УДК 519.1

L. Bedratyuk

THE DOUBLE STAR SEQUENCES AND THE GENERAL SECOND ZAGREB INDEX

L. Bedratyuk. *The double star sequences and the general second Zagreb index*, Mat. Stud. **51** (2019), 115–123.

For a simple graph we introduce notions of the double star sequence, the double star frequently sequence and prove that these sequences are inverses of each other. As a consequence, we express the general second Zagreb index in terms of the double star sequence. Also, we calculate the ordinary generating function and a linear recurrence relation for the sequence of the general second Zagreb indices.

1. Introduction. Let G be a simple graph whose vertex and edge sets are $V(G)$ and $E(G)$, respectively. Let d_v be the degree of the vertex $v \in V(G)$. For any real p the general second Zagreb index is defined by

$$M_2^{(p)}(G) = \sum_{uv \in E(G)} (d_u\, d_v)^p,$$

see [1] for more details. Put $n = |V(G)|$ and $m = |E(G)|$. The double star graph $S_{a,b}, 0 \leq a \leq b$ is a three with the following degree sequence $a+1, b+1, \underbrace{1, \dots, 1}_{a+b \text{ times}}$.

For example $S_{0,0} = P_2$, $S_{0,1} = P_3$, $S_{1,1} = P_4$ and $S_{0,a}$ is the star S_{a+1}.

Denote by $S_{a,b}(G)$ the number of subgraphs of G which are isomorphic to the double star $S_{a,b}$. For instance $S_{0,0}(G)$ is equal to the number of edges of G. It is easy to see that there exists only one connected graph G of order n such that $S_{n-2,n-2}(G) = 1$, namely, the graph with the degree sequence $n-1, n-1, \underbrace{2, \dots, 2}_{n-2 \text{ times}}$.

Also, we have that $S_{a,n-1}(G) = 0$ for all a.

The triangle

$$\begin{array}{l} S_{0,0}(G), S_{0,1}(G), \dots, S_{0,n-2}(G), \\ S_{1,2}(G), \dots, S_{1,n-2}(G), \\ \dots\dots\dots\dots \\ S_{n-2,n-2}(G), \end{array}$$

is called the *double star sequence* of a graph G. The value $S_{a,b}(G)$ can be counted by simple combinatorial techniques:

2010 *Mathematics Subject Classification:* 05C07, 05C90, 05A15.
Keywords: simple graph; double star sequence; general second Zagreb index.
doi:10.15330/ms.51.2.115-123

$$S_{a,b}(G)=\begin{cases}\sum\limits_{\{u,v\}\in E(G)}\left(\binom{d_u-1}{a}\binom{d_v-1}{b}+\binom{d_u-1}{b}\binom{d_v-1}{a}\right), & a\leq b,\\ \sum\limits_{\{u,v\}\in E(G)}\binom{d_u-1}{a}\binom{d_v-1}{a}, & a=b,\end{cases}$$

see [2], the formulas (1.7)–(1.8). For small p we have $M_2^{(0)}(G)=S_{0,0}(G)=m$, and

$$M_2^{(1)}(G)=M_2(G)=\sum_{uv\in E(G)}d_ud_v=\sum_{uv\in E(G)}1+(d_u-1)+(d_v-1)+(d_u-1)(d_v-1)=$$
$$=m+\sum_{uv\in E(G)}\binom{d_u-1}{1}\binom{d_v-1}{0}+\binom{d_u-1}{0}\binom{d_v-1}{1}+\binom{d_u-1}{1}\binom{d_v-1}{1}$$
$$=S_{0,0}(G)+S_{0,1}(G)+S_{1,1}(G).$$

Also, by using the simple identity

$$((x+1)(y+1))^2=1+3\left(\binom{x}{1}+\binom{y}{1}\right)+2\left(\binom{x}{2}+\binom{y}{2}\right)+9\binom{x}{1}\binom{y}{1}+$$
$$+6\left(\binom{x}{1}\binom{y}{2}+\binom{x}{2}\binom{y}{1}\right)+4\binom{x}{2}\binom{y}{2},$$

we have that

$$M_2^{(2)}(G)=\sum_{uv\in E(G)}(d_ud_v)^2=S_{0,0}(G)+3\,S_{0,1}(G)+2\,S_{0,2}(G)+9\,S_{1,1}(G)+$$
$$+6\,S_{1,2}(G)+4\,S_{2,2}(G).$$

