and } P'(f(z_2)) = A_1(f'(z_2))^k f''(z_2) + B_1(z_2)(f'(z_2))^{k+1}.$$

This shows that z_2 is a zero of $P(f)f'' - [K_1P'(f) - K_2P(f)]f'$, where $K_1 = \frac{-k(k+1)!}{A_1}$ and $K_2 = \frac{B_1}{A_1}$. Also we see that $K_1 \in S(f)$ and $K_2 \in S(f)$. Let

$$\Phi_1 = \frac{P(f)f'' - [K_1P'(f) - K_2P(f)]f'}{f}. \tag{20}$$

Suppose $\Phi_1(z) \not\equiv 0$. Then clearly $m(r, \Phi_1) = S(r, f)$ and since $N_{(2}(r, 0; f) + N(r, \infty; f) = S(r, f)$, we have $\Phi_1 \in S(f)$. From (20) we obtain

$$f''(z) = \alpha_1(z)f(z) + \beta_1(z)f'(z), \tag{21}$$

where

$$\alpha_1 = \frac{\Phi_1}{P(f)} \text{ and } \beta_1 = K_1\frac{P'(f)}{P(f)} - K_2. \tag{22}$$

We note that (21) is also true even for $\Phi_1(z) \equiv 0$. Actually in this case $\alpha_1(z) \equiv 0$. Also (22) yields

$$P'(f) = \Big(\frac{\beta_1}{K_1} + \frac{K_2}{K_1}\Big)P(f) \tag{23}$$

and so

$$\beta_1 = K_1\frac{P'(f)}{P(f)} - K_2 = \frac{-k(k+1)!}{A_1}\frac{P'(f)}{P(f)} - \frac{B_1}{A_1},$$

i.e.,

$$A_1\beta_1 + B_1 + k(k+1)!\frac{P'(f)}{P(f)} = 0. \tag{24}$$

Now we consider following two subcases.

Subcase 1.2.1. Suppose $e^\gamma \in S(f)$. Now from (5) we have

$$F^{(k)} - \alpha e^\gamma F = a_2 - a_1\alpha e^\gamma + \beta e^\gamma. \tag{25}$$

Suppose $a_2 - a_1\alpha e^\gamma + \beta e^\gamma \not\equiv 0$. Since $n = k+1$, from (25) we have $N_{1)}(r, 0; f) = S(r, f)$. Therefore from (18) we arrive to a contradiction. Therefore $a_2 - a_1\alpha e^\gamma + \beta e^\gamma \equiv 0$ and so from (25) we get

$$F^{(k)} \equiv \frac{a_2\alpha}{a_1\alpha - \beta}F,$$

which is the desired result.

Subcase 1.2.2. Suppose $e^\gamma \notin S(f)$. Following two subcases are immediately.

Subcase 1.2.2.1. Suppose $k = 1$. Now from (16) and (21) we have

$$P(f) = -2(f')^2 - 2\xi ff' + 2ff'' = -2(f')^2 + (2\beta_1 - 2\xi)ff' + 2\alpha_1 f^2$$

and so

$$P'(f)=(-2\beta_1-2\xi)(f')^2+(2\beta_1'-2\xi'+2\beta_1^2-2\beta_1\xi)ff'+(2\alpha_1\beta_1-2\alpha_1\xi+2\alpha_1')f^2.$$

Note that $K_1=1$ and $K_2=\xi$ and so from (23) we have

$$\big(\beta_1'-\xi'-\beta_1\xi+\xi^2\big)f'+\big(-2\alpha_1\xi+\alpha_1'\big)f\equiv 0. \tag{26}$$

