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WEAKLY WEIGHTED-SHARING AND UNIQUENESS OF HOMOGENEOUS DIFFERENTIAL POLYNOMIALS

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In 2006, S. Lin and W. Lin introduced the definition of weakly weighted-sharing of meromorphic functions which is between "CM" and "IM". In this paper, using the notion of weakly weighted-sharing, we study the uniqueness of nonconstant homogeneous differential polynomials $P[f]$ and $P[g]$ generated by meromorphic functions f and g, respectively. Our results generalize the results due to S. Lin and W. Lin, and H-Y. Xu and Y. Hu.

1. Introduction and main result. Let $\mathbb{C}$ denote the complex plane and let f be a nonconstant meromorphic function defined on $\mathbb{C}$. We assume that the reader is familiar with the standard definitions and notions used in the value distribution theory, such as $T(r,f), m(r,f), N(r,f)$ (see [2, 6, 8]). By $S(r,f)$ we denote any quantity satisfying the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set of r of finite linear measure. A meromorphic function a is called a *small function* with respect to f if either $a \equiv \infty$ or $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \in S(f)$ and $S(f)$ is a field over the set of complex numbers. For $a \in \mathbb{C} \cup \{\infty\}$ the quantities

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)}, \quad \Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

are called the deficiency and ramification index of a for the function f, respectively.

For any two nonconstant meromorphic functions f and g, and $a \in S(f) \cap S(g)$, we say that f and g share a IM (CM) provided that $f-a$ and $g-a$ have the same zeros ignoring(counting) multiplicities. If $\frac{1}{f}$ and $\frac{1}{g}$ *share* 0 IM (CM), we say that f and g share ∞ IM (CM).

Definition 1. Let l be a nonnegative integer or infinity and $a \in S(f)$. We denote by $E_l(a,f)$ the set of all zeros of $f-a$, where a zero of multiplicity m is counted m times if $m \le l$ and $l+1$ times if $m > l$. If $E_l(a,f) = E_l(a,g)$, we say that $f,\ g$ share the function a with weight l. We write f and g share (a,l) to mean that f and g share the function a with weight l. Since $E_l(a,f) = E_l(a,g)$ implies that $E_s(a,f) = E_s(a,g)$ for any integer $s\,(0 \le s < l)$, if $f,\ g$ share (a,l), then $f,\ g$ share (a,s), $(0 \le s < l)$. Moreover, we note that f and g share the function a IM or CM if and only if f and g share $(a,0)$ or (a,∞), respectively.

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Definition 2 ([4])**.** Let $N_E(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ with the same multiplicities and $N_0(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. We denote by $\overline{N}_E(r,a)$ and $\overline{N}_0(r,a)$ the reduced counting functions of f and g corresponding to the counting functions $N_E(r,a)$ and $N_0(r,a)$ respectively. If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_E(r,a)=S(r,f)+S(r,g),$$

then we say that f and g share a "CM". If

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_0(r,a)=S(r,f)+S(r,g),$$

then we say that f and g share a "IM".

Let k be a positive integer, and let f be a meromorphic function and $a\in S(f)$.
(i) $\overline{N}_{k)}(r,a;f)$ denotes the counting function of those a-points of f whose multiplicities are not greater than k, where each a-point is counted only once.
(ii) $\overline{N}_{(k}(r,a;f)$ denotes the counting function of those a-points of f whose multiplicities are not less than k, where each a-point is counted only once.
(iii) $N_k(r,a;f)$ denotes the counting function of those a-points of f, where an a-point of f with multiplicity m counted m times if $m\le k$ and k times if $m>k$.

We denote by $\delta_k(a,f)$ the quantity

$$\delta_k(a,f)=1-\limsup_{r\to\infty}\frac{N_k(r,a;f)}{T(r,f)}.$$

Clearly $\delta_k(a,f)\ge\delta(a,f)$.

Let f and g be two nonconstant meromorphic functions sharing a "IM", for $a\in S(f)\cap S(g)$, and a positive integer k or ∞.
(i) $\overline{N}^E_{k)}(r,a)$ denotes the counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than k, where each a-point is counted only once.
(ii) $\overline{N}^0_{(k}(r,a)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than k, where each a-point is counted only once.