We generalize these expressions for the general second Zagreb index $M_2^{(p)}(G)$ for any natural p. See also [3] for similar results for the general first Zagreb index. The main result of the paper is the formula for expressing of the general second Zagreb index in terms of the double star sequence:

$$M_2^{(p)}(G)=\sum_{i=0}^{n-2}\sum_{k=i}^{n-2}i!k!\left\{{p+1\atop i+1}\right\}\left\{{p+1\atop k+1}\right\}S_{i,k}(G),p\in\mathbb{N},$$

here $\left\{{p\atop i}\right\}$ are the Stirling numbers of the second kind.

Also, we calculate the ordinary generating function for the integer sequence $\{M_2^{(p)}(G)\}$. Denote by C_n the set $C_n=\{i\cdot j\mid 0\leq i,j\leq n\}$. Then

$$\sum_{p=0}^{\infty}M_2^{(p)}(G)z^p=\frac{\sum\limits_{k=0}^{|C_{n-1}|-1}\left(\sum\limits_{i=\max(0,k-|C_{n-1}|)}^{\min(k,|C_{n-1}|)}\left[{C_{n-1}\atop |C_{n-1}|-i}\right]M_2^{(k-i)}(G)\right)t^k}{\prod\limits_{c\in C_{n-1}}(1-c\cdot t)},$$

and the linear recurrence relation for the integer sequence $\{M_2^{(p)}(G)\}$:

$$M_2^{(|C_{n-1}|)}(G)=-\sum_{i=1}^{|C_{n-1}|-1}\left[{C_{n-1}\atop i}\right]M_2^{(i)}(G),$$

here $\left[{C_{n-1}\atop i}\right]$ are the Comtet numbers of the first kind associated with the set C_{n-1}.

2. Double star and double frequently sequences. Any edge $\{u,v\} \in E(G)$ is the double star $S_{d_u-1,d_v-1}, d_u \leq d_v$. Let $f_{i,j}$ denote the number of edges in G which are the double stars $S_{i,j}$. For example, $f_{0,0}$ is the number of isolated edges in G. For the graph G with the degree sequence $n-1, n-1, \underbrace{2,\ldots,2}_{n-2 \text{ times}}$, we have $f_{n-2,n-2}=1$, $f_{0,n-2}=2(n-2)$ and $f_{i,j}=0$. The integer triangle

$$\begin{aligned}
&f_{0,0}, f_{0,1}, \ldots, f_{0,n-2},\\
&f_{1,1}, \ldots, f_{1,n-2},\\
&\vdots\\
&f_{n-2,n-2},
\end{aligned}$$

is called the *double star frequently sequence* of a graph G.

Example 1. Let $G = K_4$. Then the double star sequence and the double star frequently sequence have form

$$\begin{array}{ll}
S_{0,0}(G)=6, S_{0,1}(G)=24, S_{0,2}(G)=12, & f_{0,0}=0, f_{0,1}=0, f_{0,2}=0,\\
S_{1,1}(G)=24, S_{1,2}(G)=24, & f_{1,1}=0, f_{1,2}=0,\\
S_{2,2}(G)=6, & f_{2,2}=6.
\end{array}$$

The double star sequence and the double star frequency sequence are a pair of multinomial inverse sequences, see [4, section 3.5] for more details. The following theorem holds.