We claim that $\beta_1'-\xi'-\beta_1\xi+\xi^2\not\equiv 0$. If not, suppose

$$\beta_1'-\xi'-\beta_1\xi+\xi^2\equiv 0. \tag{27}$$

Let $\beta_1\equiv\xi$. Then a simple calculation gives $2(\frac{\alpha'}{\alpha}+\gamma')=\frac{P'(f)}{P(f)}$ and so on integration we get $\alpha^2e^{2\gamma}=d_0P(f)$, where $d_0\in\mathbb{C}\setminus\{0\}$. This shows that $e^\gamma\in S(f)$, which is a contradiction. So $\beta_1\not\equiv\xi$. Now from (27) we get $\frac{\beta_1'-\xi'}{\beta_1-\xi}=\xi=\frac{\alpha'}{\alpha}+\gamma'$. So on integration we get $\alpha e^\gamma=d_1(\beta_1-\xi)$, where $d_1\in\mathbb{C}\setminus\{0\}$. This contradicts the fact that $e^\gamma\notin S(f)$. So $\beta_1'-\xi'-\beta_1\xi+\xi^2\not\equiv 0$. Since $ff'\not\equiv 0$, from (27) we conclude that $-2\alpha_1\xi+\alpha_1'\not\equiv 0$. Then from (26) we see that if z_3 is a simple zero of f, then z_3 is either a pole of $-2\alpha_1\xi+\alpha_1'$ or a zero of $\beta_1'-\xi'-\beta_1\xi+\xi^2$. Hence

$$N_{1)}(r,0;f)\le N(r,\infty;-2\alpha_1\xi+\alpha_1')+N(r,0;\beta_1'-\xi'-\beta_1\xi+\xi^2)=S(r,f).$$

So we arrive to a contradiction by (18).
Subcase 1.2.2.2. Suppose $k\ge 2$. From (12) and (14) we have

$$F^{(k)}=T_1(f),F^{(k+1)}=(k+1)!(f')^{k+1}+T_2(f),$$
$$F^{(k+2)}=\frac{(k+1)(k+2)}{2}(k+1)!(f')^kf''+T_3(f),$$

where $T_1(f)$, $T_2(f)$ and $T_3(f)$ are differential polynomials in f such that each term of $T_1(f)$, $T_2(f)$ and $T_3(f)$ contains f as a factor. Comparing (11) and (14) and noting that $F=f^n=f^{k+1}$ we have

$$\begin{aligned}\alpha Q(f)=\big(a_1\alpha-\beta\big)F^{(k+1)}-\big(a_1\alpha'+a_1'\alpha-\beta'+a_1\alpha\gamma'-\beta\gamma'\big)F^{(k)}-a_2\alpha F'+\\ +\big(a_2'\alpha-a_2\alpha\gamma'-a_2\alpha'\big)F+\delta=\big(a_1\alpha-\beta\big)\big\{(k+1)!(f')^{k+1}+T_2(f)\big\}\\ -\big(a_1\alpha'+a_1'\alpha-\beta'+a_1\alpha\gamma'-\beta\gamma'\big)T_1(f)-(k+1)a_2\alpha f^kf'+\big(a_2'\alpha-a_2\alpha\gamma'-a_2\alpha'\big)f^{k+1}+\delta,\end{aligned} \tag{28}$$

where $\delta=a_1a_2\alpha\gamma'+a_1a_2\alpha'-a_1a_2'\alpha+a_1'a_2\alpha-a_2\beta\gamma'-a_2\beta'+a_2'\beta$. First, we claim that $a_1\alpha-\beta\not\equiv 0$. If not, suppose $a_1\alpha-\beta\equiv 0$. In this case $\delta\equiv 0$. Then from (15) and (28) we have

$$\big(a_1\alpha'+a_1'\alpha-\beta'+a_1\alpha\gamma'-\beta\gamma'\big)F^{(k)}+a_2\alpha F'-\big(a_2'\alpha-a_2\alpha\gamma'-a_2\alpha'-\alpha P(f)\big)F\equiv 0, \tag{29}$$

where $a_2\alpha\not\equiv 0$. Since $n=k+1$, from (29) we have $N_{1)}(r,0;f)=S(r,f)$. Therefore from (18) we arrive to a contradiction. Hence $a_1\alpha-\beta\not\equiv 0$.
Next we claim that $\delta(z)\not\equiv 0$. If not, suppose $\delta(z)\equiv 0$. Then from (15) and (28) we have

$$\begin{aligned}\big(a_1\alpha-\beta\big)\big\{(k+1)!(f')^{k+1}+T_2(f)\big\}-\big(a_1\alpha'+a_1'\alpha-\beta'+a_1\alpha\gamma'-\beta\gamma'\big)T_1(f)-\\ -(k+1)a_2\alpha f^kf'+\big(a_2'\alpha-a_2\alpha\gamma'-a_2\alpha'-\alpha P(f)\big)f^{k+1}\equiv 0,\end{aligned} \tag{30}$$