Definition 3 ([4])**.** For $a\in S(f)\cap S(g)$, if l is a positive integer or ∞, and

$$\overline{N}_{l)}(r,a;f)+\overline{N}_{l)}(r,a;g)-2\overline{N}^E_{l)}(r,a)=S(r,f)+S(r,g),$$
$$\overline{N}_{(l+1}(r,a;f)+\overline{N}_{(l+1}(r,a;g)-2\overline{N}^0_{(l+1}(r,a)=S(r,f)+S(r,g),$$

or if $l=0$ and

$$\overline{N}(r,a;f)+\overline{N}(r,a;g)-2\overline{N}_0(r,a)=S(r,f)+S(r,g),$$

then we say f and g *weakly share* a *with weight* l. Here, we write f, g share "(a,l)" to mean that f, g weakly share a with weight l.

Obviously if f and g share "(a,l)", then f and g share "(a,s)" for any s $(0\le s<l)$. Also, we note that f and g share a "IM" or "CM" if and only if f and g share "$(a,0)$" or "(a,∞)", respectively.

Suppose F and G share 1 "IM". By $\overline{N}_L(r,1;F)$ we denote the counting function of the 1-points of F whose multiplicities are greater than 1-points of G. $\overline{N}_L(r,1;G)$ is defined similarly.

Definition 4. Let n (≥ 1) be a positive integer, p (≥ 0) be an integer and f be a nonconstant meromorphic function. An expression of the form

$$P[f] = \sum_{k=1}^{n} a_k \prod_{j=0}^{p} \left(f^{(j)}\right)^{l_{kj}}, \tag{1}$$

where $a_k \in S(f)$ for $k = 1, 2, \ldots, n$ and l_{kj} $(1 \leq k \leq n;\ 0 \leq j \leq p)$ are nonnegative integers and $d = \sum_{j=0}^{p} l_{kj}$ for $k = 1, 2, \ldots, n$, is called a *homogeneous differential polynomial of degree d generated by f*. Also we denote by Q the quantity

$$Q = \max_{1\leq k\leq n} \sum_{j=0}^{p} j l_{kj}.$$

In 2006 S. Lin and W. Lin ([4]) defined and used the concept of weakly-weighted sharing of functions to prove the uniqueness of a meromorphic function and its derivative for the first time. They proved the following theorems.

Theorem 1. *Let n (≥ 1) be a positive integer and k be a positive integer or ∞ satisfying $2 \leq k \leq \infty$. Let f be a nonconstant meromorphic function, $a \in S(f)$ such that $a \not\equiv 0, \infty$. If f and $f^{(n)}$ share "(a,k)" and*

$$4\Theta(\infty, f) + 2\delta_{2+n}(0, f) > 5,$$

then $f = f^{(n)}$.

Theorem 2. *Let n (≥ 1) be a positive integer and let f be a nonconstant meromorphic function, $a \in S(f)$ such that $a \not\equiv 0, \infty$. If f and $f^{(n)}$ share "$(a,1)$" and*

$$\left(\frac{n+9}{2}\right)\Theta(\infty, f) + \frac{5}{2}\delta_{2+n}(0, f) > \frac{n}{2} + 6,$$

then $f = f^{(n)}$.

Theorem 3. *Let n (≥ 1) be a positive integer and let f be a nonconstant meromorphic function, $a \in S(f)$ such that $a \not\equiv 0, \infty$. If f and $f^{(n)}$ share "$(a,0)$" and*

$$(7+2n)\Theta(\infty, f) + 5\delta_{2+n}(0, f) > 2n + 11,$$

then $f = f^{(n)}$.

Later in 2011, H-Y. Xu and Y. Hu ([5]) generalize Theorems 1.1–1.3 by proving the following theorems.

Theorem 4. *Let n (≥ 1) be a positive integer, let k be a positive integer or ∞ satisfying $2 \leq k \leq \infty$ and let f be a nonconstant meromorphic function, $a \in S(f)$ such that $a \not\equiv 0, \infty$. Suppose $L(f) = f^{(n)} + a_{n-1}f^{(n-1)} + \ldots + a_0 f$ where $a_j(\not\equiv 0, \infty) \in S(f)$ for $0 \leq j \leq n-1$. If f and $L(f)$ share "(a,k)" and*

$$4\Theta(\infty, f) + 2\delta_{2+n}(0, f) > 5,$$

then $f = L(f)$.

