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ON INTERPOLATION PROBLEM WITH DERIVATIVE IN A SPACE OF ENTIRE FUNCTIONS WITH FAST-GROWING INTERPOLATION KNOTS

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We obtaine the conditions on a sequence $(b_{k,1}; b_{k,2})$, $k \in \mathbb{N}$, such that the interpolation problem $g(\lambda_k) = b_{k,1}$, $g'(\lambda_k) = b_{k,2}$ has the unique solution in a subspace of entire functions g satisfying the condition $\ln M_g(r) \le c_1 \exp(N(r) + N(\rho_1 r))$, where $|\lambda_k/\lambda_{k+1}| \le \Delta < 1$, and $N(r)$ is the Nevanlinna counting function of the sequence (λ_k). These results have been applied to describe solutions of the differential equation $f'' + a_0 f = 0$ with a coefficient a_0 from the some space of entire functions.

1. Introduction. Let (λ_k), $\lambda_k \ne 0$, $k \in \mathbb{N}$, be a sequence of distinct complex numbers without finite limit points. For (λ_k), the Nevanlinna counting function is defined by $N(r) = \int_0^r \frac{n(t)}{t} dt$, where $n(r)$ is equal to the number of points of the sequence in the disk $|z| < r$. We note that $N(r) = \sum_{|\lambda_k| \le r} \log \frac{r}{|\lambda_k|}$. To characterize the growth of an entire function $g(z)$ we introduce the function $M_g(r) = \max\{|g(z)| \colon |z| = r\}$.

Various interpolation problems in spaces of entire functions were investigated in papers of many authors (see [1–15]). A. Gel'fond ([1]) and Y. Kaz'min ([4]) considered the interpolation problem $g(\lambda_k) = b_k$, $k \in \mathbb{N}$, where $\lambda_k = \beta^{k-1}$, $|\beta| > 1$. From their results the theorems follow.

Theorem H (Hel'fond). *The unique entire function such that*

$$g(\beta^{k-1}) = 0, \quad k \in \mathbb{N},$$

$$\varliminf_{r \to \infty} \left(\ln^+ M_g(r) - \frac{\ln^2 r}{2 \ln |\beta|} - \frac{\ln r}{2} \right) = -\infty$$

is $g \equiv 0$.

Theorem K (Kaz'min). *Let* $\lambda_k = \beta^{k-1}, |\beta| > 1$. *Then for every sequence* (b_k) *such that*

$$\varlimsup_{k \to \infty} |\beta|^{-\frac{(k-1)}{2}} |b_k|^{1/k} \le r_1, \quad r_1 \in (1/|\beta|; 1),$$

interpolation problem $g(\lambda_k) = b_k$ *has the unique solution in the space of entire functions* g, *that satisfy the condition*

$$\ln M_g(r) \le \frac{\ln^2 \rho_1 r}{2 \ln |\beta|} + \frac{\ln r}{2} + c_2$$

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for each $\rho_1 > r_1$.

The aim of this paper is to investigate a solution of the interpolation problem

$$g(\lambda_k) = b_{k,1}, \quad g'(\lambda_k) = b_{k,2}, \quad k \in \mathbb{R}, \tag{1}$$

in the case for some $\Delta \in (0;1)$ the sequence (λ_k) satisfies the condition

$$|\lambda_k/\lambda_{k+1}| \le \Delta, \quad k \in \mathbb{R}. \tag{2}$$

Let

$$L(z) = \prod_{j=1}^{\infty} \left(1 - \tfrac{z}{\lambda_j}\right), \quad z \in \mathbb{C}.$$

Note that condition (2) ensures the convergence of $L(z)$.

Here we directly select classes of solution existence by characteristics of the sequence (λ_k). It is important to investigate some bases (see [10], [15]). Earlier interpolation problems in similar terms were considered in papers by I. I. Ibragimov and his disciples (e.g., see [16]). Our main result is the following.