Now from (30) we see that a simple zero of f must be either a zero of $a_1\alpha-\beta$ or a pole of at least one of $a_1\alpha'+a_1'\alpha-\beta'+a_1\alpha\gamma'-\beta\gamma'$, α, $a_2'\alpha-a_2\alpha\gamma'-a_2\alpha'-\alpha P(f)$. Therefore

$$N_{1)}(r,0;f)\leq N\big(r,0;a_1\alpha-\beta\big)+N\big(r,\infty;a_1\alpha'+a_1'\alpha-\beta'+a_1\alpha\gamma'-\beta\big)\leq$$
$$\leq N\big(r,\infty;\alpha)+N(r,\infty;a_2'\alpha-a_2\alpha\gamma'-a_2\alpha'-\alpha P(f)\big)\leq S(r,f).$$

So we arrive to a contradiction from (18). Hence $\delta(z)\not\equiv 0$. We further note that $\delta\in S(f)$. Differentiating (28) we have

$$\begin{aligned}(\alpha Q(f))'&=(a_1'\alpha+a_1\alpha'-\beta')F^{(k+1)}+(a_1\alpha-\beta)F^{(k+2)}-\\&-(a_1\alpha'+a_1'\alpha-\beta'+a_1\alpha\gamma'-\beta\gamma')F^{(k+1)}-\\&-(a_1\alpha'+a_1'\alpha-\beta'+a_1\alpha\gamma'-\beta\gamma')'F^{(k)}-(a_2\alpha)'F'-a_2\alpha F''+\\&+(a_2'\alpha-a_2\alpha\gamma'-a_2\alpha')'F+(a_2'\alpha-a_2\alpha\gamma'-a_2\alpha')F'+\delta'=\\&=(a_1'\alpha+a_1\alpha'-\beta')\Big\{(k+1)!(f')^{k+1}+T_2(f)\Big\}+\\&+(a_1\alpha-\beta)\Big\{\frac{(k+1)(k+2)}{2}(k+1)!(f')^kf''+T_3(f)\Big\}-\\&-(a_1\alpha'+a_1'\alpha-\beta'+a_1\alpha\gamma'-\beta\gamma')\Big\{(k+1)!(f')^{k+1}+T_2(f)\Big\}-\\&-(a_1\alpha'+a_1'\alpha-\beta'+a_1\alpha\gamma'-\beta\gamma')'T_1(f)-(k+1)(a_2\alpha)'f^kf'-\\&-a_2\alpha\Big\{k(k+1)f^{k-1}(f')^2+(k+1)f^kf''\Big\}+(a_2'\alpha-a_2\alpha\gamma'-a_2\alpha')'f^{k+1}+\\&+(k+1)(a_2'\alpha-a_2\alpha\gamma'-a_2\alpha')f^kf'+\delta'.\end{aligned}\tag{31}$$

Let z_4 be a simple zero of $f(z)$. Then from (15), (28) and (31) we have

$$\delta(z_4)=A(z_4)(f'(z_4))^{k+1},\quad \delta'(z_4)=A_2(z_4)(f'(z_4))^kf''(z_4)+B_2(z_4)(f'(z_4))^{k+1},$$

where

$$A(z)=-(k+1)!(a_1(z)\alpha(z)-\beta(z)),\quad A_2(z)=-\frac{(k+1)(k+2)}{2}(k+1)!(a_1(z)\alpha(z)-\beta(z)),$$
$$B_2(z)=(k+1)!(a_1(z)\alpha(z)-\beta(z))\gamma'(z).$$