Theorem 1. *Let G be a simple graph. Then*

$$(i) S_{a,b}(G) = \begin{cases} \sum\limits_{i=0}^{n-2}\sum\limits_{j=i}^{n-2}\left(\binom{i}{a}\binom{j}{b}+\binom{i}{b}\binom{j}{a}\right) f_{i,j}, a<b,\\ \sum\limits_{i=0}^{n-2}\sum\limits_{j=i}^{n-2}\binom{i}{a}\binom{j}{a} f_{i,j}, a=b,\end{cases}$$

$$(ii) f_{a,b} = \begin{cases} \sum\limits_{i=0}^{n-2}\sum\limits_{j=i}^{n-2}(-1)^{a+b+i+j}\left(\binom{i}{a}\binom{j}{b}+\binom{i}{b}\binom{j}{a}\right) S_{i,j}(G), a<b,\\ \sum\limits_{i=0}^{n-2}\sum\limits_{j=i}^{n-2}(-1)^{i+j}\binom{i}{a}\binom{j}{a} S_{i,j}(G), a=b.\end{cases}$$

Proof. (i) The statement follows immediately from the definitions of $f_{i,j}$.

(ii) For simplicity, we consider only the case $a=b$. We have

$$f_{a,a} = \sum_{i=0}^{n-2}\sum_{j=i}^{n-2}(-1)^{i+j}\binom{i}{a}\binom{j}{a}S_{i,j}(G) = \sum_{i=0}^{n-2}(-1)^{i+i}\binom{i}{a}\binom{i}{a}S_{i,i}(G)+$$

$$+\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-2}(-1)^{i+j}\binom{i}{a}\binom{j}{a}S_{i,j}(G) = \sum_{i=0}^{n-2}\binom{i}{a}\binom{i}{a}\sum_{p=0}^{n-2}\sum_{q=p}^{n-2}\binom{p}{i}\binom{q}{i}f_{p,q}+$$

$$+\sum_{i=0}^{n-2}\sum_{j=i+1}^{n-2}(-1)^{i+j}\binom{i}{a}\binom{j}{a}\sum_{p=0}^{n-2}\sum_{q=p}^{n-2}\left(\binom{p}{i}\binom{q}{j}+\binom{p}{j}\binom{q}{i}\right)f_{p,q} =$$

$$= \sum_{p=0}^{n-2} \sum_{q=p}^{n-2} \sum_{i=0}^{n-2} \left(\binom{i}{a}\binom{i}{a}\binom{p}{i}\binom{q}{i} + \sum_{j=i+1}^{n-2} (-1)^{i+j} \binom{i}{a}\binom{j}{a} \left(\binom{p}{i}\binom{q}{j} + \binom{p}{j}\binom{q}{i} \right) \right) f_{p,q}.$$

Now we use the orthogonal relation (see [4])

$$\sum_{i=0}^{n} (-1)^i \binom{i}{a}\binom{b}{i} = (-1)^a \delta_{a,b},$$

and the simple identity

$$\left(\sum_{i=0}^{p} a_i \right) \left(\sum_{k=0}^{p} b_k \right) = \sum_{i=0}^{n} a_i b_i + \sum_{i=0}^{p} \sum_{k=i+1}^{p} (a_i b_k + b_i a_k).$$

Then the coefficient of $f_{p,q}$ equals

$$\sum_{i=0}^{n-2} \binom{i}{a}\binom{i}{a}\binom{p}{i}\binom{q}{i} + \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-2} (-1)^{i+j} \binom{i}{a}\binom{j}{a} \left(\binom{p}{i}\binom{q}{j} + \binom{p}{j}\binom{q}{i} \right) =$$

$$= \sum_{i=0}^{n-2} \sum_{j=0}^{n-2} (-1)^{i+j} \binom{i}{a}\binom{p}{i}\binom{j}{a}\binom{q}{j} =$$

$$= \sum_{i=0}^{n-2} (-1)^i \binom{i}{a}\binom{p}{i} \times \sum_{j=0}^{n-2} (-1)^j \binom{j}{a}\binom{q}{j} = (-1)^a \delta_{p,a} (-1)^a \delta_{q,a}.$$

Thus

$$\sum_{i=0}^{n-2} \sum_{j=i}^{n-2} (-1)^{i+j} \binom{i}{a}\binom{j}{a} S_{i,j}(G) = (-1)^a \delta_{p,a} (-1)^a \delta_{q,a} f_{p,q} = f_{a,a},$$

as required.