Theorem 5. *Let n (≥ 1) be a positive integer, let f be a nonconstant meromorphic function, $a \in S(f)$ such that $a \not\equiv 0, \infty$. Suppose $L(f)$ is defined as in Theorem 4. If f and $L(f)$ share "$(a, 1)$" and*

$$\left(\frac{7}{2} + n\right) \Theta(\infty, f) + \frac{3}{2}\delta_2(0, f) + \delta_{n+2}(0, f) > n + 5,$$

then $f = L(f)$.

Theorem 6. *Let n (≥ 1) be positive integer, let f be a nonconstant meromorphic function, $a \in S(f)$ such that $a \not\equiv 0, \infty$. Suppose $L(f)$ be defined as in Theorem 4. If f and $L(f)$ share "$(a, 0)$" and*

$$(6 + 2n)\Theta(\infty, f) + \delta_2(0, f) + 2\Theta(0, f) + 2\delta_{2+n}(0, f) > 2n + 10,$$

then $f = L(f)$.

Motivated by such uniqueness investigation, it is natural to consider the problem in a more general setting: *Let f and g be any two nonconstant meromorphic functions, $P[f]$ and $P[g]$ be nonconstant homogeneous differential polynomials of f and g respectively, and $a \in S(f) \cap S(g)$, $a \not\equiv 0, \infty$. If $P[f]$ and $P[g]$ share "(a, l)", then what will be the relation between $P[f]$ and $P[g]$?* In this paper we prove that under certain conditions either $P[f] = P[g]$ or $P[f] \cdot P[g] = a^2$.

Now, we state the main result of this paper.

Theorem 7. *Let f and g be two nonconstant meromorphic functions, a ($\not\equiv 0, \infty$) $\in S(f) \cap S(g)$. Suppose that $P[f]$ and $P[g]$, as defined by (1), are nonconstant. If $P[f]$ and $P[g]$ share "(a, l)" with one of the following conditions:*
(i) $l \geq 2$ and

$$\min\left\{2\delta(0, f) + \frac{Q+4}{d}\Theta(\infty, f), 2\delta(0, g) + \frac{Q+4}{d}\Theta(\infty, g)\right\} > \frac{Q+d+4}{d}, \tag{2}$$

(ii) $l = 1$ and

$$\min\left\{\frac{5}{2}\delta(0, f) + \frac{3Q+9}{2d}\Theta(\infty, f), \frac{5}{2}\delta(0, g) + \frac{3Q+9}{2d}\Theta(\infty, g)\right\} > \frac{3Q+3d+9}{2d}, \tag{3}$$

(iii) $l = 0$ and

$$\min\left\{5\delta(0, f) + \frac{4Q+7}{d}\Theta(\infty, f), 5\delta(0, g) + \frac{4Q+7}{d}\Theta(\infty, g)\right\} > \frac{4Q+4d+7}{d}, \tag{4}$$

then either $P[f] = P[g]$ or $P[f].P[g] = a^2$.

2. Lemmas. In this section we present some lemmas which we needed in the sequel.

Lemma 1 ([3]). *Let f be a nonconstant meromorphic function and $P[f]$ be defined by (1) then*

(i) $T(r, P) \leq dT(r, f) + Q\overline{N}(r, \infty; f) + S(r, f)$.

(ii) $N(r, 0; P) \leq T(r, P) - dT(r, f) + dN(r, 0; f) + S(r, f) \leq\leq Q\overline{N}(r, \infty; f) + dN(r, 0; f) + S(r, f)$.