Theorem 1. *Let (λ_k) be a sequence of complex numbers satisfying condition (2) for some $\Delta < 1$. Then for every sequences $(b_{k,1})$ and $(b_{k,2})$ such that for some $q \in (\Delta; 1)$*

$$|b_{k,1}| \le c_1 \exp\left(2N\left(q|\lambda_k|\right)\right), \ |\lambda_k||b_{k,2}| \le c_5 \exp\left(N\left(|\lambda_k|\right) + N\left(q|\lambda_k|\right)\right), \ k \in \mathbb{N}, \tag{3}$$

(henceforth, c_i is a positive constant) interpolation problem (1) has the unique solution in the space of entire functions g satisfying the condition

$$M_g(r) \le c_3 \exp\left(N(r) + N(\rho_1 r)\right) \tag{4}$$

for each $\rho_1 \in (q;1)$. The following function

$$g(z) = \sum_{k=1}^{\infty} \left(-\frac{L''(\lambda_k) b_{k,1}}{L'^3(\lambda_k)} \frac{L^2(z)}{z-\lambda_k} + \frac{b_{k,2}}{L'^2(\lambda_k)} \frac{L^2(z)}{z-\lambda_k} + \frac{b_{k,1}}{L'^2(\lambda_k)} \frac{L^2(z)}{(z-\lambda_k)^2} \right) \tag{5}$$

is its solution.

This theorem can be applied to study zeros of solutions of the differential equation $f'' + a_0 f = 0$. It is considered in Section 3.

2. Proof of Theorem 1. First, we shall prove the uniqueness of the solution. On the contrary, we assume that for the sequences $(b_{k,1})$ and $(b_{k,2})$ with properties (3) there exist two different entire functions $g = f_1$ and $g = f_2$, satisfying (4) and solving problem (1). Then the function $f = f_2 - f_1$ has zeros of order $m_k \ge 2$ in all points λ_k and satisfies the condition (4) for some $\rho_1 < 1$. Therefore, from the Jensen inequality we obtain $N(r) \le N(\rho_1 r) + c_0$. But it contradicts to the assumption of the theorem, because $N(r) - N(\rho_1 r) \to +\infty$, if $\rho_1 < 1$ and $r \to +\infty$. Thus, the uniqueness is proved.

Next, for given $F(z) = L^2(z)$ we denote

$$Q_k(z) = \sum_{m=1}^{2} A_{km} \left(\frac{1}{z-\lambda_k}\right)^{(m-1)}, \quad \gamma_{k,j} = \left(\frac{(z-\lambda_k)^2}{F(z)}\right)^{(j)}\Bigg|_{z=\lambda_k}, \quad j = 1,\ 2,$$

$$A_{k,m} = \frac{(-1)^{m-1}}{(p_m - m)!(m-1)!} \sum_{l=0}^{2-m} C^l_{2-m} \gamma_{k,2-m+1-l} b_{k,l+1}.$$

Then the series (see, for example, [4])

$$g(z) = F(z) \sum_{k=1}^{\infty} Q_k(z)$$

satisfies condition (1). Let us show that g also satisfies condition (4). One should observe that $\gamma_{k,1} = \frac{1}{L'^2(\lambda_k)}$, $\gamma_{k,2} = -\frac{L''(\lambda_k)}{L'^3(\lambda_k)}$ and therefore g admits representation (5). Thus,

$$|g(z)| \le \sum_{k=1}^{\infty} \left| \frac{b_{k,2}}{L'^2(\lambda_k)} \frac{L^2(z)}{z-\lambda_k} \right| + \left| \frac{b_{k,1}}{L'^2(\lambda_k)} \frac{L^2(z)}{(z-\lambda_k)^2} \right| + \left| \frac{L''(\lambda_k)}{L'(\lambda_k)} \frac{b_{k,1}}{L'^2(\lambda_k)} \frac{L^2(z)}{z-\lambda_k} \right| = \sum_1 + \sum_2 + \sum_3 .$$