The proof of the case $a \neq b$ is almost identical to that of the case $a = b$ and is omitted. □

As a consequence, we obtain the double star variant of the Handshaking lemma

$$\sum_{i=0}^{n-2} \sum_{j=i}^{n-2} f_{i,j} = \sum_{i=0}^{n-2} \sum_{j=i}^{n-2} \binom{i}{0}\binom{j}{0} f_{i,j} = S_{0,0}(G) = m.$$

From [5] we know that

$$\sum_{uv \in E(G)} \left(\frac{1}{d_u} + \frac{1}{d_v} \right) = n - n_0,$$

here n_0 is the number of isolated vertices of G. Then by the definition of the double star frequently sequence we get that

$$\sum_{i=0}^{n-2} \sum_{j=i}^{n-2} \left(\frac{1}{i+1} + \frac{1}{j+1} \right) f_{i,j} = n - n_0.$$

3. The second general Zagreb index. For the second general Zagreb index we have

$$M_2^{(p)}(G) = \sum_{i=0}^{n-2} \sum_{j=i}^{n-2} ((i+1)(j+1))^p f_{i,j}.$$

Now we can express the second general Zagreb index $M_2^p(G)$ in terms of double star sequence:

Theorem 2.

$$M_2^p(G) = \sum_{i=0}^{n-2} \sum_{k=i}^{n-2} i!k! \left\{ {p+1 \atop i+1} \right\} \left\{ {p+1 \atop k+1} \right\} S_{i,k}(G),$$

for any natural number p.

Proof. We have

$$\sum_{i=0}^{n-2} \sum_{k=i}^{n-2} i!k! \left\{ {p+1 \atop i+1} \right\} \left\{ {p+1 \atop k+1} \right\} S_{i,k}(G) = \sum_{i=0}^{n-2} \sum_{k=i+1}^{n-2} i!k! \left\{ {p+1 \atop i+1} \right\} \left\{ {p+1 \atop k+1} \right\} S_{i,k}(G) + \\ + \sum_{i=0}^{n-2} \left(i! \left\{ {p+1 \atop i+1} \right\} \right)^2 S_{i,i}(G).$$

Simplify the second sum

$$\sum_{i=0}^{n-2} \left(i! \left\{ {p+1 \atop i+1} \right\} \right)^2 S_{i,i}(G) = \sum_{i=0}^{n-2} \left(i! \left\{ {p+1 \atop i+1} \right\} \right)^2 \sum_{s=i}^{n-2} \sum_{t=s}^{n-2} \binom{s}{i} \binom{t}{i} f_{s,t} = \\ = \sum_{s=0}^{n-2} \sum_{t=s}^{n-2} \left(\sum_{i=0}^{p} \left(i! \left\{ {p+1 \atop i+1} \right\} \right)^2 \binom{s}{i} \binom{t}{i} \right) f_{s,t}.$$

Simplify the first sum

$$\sum_{i=0}^{n-2} \sum_{k=i+1}^{n-2} i!k! \left\{ {p+1 \atop i+1} \right\} \left\{ {p+1 \atop k+1} \right\} S_{i,k}(G) = \\ = \sum_{i=0}^{n-2} \sum_{k=i+1}^{n-2} i!k! \left\{ {p+1 \atop i} \right\} \left\{ {p+1 \atop k+1} \right\} \sum_{s=i}^{n-2} \sum_{t=s}^{n-2} \left(\binom{s}{i} \binom{t}{k} + \binom{s}{k} \binom{t}{i} \right) f_{s,t} = \\ = \sum_{s=0}^{n-2} \sum_{t=s}^{n-2} \left(\sum_{i=0}^{p} \sum_{k=i+1}^{p} i!k! \left\{ {p+1 \atop i+1} \right\} \left\{ {p+1 \atop k+1} \right\} \left(\binom{s}{i} \binom{t}{k} + \binom{s}{k} \binom{t}{i} \right) \right) f_{s,t}.$$