Lemma 2 ([4]). *Let l be a nonnegative integer or infinity, F and G be nonconstant meromorphic functions, F and G share "$(1,l)$". Let*

$$H=\left(\frac{F^{(2)}}{F^{(1)}}-2\frac{F^{(1)}}{F-1}\right)-\left(\frac{G^{(2)}}{G^{(1)}}-2\frac{G^{(1)}}{G-1}\right).$$

If $H\not\equiv 0$, then
(i) If $2\le l\le\infty$, then

$$T(r,F)\le N_2(r,\infty;F)+N_2(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G)+S(r,F)+S(r,G).$$

(ii) If $l=1$, then

$$T(r,F)\le N_2(r,\infty;F)+N_2(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G)+\overline{N}_L(r,1;F)+S(r,F)+S(r,G).$$

(iii) If $l=0$, then

$$\begin{aligned}T(r,F)\le{}& N_2(r,\infty;F)+N_2(r,\infty;G)+N_2(r,0;F)+N_2(r,0;G)\\&+2\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+S(r,F)+S(r,G).\end{aligned}$$

The same inequality holds for $T(r,G)$.

Lemma 3 ([5]). *Let F and G be nonconstant meromorphic functions such that F and G share "$(1,1)$". Then*

$$\overline{N}_L(r,1;F)\le\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,\infty;F)+S(r,F).$$

Lemma 4 ([5]). *Let F and G be nonconstant meromorphic functions such that F and G share "$(1,0)$". Then*

$$\overline{N}_L(r,1;F)\le\overline{N}(r,\infty;F)+\overline{N}(r,0;F)+S(r,F).$$

Lemma 5 ([7]). *Let f be a nonconstant meromorphic function and let*

$$p(f)=a_nf^n+a_{n-1}f^{n-1}+...+a_1f+a_0,$$

where $a_i\in S(f)$ for $i=0,\ 1\ldots\ n$, $a_n\ne 0$ be a polynomial of degree n. Then $T(r,p(f))=nT(r,f)+S(r,f)$.

3. Proof of the main Theorem.

Proof of Theorem 7. Let

$$F=\frac{P[f]}{a},\ G=\frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share "(a,l)", it follows that F, G share "$(1,l)$" except at the zeros and poles of a.

Let H be same as in Lemma 2. Suppose that $H\not\equiv 0$. Now we consider the following three cases:

Case 1: $2\le l\le\infty$.

From (i) of Lemma 2, we have

$$T(r,F) \le 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + N(r,0;F) + N(r,0;G) + S(r,F) + S(r,G).$$

Using Lemma 1 in the above inequality, we obtain

$$T(r,F) \le 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + T(r,F) - dT(r,f) + dN(r,0;f) \\ +Q\overline{N}(r,\infty;g) + dN(r,0;g) + S(r,f) + S(r,g)$$

and so

$$dT(r,f) \le 2\overline{N}(r,\infty;f) + dN(r,0;f) + (2+Q)\overline{N}(r,\infty;g) + \\ +dN(r,0;g) + S(r,f) + S(r,g). \tag{5}$$

Similarly,

$$dT(r,g) \le 2\overline{N}(r,\infty;g) + dN(r,0;g) + (2+Q)\overline{N}(r,\infty;f) + \\ +dN(r,0;f) + S(r,f) + S(r,g). \tag{6}$$

Adding (5) and (6), we subsequently get

$$T(r,f) + T(r,g) \le 2N(r,0;f) + \frac{Q+4}{d}\overline{N}(r,\infty;f) + 2N(r,0;g) + \\ +\frac{Q+4}{d}\overline{N}(r,\infty;g) + S(r,f) + S(r,g), \\ \left\{2\delta(0,f) + \frac{Q+4}{d}\Theta(\infty,f) - \frac{Q+d+4}{d}\right\} T(r,f) + \\ +\left\{2\delta(0,g) + \frac{Q+4}{d}\Theta(\infty,g) - \frac{Q+d+4}{d}\right\} T(r,g) \le S(r,f) + S(r,g),$$

which contradicts our hypothesis (2). Therefore $H \equiv 0$ and integrating twice we get

$$\frac{1}{G-1} = \frac{A}{F-1} + B,$$

where A ($\neq 0$) and B are constants. This gives

$$G = \frac{(B+1)F + (A-B-1)}{BF + (A-B)} \tag{7}$$

and

$$F = \frac{(B-A)G + (A-B-1)}{BG - (B+1)}. \tag{8}$$

Next we consider following three subcases: Subcase 1: $B \neq 0, -1$. Then from (8) we have

$$\overline{N}\Big(r, \frac{B+1}{B}; G\Big) = \overline{N}(r,\infty;F).$$

By using the Second Fundamental Theorem of Nevanlinna and (ii) of Lemma 1 we get

$$T(r,G) < \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\Big(r, \frac{B+1}{B}; G\Big) + S(r,G) \le \\ \le \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + S(r,G) \le \\ \le \overline{N}(r,\infty;G) + T(r,G) - dT(r,g) + dN(r,0;g) + \overline{N}(r,\infty;F) + S(r,g), \\ \Rightarrow dT(r,g) < \overline{N}(r,\infty;f) + dN(r,0;g) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g). \tag{9}$$

