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WEAKENED PROBLEM ON EXTREMAL DECOMPOSITION OF THE COMPLEX PLANE

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The paper deals with the problem of the maximum of the functional

$$r^{\gamma}\left(B_0,0\right)\prod_{k=1}^{n} r\left(B_k,a_k\right),$$

where B_0,..., B_n, $n\geqslant 2$, are pairwise disjoint domains in $\overline{\mathbb{C}}$, $a_0=0$, $|a_k|=1$, $k\in\{1,\ldots,n\}$ and $\gamma\in(0,n]$ ($r(B,a)$ is the inner radius of the domain $B\subset\overline{\mathbb{C}}$ with respect to a). We show that the functional attains its maximum at a configuration of the domains B_k and the points a_k possessing rotational n-symmetry. The proof is due to Dubinin [1] for $\gamma=1$ and to Kuz'mina [3] for $0<\gamma<1$. Subsequently, Kovalev [4] solved this problem for $n\geqslant 5$ under the additional assumption that the angles between neighbouring line segments $[0,a_k]$ do not exceed $2\pi/\sqrt{\gamma}$. In the paper, we obtain some estimate of the functional for $\gamma\in(1,\,n]$.

In the geometric function theory of complex variable extremal problems on non-overlapping domains are well-known classic direction. A lot of such problems are reduced to determination of the maximum of the product of inner radii on the system of non-overlapping domains satisfying certain conditions (see, for example, [1–18]). Let $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ be a point compactification, $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, and $\mathbb{R}^+=(0,\infty)$. Let $r(B,a)$ be the inner radius of the domain $B\subset\overline{\mathbb{C}}$ relative to a point $a\in B$.

The system of points $A_n:=\{a_k\in\mathbb{C}, k=\overline{1,n}\}$, $n\in\mathbb{N}$, $n\geqslant 2$ is called n-radial, if $|a_k|\in\mathbb{R}^+$ for $k=\overline{1,n}$ and $0=\arg a_1<\arg a_2<\ldots<\arg a_n<2\pi$. Denote $a_{n+1}:=a_1$, $\alpha_k:=\frac{1}{\pi}\arg\frac{a_{k+1}}{a_k}$, $\alpha_{n+1}:=\alpha_1$, $k=\overline{1,n}$, $\sum_{k=1}^{n}\alpha_k=2$.

Let us consider an open extremal problem which was formulated in [1] in the list of unsolved problems. Later, it was repeated in monograph [8].

Problem. *For any fixed value of $\gamma\in(0,n]$ to find the maximum of the functional*

$$I_n(\gamma)=r^{\gamma}\left(B_0,0\right)\prod_{k=1}^{n} r\left(B_k,a_k\right), \tag{1}$$

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where B_0, B_1, B_2,..., B_n, $n \geqslant 2$, *are mutually non-overlapping domains,* $a_0 = 0$, $|a_k| = 1$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$.

Nowadays, this task is not completely solved, its solutions for are only known certain particular cases (see, for example, [1–17]). In [1, 2] the problem was solved for any $n \geqslant 2$ and $\gamma = 1$. In 1996 L. V. Kovalev [4] got the solution of this problem with some restrictions on the location of points on the unit circle and, namely, for $n \geqslant 5$ and the subclass of point systems satisfying the following condition

$$0 < \alpha_k \leqslant 2/\sqrt{\gamma},\ k = \overline{1,n}.$$

It is clear that these conditions are sufficiently stringent conditions. They significantly narrow the set of feasible configurations. In 2003, in paper of G.V. Kuz'mina [3] in the case of simply connected domains this problem was also been studied for $\gamma \in (0,1]$ by another method. In 2008, in the monograph [9, p. 255] it was shown that the result of V.N. Dubinin holds for an arbitrary $\gamma \in \mathbb{R}^+$ and some number $n_0(\gamma)$. The problem was solved in [12] for $n \geqslant 8$ and $1 < \gamma \leqslant \sqrt[4]{n}$, and in [13] for $n \geqslant 12$ and $1 < \gamma \leqslant n^{0,45}$.

It should be noted that in the case $\gamma > 1$ the method developed in paper of V.N. Dubinin [1] can not be applied. We note that the maximum of the functional $I_n(\gamma)$ was studying in papers [1, 2, 4, 8, 9, 12, 13]. In particular cases for certain values of γ it is shown that the following inequality holds

$$I_n(\gamma) \leqslant \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}.$$

The equality in this inequality is achieved when 0, a_k and B_0, B_k, $k = \overline{1,n}$, are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\, dw^2.$$

The goal of this paper is to obtain some estimate of the functional $I_n(\gamma)$ for all $\gamma \in (1,\, n]$. Thus, we obtain the following statement.