Thus the coefficient of $f_{s,t}$ in

$$\sum_{i=0}^{p} \sum_{k=i}^{p} i!k! \left\{ {p+1 \atop i+1} \right\} \left\{ {p+1 \atop k+1} \right\} S_{i,k}(G),$$

equals

$$\sum_{i=0}^{p}\left(i!\left\{{p+1 \atop i+1}\right\}\right)^{2}\binom{s}{i}\binom{t}{i}+\sum_{i=0}^{p}\sum_{k=i+1}^{p} i!k!\left\{{p+1 \atop i+1}\right\}\left\{{p+1 \atop k+1}\right\}\left(\binom{s}{i}\binom{t}{k}+\binom{s}{k}\binom{t}{i}\right)=$$
$$=\sum_{i=0}^{p}\binom{s}{i} i!\left\{{p+1 \atop i+1}\right\}\sum_{k=0}^{p}\binom{t}{k} k!\left\{{p+1 \atop k+1}\right\}=((s+1)(t+1))^{p}.$$

We are using here the identity

$$\sum_{i=0}^{p}\binom{s}{i} i!\left\{{p+1 \atop i+1}\right\}=(s+1)^{p},$$

which can be derived from the formula (see [8])

$$\left\{{p+1 \atop i+1}\right\}=\sum_{j=i}^{p}\binom{p}{j}\left\{{j \atop i}\right\}.$$

In fact, using the summation interchange formula $\sum_{i=0}^{n} a_i \sum_{j=i}^{n} b_j=\sum_{j=0}^{n} b_i \sum_{i=0}^{j} a_i$. we have

$$\sum_{i=0}^{p}\binom{s}{i} i!\left\{{p+1 \atop i+1}\right\}=\sum_{i=0}^{p}\binom{s}{i} i!\sum_{j=i}^{p}\binom{p}{j}\left\{{j \atop i}\right\}=$$
$$=\sum_{j=0}^{p}\binom{p}{j}\sum_{i=0}^{j}\binom{s}{i} i!\left\{{j \atop i}\right\}=\sum_{j=0}^{p}\binom{p}{j} s^{j}=(s+1)^{p}.$$

Thus

$$\sum_{i=0}^{n-2}\sum_{k=i}^{n-2} i!k!\left\{{p+1 \atop i+1}\right\}\left\{{p+1 \atop k+1}\right\} S_{i,k}(G)=\sum_{s=0}^{p}\sum_{t=s}^{p}((s+1)\,(t+1))^{p} f_{s,t}=M_2^{(p)}(G),$$

as required. □

Example 2. For any simple graph G we have

$$\begin{aligned}
&M_2^{(0)}(G)=S_{0,0}(G),\\
&M_2^{(1)}(G)=S_{0,0}(G)+S_{0,1}(G)+S_{1,1}(G),\\
&M_2^{(2)}(G)=S_{0,0}(G)+3\,S_{0,1}(G)+2\,S_{0,2}(G)+9\,S_{1,1}(G)+6\,S_{1,2}(G)+4\,S_{2,2}(G),\\
&M_2^{(3)}(G)=S_{0,0}(G)+7\,S_{0,1}(G)+12\,S_{0,2}(G)+6\,S_{0,3}(G)+49\,S_{1,1}(G)+84\,S_{1,2}(G)\\
&42\,S_{1,3}(G)+144\,S_{2,2}(G)+72\,S_{2,3}(G)+36\,S_{3,3}(G).
\end{aligned}$$

Similar formulas for triangle free graphs are derived in [2].

4. Recurrence relations for the generalized second Zagreb indices. The following generalization of the Srirling numbers of the first kind is well known, see [6], [7]. Let S be an arbitrary set of natural numbers of cardinality $|S|$. Then the *Comtet numbers of the first kind* $\left[{S \atop i}\right]$ associated with the set S are defined by

$$(z)_S = \prod_{s\in S}(z-s) = \sum_{i=0}^{|S|} \begin{bmatrix} S \\ i \end{bmatrix} z^i,$$

here $(z)_S$ is the generalized Pochhamer symbols.