This shows that z_4 is a zero of $\delta f''-[K_3\delta'-K_4\delta]f'$, where $K_3=\frac{A}{A_2}$ and $K_4=\frac{B_2}{A_2}$. Also we see that $K_3\in S(f)$ and $K_4\in S(f)$. Let

$$\Phi_2=\frac{\delta f''-[K_3\delta'-K_4\delta]f'}{f}.\tag{32}$$

Suppose $\Phi_2(z)\not\equiv 0$. Then clearly $m(r,\Phi_2)=S(r,f)$ and since $N_{(2}(r,0;f)+N(r,\infty;f)=S(r,f)$, we have $\Phi_2\in S(f)$. From (32) we obtain

$$f''=\phi_1 f+\psi_1 f',\tag{33}$$

where

$$\phi_1=\frac{\Phi_2}{\delta}\text{ and }\psi_1=K_3\frac{\delta'}{\delta}-K_4.\tag{34}$$

We note that (33) is also true even for $\Phi_2(z) \equiv 0$. Actually in this case $\phi_1(z) \equiv 0$.

Now we claim that $\psi_1 \not\equiv \beta_1$. If $\psi_1 \equiv \beta_1$ then from (22) and (34) we have

$$\frac{2}{(k+1)(k+2)}\frac{\delta'}{\delta} + \frac{2}{(k+1)(k+2)}\gamma' \equiv \frac{4}{(k+1)^2}\frac{P(f)'}{P(f)} - \frac{4}{k(k+1)^2}\xi,$$

i.e.,

$$2k(k+2)\frac{P(f)'}{P(f)} - k(k+1)\frac{\delta'}{\delta} \equiv (k^2+3k+4)\gamma' + 2(k+2)\frac{\alpha'}{\alpha}.$$

On integration we have

$$\alpha^{2(k+2)}e^{(k^2+3k+4)\gamma} \equiv \frac{d_3 P(f)^{2k(k+2)}}{\delta^{k(k+1)}}$$

where $d_3 \in \mathbb{C}\setminus\{0\}$ and so from (17) we have $e^{\gamma} \in S(f)$, a contradiction. So we suppose that $\psi_1 \not\equiv \beta_1$.

Now from (33) we have

$$f^{(i)} = \phi_{i-1}f + \psi_{i-1}f', \tag{35}$$

where $i \geq 2$ and $\phi_{i-1} \in S(f)$, $\psi_{i-1} \in S(f)$. Also from (16), (19) and (35) we have respectively

$$P(f) = -k(k+1)!(f')^{k+1} + \sum_{j=1}^{k+1} T_j f^j (f')^{k+1-j}, \tag{36}$$

$$P'(f) = (A_1\psi_1 + B_1)(f')^{k+1} + \sum_{j=1}^{k+1} S_j f^j (f')^{k+1-j}, \tag{37}$$

where $T_j \in S(f)$ and $S_j \in S(f)$. Multiplying (36) by $P'(f)$ and (37) by $P(f)$ and then subtracting we get

$$H_0(f')^{k+1} + H_1 f(f')^k + \ldots + H_{k+1}f^{k+1} \equiv 0, \tag{38}$$

where

$$H_0 = P(f)\Big[A_1\psi_1 + B_1 + k(k+1)!\frac{P'(f)}{P(f)}\Big] \tag{39}$$

and $H_j = P(f)S_j - P'(f)T_j$ for $j = 1, 2, \ldots, k+1$. Note that $H_0 \in S(f)$ and $H_j \in S(f)$ for $j = 1, 2, \ldots, k+1$. Since $\beta_1 \not\equiv \psi_1$, it follows from (24) and (39) that $H_0 \not\equiv 0$. Now from (38) we see that a simple zero of f must be either a zero of H_0 or a pole of at least one of $H_1, H_2, \ldots H_{k+1}$. Therefore

$$N_{1)}(r, 0; f) \leq N(r, 0; H_0) + \sum_{i=1}^{k+1} N(r, \infty; H_i) + S(r, f) = S(r, f),$$

because $H_i \in S(f)$ for $i = 0, 1, 2, \ldots, k+1$. So we arrive to a contradiction from (18).