Since

$$\exp\left(N(\lambda_k)\right) = \frac{|\lambda_k|^k}{\prod\limits_{n=1}^{k} |\lambda_n|}, \quad \exp\left(N\left(q|\lambda_k|\right)\right) = \frac{\left(q|\lambda_k|\right)^{k-1}}{\prod\limits_{n=1}^{k-1} |\lambda_n|} = q^{k-1} \frac{|\lambda_k|^k}{\prod\limits_{n=1}^{k} |\lambda_n|}$$

for any $q \in (\Delta; 1)$, the relationships

$$\ln M_L(r) = N(r) + O(1), \quad r \in [0; +\infty), \tag{6}$$

$$\ln |\lambda_k L'(\lambda_k)| = N\left(|\lambda_k|\right) + O(1), \quad k \in \mathbb{R}, \tag{7}$$

$$\max\left\{ \left| \frac{L(z)}{z-\lambda_k} \right| : |z| \le r \right\} \le c(\varepsilon) \frac{M_L((1+\varepsilon)r)}{r + |\lambda_k|}, \ k \in \mathbb{R}, \ r \in [0; +\infty), \ \varepsilon > 0, \tag{8}$$

hold for some function

$$L(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{\lambda_j}\right)$$

see [14, 15]. Using (3), (6)–(8) and denoting $\rho_2 := (1+\varepsilon)$, we obtain

$$\left| \frac{b_{k,2}}{L'^2(\lambda_k)} \right| \left| \frac{L^2(z)}{z-\lambda_k} \right| \le c_4 \exp\left(N(r) + N(\rho_2 r)\right) \frac{|\lambda_k| \exp\left(N(|\lambda_k|) + N\left(q|\lambda_k|\right)\right)}{\exp\left(2N\left(|\lambda_k|\right)\right)(r + |\lambda_k|)} =$$

$$= c_4 \exp\left(N(r) + N(\rho_2 r)\right) \frac{|\lambda_k| q^{k-1}}{(r + |\lambda_k|)},$$

if $|z| \le r$ and $k \in \mathbb{N}$.

Further, we suppose $|\lambda_m| \le r < |\lambda_{m+1}|$, $|z| \le r$ and $\rho_1 = \rho_2 q$. Then $m = n(r)$ and

$$\sum_{k=1}^{\infty} \left| \frac{b_{k,2} L^2(z)}{(z-\lambda_k)\, L'^2(\lambda_k)} \right| \le c_4 \exp\left(N(r)\right) \exp\left(N(\rho_2 r)\right) \sum_{k=1}^{\infty} \frac{|\lambda_k| q^{k-1}}{(r + |\lambda_k|)} \le$$

$$\le c_4 \exp\left(N(r)\right) \exp\left(N(\rho_2 r)\right) \left(\sum_{k=1}^{m} q^{k-1} \frac{|\lambda_k|}{r} + \sum_{k=m+1}^{\infty} q^{k-1} \right) \le$$

$$\le c_4 \exp\left(N(r)\right) \exp\left(N(\rho_2 r)\right) \left(\sum_{k=1}^{m} q^{k-1} \frac{|\lambda_k|}{|\lambda_m|} + \sum_{k=m+1}^{\infty} q^{k-1} \right) \le$$

$$\leq c_5 \exp\left(N(r)\right) \exp\left(N(\rho_2 r)\right) \left(\frac{\Delta^m}{q} \sum_{n=1}^{m} \left(\frac{q}{\Delta}\right)^n + \frac{q^m}{1-q} \right) \leq$$

$$\leq c_5 \exp\left(N(r)\right) \exp\left(N(\rho_2 r)\right) \left(\frac{\Delta^m}{q} \frac{q}{\Delta} \frac{\left(\frac{q}{\Delta}\right)^m - 1}{\left(\frac{q}{\Delta}\right) - 1} + \frac{q^m}{1-q} \right) \leq$$

$$\leq c_5 \exp\left(N(r)\right) \exp\left(N(\rho_2 r)\right) \left(\frac{q^m}{q-\Delta} + \frac{q^m}{1-q} \right) \leq$$

$$\leq c_5 \exp\left(N(r)\right) \exp\left(N(\rho_2 r)\right) q^m = c_5 \exp\left(N(r)\right) \exp\left(N(\rho_2 r) - m \ln \frac{1}{q} \right).$$

Moreover, there exists a constant $c_1 \geq 0$ such that for each $\rho_2 \geq 1$ with $\rho_2 \Delta \leq 1$ (and for every ρ_1, $0 < \rho_1 \leq \rho_2$), that inequality $n(\rho_2 r) \leq n(r) + c_1$ is fulfilled.