The Comtet numbers of the first kind associated with the set $S = \{0, 1, \cdots, n-1\}$ coincides with the usual Stirling numbers of the first kind.

Example 3. Denote by C_n the set $C_n = \{i \cdot j \mid 0 \le i, j \le n\}$. For $n = 4$ we have $C_n = \{0, 1, 2, 3, 4, 6, 8, 9, 12, 16\}$, and

$$\begin{aligned}(z)_{C_4} &= z\,(z-1)\,(z-2)\,(z-3)\,(z-4)\,(z-6)\,(z-8)\,(z-9)\,(z-12)\,(z-16) = \\ &= z^{10} - 61\,z^9 + 1555\,z^8 - 21655\,z^7 + 180628\,z^6 - 929908\,z^5 + 2932320\,z^4 - \\ &\quad -5411520\,z^3 + 5239296\,z^2 - 1990656\,z.\end{aligned}$$

Thus, the corresponding Comtet numbers of the first kind are as follows

$$\begin{bmatrix} C_4 \\ 1 \end{bmatrix} = -1990656, \begin{bmatrix} C_4 \\ 2 \end{bmatrix} = 5239296, \begin{bmatrix} C_4 \\ 3 \end{bmatrix} = -5411520, \begin{bmatrix} C_4 \\ 4 \end{bmatrix} = 2932320,$$
$$\begin{bmatrix} C_4 \\ 5 \end{bmatrix} = -929908, \begin{bmatrix} C_4 \\ 6 \end{bmatrix} = 180628, \begin{bmatrix} C_4 \\ 7 \end{bmatrix} = -21655, \begin{bmatrix} C_4 \\ 8 \end{bmatrix} = 1555,$$
$$\begin{bmatrix} C_4 \\ 9 \end{bmatrix} = -61, \begin{bmatrix} C_4 \\ 10 \end{bmatrix} = 1, \begin{bmatrix} C_4 \\ 0 \end{bmatrix} = 0.$$

Let $\mathcal{G}(M_2, t) = \sum_{p=0}^{\infty} M_2^{(p)}(G)t^p$, be the ordinary generating functions of the sequence of the second general Zagreb indices. Let us express the generating functions $\mathcal{G}(M_2, t)$ in terms of the double star sequences. The following theorem holds.

Theorem 3. *Let $\mathcal{G}(M_2, t)$ be the ordinary generating functions of the sequence of the second general Zagreb indices. Then*

$$(i)\;\mathcal{G}(M_2, t) = \frac{\displaystyle\sum_{k=0}^{|C_{n-1}|-1} \left(\sum_{i=\max(0,k-|C_{n-1}|)}^{\min(k,|C_{n-1}|)} \begin{bmatrix} C_{n-1} \\ |C_{n-1}| - i \end{bmatrix} M_2^{(k-i)}(G) \right) t^k}{\displaystyle\prod_{c\in C_{n-1}} (1-ct)},$$

$$(ii)\; M_2^{(|C_{n-1}|)}(G) = -\sum_{i=1}^{|C_{n-1}|-1} \begin{bmatrix} C_{n-1} \\ i \end{bmatrix} M_2^{(i)}(G),$$

where $\left[{S \atop i}\right]$ is the Comtet numbers of the first kind associated with the set S.

Proof. We have

$$\mathcal{G}(M_2, t) = \sum_{p=0}^{\infty} M_2^{(p)}(G)t^p = \sum_{p=0}^{\infty} \left(\sum_{i=1}^{n-2} \sum_{k=i}^{n-2} i!k! \left\{ {p+1 \atop i+1} \right\} \left\{ {p+1 \atop k+1} \right\} S_{i,k}(G) \right) t^p =$$

$$= \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \left(\sum_{p=0}^{\infty} i!k! \left\{ {p+1 \atop i+1} \right\} \left\{ {p+1 \atop k+1} \right\} t^p \right) S_{i,k}(G) =$$

$$= \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \left(\sum_{p=i}^{\infty} i!k! \left\{ {p+1 \atop i+1} \right\} \left\{ {p+1 \atop k+1} \right\} t^p \right) S_{i,k}(G).$$