Case 2. Suppose $F^{(k+1)}F - \xi F^{(k)}F - F^{(k)}F' \equiv 0$. Then we have

$$\frac{F^{(k+1)}}{F^{(k)}} \equiv \frac{\alpha'}{\alpha} + \gamma' + \frac{F'}{F}.$$

On integration, we have $F^{(k)} = d_4\alpha e^{\gamma}F$, where $d_4 \in \mathbb{C}\setminus\{0\}$. Now from (5) we have

$$(d_4 - 1)\alpha e^{\gamma} f^n \equiv a_2 - (a_1\alpha - \beta)e^{\gamma}. \tag{40}$$

If $d_4 = 1$, then from (40) we have $a_2 - (a_1\alpha - \beta)e^\gamma \equiv 0$. Now from (5) we get $F^{(k)} \equiv \frac{a_2\alpha}{a_1\alpha-\beta}F$, which is the desired result.

Next we suppose $d_4 \neq 1$. Now we consider following two subcases.

Subcase 2.1. Suppose $e^\gamma \in S(f)$. Clearly from (40) and Lemma 3 we have $n\ T(r, f) = S(r, f)$, which is a contradiction.

Subcase 2.2. Suppose $e^\gamma \notin S(f)$. Now from (40) we have $e^{-\gamma} \equiv (d_4 - 1)\frac{\alpha}{a_2}f^n + \frac{1}{a_2}(a_1\alpha - \beta)$. By Lemma 3 we get

$$T(r, e^\gamma) = T(r, e^{-\gamma}) + O(1) = nT(r, f) + S(r, f)$$

and so $S(r, f) = S(r, e^\gamma)$. Therefore $a_1, a_2, \alpha, \beta \in S(e^\gamma)$. Also from (40) we have

$$\overline{N}\Big(r, \frac{a_2}{a_1\alpha - \beta}; e^\gamma\Big) \leq \frac{1}{n}N\Big(r, \frac{a_2}{a_1\alpha - \beta}; e^\gamma\Big) \leq \frac{1}{n}T(r, e^\gamma).$$

Now by Lemma 2 we have

$$T(r, e^\gamma) \leq \overline{N}(r, \infty; e^\gamma) + \overline{N}(r, 0; e^\gamma) + \overline{N}\Big(r, \frac{a_2}{a_1\alpha - \beta}; e^\gamma\Big) + S(r, e^\gamma) \leq \frac{1}{n}T(r, e^\gamma) + S(r, e^\gamma),$$

which is impossible since $n \geq k + 1 \geq 2$. □

REFERENCES

1. Brück R., *On entire functions which share one value CM with their first derivative*, Results Math., **30** (1996), 21–24.
2. Cao T.B., *On the Brück conjecture*, Bull. Aust. Math. Soc., **93** (2016), 248–259.
3. Chen Z.X., Shon K.H., *On conjecture of R. Brück concerning the entire function sharing one value CM with its derivative*, Taiwanese J. Math., **8** (2004), №2, 235–244.
4. Clunie J., *On integral and meromorphic functions*, J. London Math. Soc., **37** (1962), 17–22.
5. Gundersen G.G., Yang L.Z., *Entire functions that share one value with one or two of their derivatives*, J. Math. Anal. Appl., **223** (1998), 88–95.
6. Hayman W.K., Meromorphic Functions, Clarendon Press, Oxford, 1964.
7. Lü F., Yi H.X., *The Brück conjecture and entire functions sharing polynomials with their k-th derivatives*, J. Korean Math. Soc., **48** (2011), №3, 499–512.
8. Lü F., Li Q., Yang C., *On the transcendental entire solutions of a class of differential equations*, Bull. Korean Math. Soc., **51** (2014), №5, 1281–1289.
9. Yang L.Z., Zhang J.L., *Non-existence of meromorphic solutions of Fermat type functional equation*, Aequations Math., **76** (2008), №1-2, 140–150.
10. Yang C.C., *On deficiencies of differential polynomials II*, Math. Z., **125** (1972), 107–112.
11. Zhang J.L., *Meromorphic functions sharing a small function with their derivatives*, Kyungpook Math. J., **49** (2009), 143–154.
12. Zhang J.L., Yang L.Z., *A power of a meromorphic function sharing a small function with its derivative*, Annales Academiæ Scientiarum Fennicæ Mathematica, **34** (2009), 249–260.
13. Zhang J.L., Yang L.Z., *A power of an entire function sharing one value with its derivative*, Comput. Math. Appl, **60** (2010), 2153–2160.

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