Indeed, assume that $|\lambda_m| \leq r < |\lambda_{m+1}|$ and $1/\Delta^{s-1} \leq \rho_2 < 1/\Delta^s$ for some $s \in \mathbb{R}$. Then $n(r) = m$ and

$$\rho_2 r < \rho_2 |\lambda_{m+1}| \leq \frac{1}{\Delta^s} |\lambda_{m+1}| = |\lambda_{m+s+1}| \frac{1}{\Delta^s} \frac{|\lambda_{m+1}|}{|\lambda_{m+s+1}|} \leq |\lambda_{m+s+1}|.$$

Hence, $n(\rho_2 r) \leq m + s + 1 = n(r) + s + 1$ and

$$N(\rho_2 r) - N(\rho_1 r) = \int_{\rho_1 r}^{\rho_2 r} \frac{n(t)}{t} dt \leq n(\rho_2 r) \log \frac{\rho_2}{\rho_1} = (s+1) \log \frac{\rho_2}{\rho_1} + n(r) \log \frac{\rho_2}{\rho_1}.$$

Thus, $N(\rho_2 r) \leq N(\rho_1 r) + c_1 \log \frac{\rho_2}{\rho_1} + n(r) \log \frac{\rho_2}{\rho_1}$. Hence,

$$\sum_1 := \sum_{k=1}^{\infty} \left| \frac{b_{k,2} L^2(z)}{(z - \lambda_k) L'^2(\lambda_k)} \right| \leq c_9 \exp\left(N(\rho_1 r)\right) \exp\left(N(r)\right), \quad |z| \leq r.$$

Similarly,

$$\left| \frac{b_{k,1}}{L'^2(\lambda_k)} \frac{L^2(z)}{(z-\lambda_k)^2} \right| \leq c_4 \exp\left(2N(\rho_2 r)\right) \frac{|\lambda_k|^2 \exp\left(2N\left(q|\lambda_k|\right)\right)}{\exp\left(2N\left(|\lambda_k|\right)\right) (r + |\lambda_k|)^2} \leq$$

$$\leq c_4 \exp\left(2N(\rho_2 r)\right) \frac{|\lambda_k|^2 \exp\left(2N\left(q|\lambda_k|\right)\right)}{\exp\left(2N\left(|\lambda_k|\right)\right) (r + |\lambda_k|)^2} \leq c_4 \exp\left(2N(\rho_2 r)\right) \frac{|\lambda_k|^2 q^{2(k-1)}}{(r + |\lambda_k|)^2},$$

$$\sum_2 := \sum_{k=1}^{\infty} \left| \frac{b_{k,1} L^2(z)}{(z-\lambda_k)^2 L'^2(\lambda_k)} \right| \leq c_4 \exp\left(2N(\rho_2 r)\right) \sum_{k=1}^{\infty} \frac{|\lambda_k|^2 q^{2(k-1)}}{(r + |\lambda_k|)^2} \leq$$

$$\leq c_4 \exp\left(2N(\rho_2 r)\right) \left(\sum_{k=1}^{m} q^{2(k-1)} \left(\frac{|\lambda_k|}{r}\right)^2 + \sum_{k=m+1}^{\infty} q^{2(k-1)} \right) \leq$$

$$\leq c_4 \exp\left(2N(\rho_2 r)\right) \left(\sum_{k=1}^{m} \left(\Delta^{m-k} q^{k-1}\right)^2 + \sum_{k=m+1}^{\infty} \left(q^2\right)^{k-1} \right) \leq$$

$$\leq c_4 \exp\left(N(\rho_2 r)\right) \left(\frac{\Delta^{2m}}{q^2} \sum_{k=1}^{m} \left(\frac{q}{\Delta}\right)^{2k} + \frac{q^{2m}}{1-q^2} \right) \leq$$

$$\leq c_5 \exp\left(2N(\rho_2 r)\right) \left(\frac{\Delta^{2m}}{q^2} \frac{q^2}{\Delta^2} \frac{\left(\frac{q}{\Delta}\right)^{2m} - 1}{\left(\frac{q}{\Delta}\right)^2 - 1} + \frac{q^{2m}}{1-q^2} \right) \leq$$