Since $\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^n$, we get

$$\sum_{p=i}^{\infty} i!k! \left\{ {p+1 \atop i+1} \right\} \left\{ {p+1 \atop k+1} \right\} t^p =$$

$$\sum_{p=i}^{\infty} \left(\sum_{r=0}^{i+1} \frac{(-1)^{i+1-r}}{i+1} \binom{i+1}{r} r^{p+1} \sum_{j=0}^{k+1} \frac{(-1)^{k+1-j}}{k+1} \binom{k+1}{j} j^{p+1} \right) t^p =$$

$$= \frac{1}{(i+1)(k+1)} \left(\sum_{r=0}^{i+1} (-1)^{i+1-r} r \binom{i+1}{r} \sum_{j=0}^{k+1} (-1)^{k+1-j} j \binom{k+1}{j} \right) \sum_{p=i}^{\infty} (rjt)^p =$$

$$= \frac{1}{(i+1)(k+1)} \sum_{r=0}^{i+1} \sum_{j=0}^{k+1} (-1)^{i+1-r} (-1)^{k+1-j} rj \binom{i+1}{r} \binom{k+1}{j} \frac{(rjt)^i}{1-(rj)t}.$$

It follows that the partial fraction decomposition of the generating function $\mathcal{G}(M_2, t)$ is a linear combination of fractions of the form $\frac{1}{1-(rj)t}$, $0 \le r, j \le n-1$. After simplification we get

$$\mathcal{G}(M_2, t) = \frac{a_0 + a_1 t + a_2 t^2 + \cdots + a_{|C_{n-1}|-1} t^{|C_{n-1}|-1}}{\prod_{c \in C_{n-1}} (1 - ct)}$$

for some unknown numbers $a_0, a_1, \ldots, a_{n-1}$. To define these numbers let us observe that

$$\prod_{c \in C_{n-1}} (1 - c\,t) = t^{|C_{n-1}|} \prod_{c \in C_{n-1}} \left(\frac{1}{t} - c \right) = t^{|C_{n-1}|} \left(\frac{1}{t} \right)_{C_{n-1}} = \sum_{i=0}^{C_{n-1}} \left[{C_{n-1} \atop i} \right] t^{|C_{n-1}|-i}.$$

Now

$$a_0 + a_1 t + a_2 t^2 + \cdots + a_{|C_{n-1}|-1} t^{|C_{n-1}|-1} = \left(\sum_{i=0}^{|C_{n-1}|} \left[{C_{n-1} \atop i} \right] t^{|C_{n-1}|-i} \right) \left(\sum_{p=0}^{\infty} M_2^{(p)}(G) t^p \right) =$$

$$= \sum_{p=0}^{\infty} \left(\sum_{i=\max(0,p-|C_{n-1}|)}^{\min(p,|C_{n-1}|)} \left[{C_{n-1} \atop |C_{n-1}|-i} \right] M_2^{(p-i)}(G) \right) t^p.$$

Equating coefficients of t^p yelds $a_p = \sum_{i=\max(0,p-|C_{n-1}|)}^{\min(p,|C_{n-1}|)} \left[{C_{n-1} \atop |C_{n-1}|-i} \right] M_2^{(p-i)}(G)$. Therefore

$$\mathcal{G}(M_2, t) = \frac{\sum_{p=0}^{|C_{n-1}|-1} \left(\sum_{i=\max(0,p-|C_{n-1}|)}^{\min(p,|C_{n-1}|)} \left[{C_{n-1} \atop |C_{n-1}|-i} \right] M_2^{(p-i)}(G) \right) t^p}{\prod_{c \in C_{n-1}} (1 - ct)}.$$