If $A-B-1\neq 0$, then it follows from (7) that

$$N\Big(r,\frac{-A+B+1}{B+1};F\Big)=N(r,0;G).$$

Applying Second Fundamental Theorem of Nevanlinna and Lemma 1, we subsequently obtain

$$T(r,F)<\overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}\Big(r,\frac{-A+B+1}{B+1};F\Big)+S(r,F),$$
$$dT(r,f)<\overline{N}(r,\infty;f)+dN(r,0;f)+Q\overline{N}(r,\infty;g)+dN(r,0;g)+S(r,f)+S(r,g). \quad (10)$$

From (9) and (10), we get

$$T(r,f)+T(r,g)\leq N(r,0;f)+\frac{2}{d}\overline{N}(r,\infty;f)+2N(r,0;g)+$$
$$+\frac{Q+1}{d}\overline{N}(r,\infty;g)+S(r,f)+S(r,g),$$

which again contradict (2).

Therefore $A-B-1=0$. Then from (7) it follows that

$$\overline{N}\Big(r,0;F+\frac{1}{B}\Big)=\overline{N}(r,\infty;G).$$

By applying the Second Fundamental Theorem of Nevanlinna and (ii) of Lemma 1 we have

$$T(r,F)<\overline{N}(r,\infty;F)+\overline{N}(r,0;F)+\overline{N}\Big(r,0;F+\frac{1}{B}\Big)+S(r,F)\leq$$
$$\leq\overline{N}(r,\infty;f)+T(r,F)-dT(r,f)+dN(r,0;f)+\overline{N}(r,\infty;g)+S(r,f)+S(r,g).$$

Hence

$$dT(r,f)<\overline{N}(r,\infty;f)+dN(r,0;f)+\overline{N}(r,\infty;g)+S(r,f)+S(r,g). \quad (11)$$

Combining (9) and (11) we get

$$T(r,f)+T(r,g)\leq N(r,0;f)+\frac{2}{d}\overline{N}(r,\infty;f)+N(r,0;g)+\frac{2}{d}\overline{N}(r,\infty;g)+S(r,f)+S(r,g),$$

which violates assumption (2).

Subcase 2: $B=-1$. Then (7) and (8) we get

$$G=\frac{A}{A+1-F},\quad F=\frac{(1+A)G-A}{G}.$$

If $A+1\neq 0$, then

$$\overline{N}(r,A+1;F)=\overline{N}(r,\infty;G),\quad \overline{N}\Big(r,\frac{A}{A+1};G\Big)=\overline{N}(r,0;F).$$

By similar argument as in Subcase 1, we get a contradiction. Therefore $A+1=0$, then $FG=1\Rightarrow P[f]\cdot P[g]=a^2$.

Subcase 3: $B = 0$. Then (7) and (8) gives $G = \frac{F+A-1}{A}$ and $F = AG + 1 - A$ If $A - 1 \neq 0$, $N(r, 0; A - 1 + F) = N(r, 0; G)$ and $N(r, \frac{A-1}{A}; G) = N(r, 0; F)$. Proceeding similarly as in Subcase 1 we get a contradiction. Therefore $A - 1 = 0$, then $F = G$ i.e., $P[f] = P[g]$. This completes the proof of Case 1.