$$\leq c_4 \exp\left(2N(\rho_2 r)\right)\left(\frac{q^{2m}}{q^2-\Delta^2}+\frac{q^{2m}}{1-q^2}\right) = c_5 \exp\left(2N(\rho_2 r) - 2m\ln\frac{1}{q}\right) =$$
$$= c_5 \exp\left(2N(\rho_2 r) - 2n(r)\ln\frac{\rho_2}{\rho_1}\right) \leq c_6 \exp\left(2N(\rho_1 r)\right) \leq c_6 \exp\left(N(r)\right)\exp\left(N(\rho_1 r)\right), |z| \leq r.$$

Since (see [19], [20]),

$$\frac{L''(\lambda_k)}{L'(\lambda_k)} = -2\sum_{n=1,\ n\neq k}^{\infty}\frac{1}{\lambda_n-\lambda_k},$$

it follows that

$$\left|\frac{L''(\lambda_k)}{L'(\lambda_k)}\right| \leq \frac{c_1 k}{|\lambda_k|}.$$

Hence,

$$\left|-\frac{L''(\lambda_k)}{L'^3(\lambda_k)}b_{k,1}\frac{L^2(z)}{z-\lambda_k}\right| \leq c_5\frac{k}{|\lambda_k|}\exp\left(N(r)\right)\exp\left(N(\rho_2 r)\right)\frac{|\lambda_k|^2\exp\left(2N\left(q\,|\lambda_k|\right)\right)}{(r+|\lambda_k|)\exp\left(2N\left(|\lambda_k|\right)\right)} \leq$$
$$\leq c_5 \exp\left(N(r)\right)\exp\left(N(\rho_2 r)\right)\frac{k|\lambda_k|q^{2(k-1)}}{(r+|\lambda_k|)}.$$

Consequently,

$$\sum_3 := \sum_{k=1}^{\infty}\left|\frac{b_{k,1}L(z)}{(z-\lambda_k)\,L'(\lambda_k)}\right| \leq c_5 \exp\left(N(r)\right)\exp\left(N(\rho_2 r)\right)\sum_{k=1}^{\infty}\frac{k|\lambda_k|q^{2(k-1)}}{r+|\lambda_k|} \leq$$
$$\leq c_5 \exp\left(N(r)\right)\exp\left(N(\rho_2 r)\right)\sum_{k=1}^{\infty}\frac{|\lambda_k|q^{k-1}}{r+|\lambda_k|} \leq c_6 \exp\left(N(r)\right)\exp\left(N(\rho_1 r)\right),\ |z| \leq r.$$

From the previous considerations it follows that series (5) converges uniformly on the every disk $|z| \leq r$, $0 < r < +\infty$, and g is an entire function satisfying (4). The proof is complete.

Remark 1. If $\lambda_k = \beta^{k-1}$ then $N(r) = \frac{\ln^2 r}{2\ln\beta} + \frac{\ln r}{2} + O(1)$, $r \in [1;+\infty)$, (4) is equivalent to

$$\varlimsup_{r\to+\infty}\left(\ln M_g(r) - \frac{\ln^2 r}{2\ln\beta} - \frac{\ln^2(\rho_1 r)}{2\ln|\beta|} - \ln r\right) < +\infty,$$

and the condition $|b_{k,1}| \leq c_1 \exp\left(2N\left(q|\lambda_k|\right)\right)$ is equivalent to

$$\sup\left\{|\beta|^{-\frac{(k-1)}{2}}|b_{k,1}|^{1/2k} \colon k \in \mathbb{N}\right\} < +\infty.$$

Therefore, Theorem 1 contains (see also [13]) some result from [2].

It is not difficult (see [1, 2]) to prove the next assertion.