(ii) Since $a_p = 0$ for $p \geq |C_{n-1}|$ we have the identity

$$a_{|C_{n-1}|} = \sum_{i=\max(0,p-|C_{n-1}|)}^{\min(p,|C_{n-1}|)} \begin{bmatrix} C_{n-1} \\ |C_{n-1}| - i \end{bmatrix} M_2^{(k-i)}(G) = 0,$$

or $\displaystyle\sum_{i=0}^{|C_{n-1}|} \begin{bmatrix} C_{n-1} \\ |C_{n-1}| - i \end{bmatrix} M_2^{(|C_{n-1}|-i)}(G) = \sum_{i=0}^{|C_{n-1}|} \begin{bmatrix} C_{n-1} \\ i \end{bmatrix} M_2^{(i)}(G) = 0$. Tacking into account $\begin{bmatrix} C_{n-1} \\ 0 \end{bmatrix} = 0, \begin{bmatrix} C_{n-1} \\ |C_{n-1}| \end{bmatrix} = 1$ we can rewrite the last expression in the form

$$M_2^{(|C_{n-1}|)}(G) = -\sum_{i=1}^{|C_{n-1}|-1} \begin{bmatrix} C_{n-1} \\ i \end{bmatrix} M_2^{(i)}(G).$$

□

Example 4. For $n = 4$ we have $C_3 = \{0, 1, 2, 3, 4, 6, 9\}$. Then

$$(z)_{C_3} = z\,(z-1)\,(z-2)\,(z-3)\,(z-4)\,(z-6)\,(z-9)$$
$$= z^7 - 25\,z^6 + 239\,z^5 - 1115\,z^4 + 2664\,z^3 - 3060\,z^2 + 1296\,z.$$

The Comtet numbers of the first kind are as follows:

$$\begin{bmatrix} C_3 \\ 0 \end{bmatrix} = 0, \begin{bmatrix} C_3 \\ 1 \end{bmatrix} = 1296, \begin{bmatrix} C_3 \\ 2 \end{bmatrix} = -3060, \begin{bmatrix} C_3 \\ 3 \end{bmatrix} = 2664,$$
$$\begin{bmatrix} C_3 \\ 4 \end{bmatrix} = -1115, \begin{bmatrix} C_3 \\ 5 \end{bmatrix} = 239, \begin{bmatrix} C_3 \\ 6 \end{bmatrix} = -25, \begin{bmatrix} C_3 \\ 7 \end{bmatrix} = 1.$$

Then for a simple graph G with 4 vertices the following recurrence relation holds

$$M_2^{(7)}(G) = -1296\,M_2^{(1)}(G) + 3060\,M_2^{(2)}(G) - 2664M_2^{(3)}(G) + 1115M_2^{(4)}(G) -$$
$$-239M_2^{(5)}(G) + 25M_2^{(6)}(G).$$

REFERENCES

1. B. Bollobas, P. Erdős, *Graphs of extremal weights*, Ars Comb. **50** (1998), 225–233.
2. G. Xavier, E. Suresh, I. Gutman, *Counting relations for general Zagreb indices*, Kragujevac Journal of Mathematics, **38** (2014), №1, 95–103.
3. L. Bedratyuk, O. Savenko, *The star sequence and the general first Zagreb index*, MATCH Communications in Mathematical and in Computer Chemistry, **79** (2018), №2, 407–414.
4. J. Riordan, Combinatorial Identities, New York, Wiley, 1979.
5. T. Doslic, B. Furtula, A. Graovac, I. Gutman, S. Moradi, Z. Yarahmadi, *On vertex-degree-based molecular structure descriptors*, MATCH Commun. Math. Comput. Chem., **66** (2011), 613–626.
6. L. Comtet, *Nombres de Stirling généraux et fonctions symetriques*, C. R. Acad. Sci. Paris. Ser. A, **275** (1972), 747—750.
7. T. Mansour, M. Schork, Commutation Relations, Normal Ordering, and Stirling Numbers, CRC Press, 2015.
8. R. Graham, D. Knuth, O. Patashnik, Concrete Mathematics, Addison-Wesley, Reading, 1989.

Khmelnytsky National University
leonid.uk@gmail.com

Received 01.03.2019