Case 2: $l = 1$. From (ii) of Lemma 2 we have

$$T(r,F) \leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N(r,0;F)+N(r,0;G)+\overline{N}_L(r,1;F)+S(r,F)+S(r,G).$$

Using Lemma 1 and Lemma 3, we get

$$\begin{aligned} T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + T(r,F) - dT(r,f) + dN(r,0;f) \\ +Q\overline{N}(r,\infty;g) + dN(r,0;g) + \overline{N}_L(r,1;F) + S(r,f) + S(r,g), \end{aligned}$$

i.e.,

$$\begin{aligned} dT(r,f) &\leq 2\overline{N}(r,\infty;f) + dN(r,0;f) + (2+Q)\overline{N}(r,\infty;g) \\ &+dN(r,0;g) + \frac{1}{2}\overline{N}(r,\infty;f) + \frac{1}{2}Q\overline{N}(r,\infty;f) + \frac{1}{2}dN(r,0;f) + S(r,f) + S(r,g) \\ &\leq \frac{5+Q}{2}\overline{N}(r,\infty;f) + \frac{3d}{2}N(r,0;f) + (2+Q)\overline{N}(r,\infty;g) + dN(r,0;g) + S(r,f) + S(r,g), \end{aligned}$$

which implies

$$\begin{aligned} \Rightarrow dT(r,f) \leq \frac{5+Q}{2}\overline{N}(r,\infty;f) + \frac{3d}{2}N(r,0;f)+ \\ +(2+Q)\overline{N}(r,\infty;g) + dN(r,0;g) + S(r,f) + S(r,g). \end{aligned} \tag{12}$$

Similarly,

$$dT(r,g) \leq \frac{5+Q}{2}\overline{N}(r,\infty;g)+\frac{3d}{2}N(r,0;g)+(2+Q)\overline{N}(r,\infty;f)+dN(r,0;f)+S(r,f)+S(r,g). \tag{13}$$

Adding (12) and (13), we get

$$\begin{aligned} T(r,f) + T(r,g) \leq \frac{3Q+9}{2d}\overline{N}(r,\infty;f) + \frac{5}{2}N(r,0;f) + \frac{3Q+9}{2d}\overline{N}(r,\infty;g)+ \\ +\frac{5}{2}N(r,0;g) + S(r,f) + S(r,g), \end{aligned}$$

which contradicts our hypothesis (3).

Proceeding similarly as in Case 1 we get the result for this case.

Case 3: $l = 0$. From (iii) of Lemma 2 we have

$$\begin{aligned} T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + N(r,0;F)+ \\ +N(r,0;G) + 2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,F) + S(r,G). \end{aligned}$$

Using Lemma 1 and Lemma 4 in the above inequality we obtain

$$\begin{aligned} T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + T(r,F) - dT(r,f) + dN(r,0;f)+ \\ +Q\overline{N}(r,\infty;g) + dN(r,0;g) + 2\overline{N}(r,\infty;F) + 2\overline{N}(r,0;F)+ \\ +\overline{N}(r,\infty;G) + \overline{N}(r,0;G) + S(r,f) + S(r,g), \end{aligned}$$

i.e.,

$$dT(r,f) \leq 4\overline{N}(r,\infty;f) + dN(r,0;f) + (3+Q)\overline{N}(r,\infty;g) + dN(r,0;g) + 2Q\overline{N}(r,\infty;f)+$$
$$+2dN(r,0;f) + Q\overline{N}(r,\infty;g) + dN(r,0;g) + S(r,f) + S(r,g),$$
$$\Rightarrow dT(r,f) \leq (4+2Q)\overline{N}(r,\infty;f) + 3dN(r,0;f)+$$
$$+(3+2Q)\overline{N}(r,\infty;g) + 2dN(r,0;g) + S(r,f) + S(r,g). \quad (14)$$

Similarly,

$$dT(r,g) \leq (4+2Q)\overline{N}(r,\infty;g) + 3dN(r,0;g)+$$
$$+(3+2Q)\overline{N}(r,\infty;f) + 2dN(r,0;f) + S(r,f) + S(r,g). \quad (15)$$

Combining (14) and (15), we obtain

$$T(r,f) + T(r,g) \leq \frac{4Q+7}{d}\overline{N}(r,\infty;f) + 5N(r,0;f)+$$
$$+\frac{4Q+7}{d}\overline{N}(r,\infty;g) + 5N(r,0;g) + S(r,f) + S(r,g),$$

which contradicts our hypothesis (4).

Approaching similarly as in Case 1 we get the required conclusion for this case. □

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