Remark 2. *If* (λ_k) *satisfies (2), then*

$$N(r) \leq \frac{\ln^2 r}{2\ln 1/\Delta} + \frac{\ln r}{2} + c_1, \quad r \in (1;+\infty).$$

Remark 3. If (λ_k) satisfies (2) and condition (4) is fulfilled then

$$M_g(z) \leq c_3 \exp\left(2N(\rho_3 r)\right) \tag{9}$$

for each $\rho_3 \in (\sqrt{\rho_1}; 1)$. Indeed, let $|\lambda_m| \le r < |\lambda_{m+1}|$ for some $m \in \mathbb{N}$. Then $|\lambda_{m-1}| \le \Delta|\lambda_m| < \rho_1|\lambda_m| \le \rho_1 r < \rho_1|\lambda_{m+1}| < |\lambda_{m+1}|$. So $|\lambda_{m-1}| < \rho_1 r < \rho_3 r < \rho_3|\lambda_{m+1}| < |\lambda_{m+1}|$. Thus, if 1) $|\lambda_m| \le \rho_1 r < \rho_3 r < |\lambda_{m+1}|$, then

$$\exp(N(r)) = \frac{r^m}{\prod\limits_{n=1}^{m}|\lambda_n|}, \quad \exp(N(\rho_1 r)) = \frac{(\rho_1 r)^m}{\prod\limits_{n=1}^{m}|\lambda_n|}, \quad \exp(N(\rho_3 r)) = \frac{(\rho_3 r)^m}{\prod\limits_{n=1}^{m}|\lambda_n|}$$

and

$$\frac{\exp(N(r)+N(\rho_1 r))}{\exp(2N(\rho_3 r))} = \frac{\frac{r^m(\rho_1 r)^m}{\prod_{n=1}^{m}|\lambda_n|\cdot\prod_{n=1}^{m}|\lambda_n|}}{\frac{(\rho_3 r)^{2m}}{\prod_{n=1}^{m}|\lambda_n|^2}} = \left(\frac{\rho_1}{\rho_3^2}\right)^m < 1.$$

2) If $|\lambda_{m-1}| \le \rho_1 r < |\lambda_m|$, $|\lambda_m| \le \rho_3 r < |\lambda_{m+1}|$, then

$$\exp(N(\rho_1 r)) = \frac{(\rho_1 r)^{m-1}}{\prod\limits_{n=1}^{m-1}|\lambda_n|}$$

and

$$\frac{\exp(N(r)+N(\rho_1 r))}{\exp(2N(\rho_3 r))} = \frac{\frac{r^m(\rho_1 r)^{m-1}}{\prod_{n=1}^{m}|\lambda_n|\cdot\prod_{n=1}^{m-1}|\lambda_n|}}{\frac{(\rho_3 r)^{2m}}{\prod_{n=1}^{m}|\lambda_n|^2}} = \left(\frac{\rho_1}{\rho_3^2}\right)^m \frac{|\lambda_m|}{\rho_1 r} = \left(\frac{\rho_1}{\rho_3^2}\right)^m \frac{1}{\Delta}\frac{\Delta}{\rho_1}\frac{|\lambda_m|}{r} \le \frac{1}{\Delta}.$$

3) If $|\lambda_{m-1}| \le \rho_1 r < \rho_3 r < |\lambda_m|$, then

$$\exp(N(\rho_1 r)) = \frac{(\rho_1 r)^{m-1}}{\prod\limits_{n=1}^{m-1}|\lambda_n|}, \exp(N(\rho_3 r)) = \frac{(\rho_3 r)^{m-1}}{\prod\limits_{n=1}^{m-1}|\lambda_n|}$$

and

$$\frac{\exp(N(r)+N(\rho_1 r))}{\exp(2N(\rho_3 r))} = \frac{\frac{r^m(\rho_1 r)^{m-1}}{\prod_{n=1}^{m}|\lambda_n|\cdot\prod_{n=1}^{m-1}|\lambda_n|}}{\frac{(\rho_3 r)^{2m-2}}{\prod_{n=1}^{m-1}|\lambda_n|^2}} = \left(\frac{\rho_1}{\rho_3^2}\right)^{m-1} \frac{r}{|\lambda_m|} \le \left(\frac{\rho_1}{\rho_3^2}\right)^{m-1} \frac{1}{\rho_1}\frac{\rho_1 r}{|\lambda_m|} \le \frac{1}{\rho_1}.$$

On the other hand, (2) and (9) yield validity condition (4) for any $\rho_1 \in (\rho_3^2; 1)$.

3. Application to differential equation. Interpolation theorems can be applied in the analytic theory of differential equations. Let us consider the equation

$$f'' + a_0 f = 0. \tag{10}$$

Problem. *Let (λ_k) be a given sequence of distinct complex numbers without finite limit points. Does there exist an entire function a_0 such that the differential equation (10) has the solution f with the zero-sequence (λ_k)?*

One of the first answer to this question was given by V. Šeda. He proved the next statement ([17], [18]).

Theorem S. *For every sequence (λ_k) of distinct complex numbers without finite limit points there exist the entire function a_0 such that differential equation (10) has an entire solution f with the zeros exactly at the points (λ_k).*

This result was developed in many papers [18–24]. S. Bank ([19]) proved the following theorem.

Theorem B. *Let $d > 1$ be a real number and let (λ_k) be a sequence of nonzero complex numbers satisfying $|\lambda_{k+1}| > d|\lambda_k|$, $k \in \mathbb{N}$. Then there exists an entire transcendental function a_0 of order zero, such that (10) has a solution f with the zero-sequence (λ_k).*

In that paper the solution was given in the form $f(z) = L(z)e^{g(z)}$. Then

$$a_0(z) = -\frac{f''(z)}{f(z)} = -\frac{L''(z) + 2L'(z)g'(z)}{L(z)} - \left(g^2(z) + g''(z)\right). \tag{11}$$

Later J. Gröhn, J. Heittokangas [20] supplemented Bank's result and obtained estimate for the growth of a_0:

$$\ln M_{a_0}(r) = O(\ln^2 r), \quad r \to \infty.$$

Theorem 1 implies the next assertion.

Theorem 2. *Let (λ_k) be a sequence of complex numbers satisfying condition (2) for some $\Delta < 1$ and sequences $(b_{k,1})$, $(b_{k,2})$ satisfy the conditions (3) for some $q \in (\Delta; 1)$. Then there exists an entire function a_0 such that (10) has a solution f with the zero-sequence (λ_k) and*

$$\ln M_f(r) \leq N(r) + c_3 \exp\left(N(r) + N(\rho_1 r)\right) \tag{12}$$

for each $\rho_1 \in (q; 1)$ and for all $r > 0$.

Proof. Let $g(z)$ be an entire function of form (5), that is a solution of (1) with $b_{k,1} \equiv 0$, $b_{k,2} = -\frac{L''(\lambda_k)}{2L'(\lambda_k)}$. So, if $f(z) = L(z)e^{g(z)}$, then a_0 of the form (11) is an entire function and by Theorem 1 it follows that $f(z)$ satisfies condition (12). □

Corollary 1. *Let (λ_k) be a sequence of complex numbers satisfying condition (2) for some $\Delta < 1$. Then there exists an entire function a_0 such that (10) possesses a solution f with the zero sequence (λ_k) and*

$$\ln M_f(r) \leq N(r) + c_3 \exp\left(2N(\rho_3 r)\right)$$

for each $\rho_3 \in (\sqrt{\rho_1}; 1)$.

Corollary 2. *If (λ_k) satisfies condition (2) for some $\Delta < 1$, then there exists an entire function a_0 such that (10) has a solution f with the zeros sequence (λ_k) and*

$$|a_0(z)| \leq c_2 \exp\left(4N(\rho_3 r)\right) \tag{13}$$

for each $R_1 \in (\rho_3; 1)$ and all $r > 0$.

The statement is true because

$$|a_0(z)| \leqslant \left|\frac{L''(z)}{L(z)}\right| + 2|g'(z)|\left|\frac{L'(z)}{L(z)}\right| + |g'(z)|^2 + |g''(z)|$$

and $M_{g'}(r) \le cM_g((1+\varepsilon)r)$, $\varepsilon > 0$. Next, we use (see [21, Col. 3]) the estimate

$$\left|\frac{L^{(k)}(z)}{L^{(j)}(z)}\right| \leqslant |z|^{\varepsilon_1(k-j)}, \quad k > j \ge 0,$$

which holds for some $\varepsilon_1 > 0$ and for all $z \in \mathbb{C}$ outside an exceptional set of disks of finite sum of radii. Then following [19, Lemma C], we get (13).

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