Theorem 1. *Let* $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (1,\, n]$. *Then, for any system of distinct points* $\{a_k\}_{k=1}^n$ *of the unit circle and any mutually non-overlapping domains* B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $a_0 = 0$, *the following inequality holds*

$$r^\gamma(B_0, 0) \prod_{k=1}^n r(B_k, a_k) \leqslant n^{-\frac{\gamma}{2}} \left(\prod_{k=1}^n r(B_k, a_k)\right)^{1-\frac{\gamma}{n}}. \tag{2}$$

Proof. Let Δ be the maximum of the functional $I_n(\gamma)$. In papers [1, 4, 8, 9], the authors reviewed the case when $I_n(\gamma) \leqslant \Delta$. Consider the case if $\Delta \leqslant I_n(\gamma)$. Let $d(E)$ be the transfinite diameter of a compact set $E \subset \mathbb{C}$. Then the following relation holds

$$r(B_0, 0) = r\left(B_0^+, \infty\right) = \frac{1}{d(\overline{\mathbb{C}} \setminus B_0^+)} \leqslant \frac{1}{d(\bigcup_{k=1}^n \overline{B}_k^+)}, \tag{3}$$

where $B^+ = \{z\colon \frac{1}{z} \in B\}$. In view of the well-known Pólya theorem [6, p. 34], [18, p. 28], the inequality

$$\mu E \leqslant \pi d^2(E)$$

is valid, where μE denotes the Lebesgue measure of a compact set E. Hence, we get

$$d(E) \geqslant \left(\frac{1}{\pi}\mu E\right)^{\frac{1}{2}}.$$

Then relation (3) yields

$$r\left(B_0, 0\right) \leqslant \frac{1}{d(\bigcup_{k=1}^{n} \overline{B}_k^+)} \leqslant \frac{1}{\sqrt{\frac{1}{\pi}\mu(\bigcup_{k=1}^{n} \overline{B}_k^+)}} = \left[\frac{1}{\pi}\sum_{k=1}^{n} \mu\overline{B}_k^+\right]^{-\frac{1}{2}}. \tag{4}$$

Let B be a bounded domain, $a \in B$. We consider the class of all regular functions $\varphi(z)$, given in the domain B and such that $\varphi(a) = 0$, $\varphi'(a) = 1$. Here we estimate the area of an image of the domain B at the mapping by an arbitrary function $\varphi(z)$. It follows from the theorem of minimization of areas [6, p.34] that

$$\iint\limits_{B} |\varphi'(z)|^2 dxdy \geqslant \pi r^2\left(B, a\right). \tag{5}$$

Let us set $\varphi_1(z) = (z - a)$, then relation (5) yields

$$S(B) = \mu(B) \geqslant \pi r^2\left(B, a\right).$$

Inequality (4) implies directly that

$$r\left(B_0, 0\right) \leqslant \left[\frac{1}{\pi}\sum_{k=1}^{n} \mu\overline{B}_k^+\right]^{-\frac{1}{2}} \leqslant \left[\frac{1}{\pi}\sum_{k=1}^{n} \mu B_k^+\right]^{-\frac{1}{2}} \leqslant \left[\sum_{k=1}^{n} r^2\left(B_k^+, a_k^+\right)\right]^{-\frac{1}{2}}.$$

Hence we get the inequality

$$r\left(B_0, 0\right) \leqslant \frac{1}{\left[\sum\limits_{k=1}^{n} r^2\left(B_k^+, a_k^+\right)\right]^{\frac{1}{2}}}.$$

Taking into account the relation

$$r\left(B_k^+, a_k^+\right) = \frac{r\left(B_k, a_k\right)}{|a_k|^2}$$

we arrive at the inequality

$$r\left(B_0, 0\right) \leqslant \left[1 \Big/ \sum_{k=1}^{n} \frac{r^2\left(B_k, a_k\right)}{|a_k|^4}\right]^{\frac{1}{2}}.$$

This result and the assumption of Theorem 1 yield the relation

$$\Delta \leqslant r^\gamma\left(B_0, 0\right)\prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \frac{\prod\limits_{k=1}^{n} r\left(B_k, a_k\right)}{\left[\sum\limits_{k=1}^{n} \frac{r^2(B_k, a_k)}{|a_k|^4}\right]^{\frac{\gamma}{2}}},$$

Δ is the maximum of functional $I_n(\gamma)$. The Cauchy inequality yields automatically the relation

$$\frac{1}{n}\sum_{k=1}^{n}\frac{r^2\left(B_k,a_k\right)}{|a_k|^4}\geqslant\left[\prod_{k=1}^{n}\frac{r^2\left(B_k,a_k\right)}{|a_k|^4}\right]^{\frac{1}{n}}.$$

And using the equality

$$\prod_{k=1}^{n}|a_k|=1,$$

we get easily

$$\left[\sum_{k=1}^{n}\frac{r^2\left(B_k,a_k\right)}{|a_k|^4}\right]^{\frac{\gamma}{2}}\geqslant\left[n\left[\prod_{k=1}^{n}\frac{r^2\left(B_k,a_k\right)}{|a_k|^4}\right]^{\frac{1}{n}}\right]^{\frac{\gamma}{2}}\geqslant n^{\frac{\gamma}{2}}\left[\prod_{k=1}^{n}r\left(B_k,a_k\right)\right]^{\frac{\gamma}{n}}.$$

Eventually,

$$r^{\gamma}\left(B_0,0\right)\prod_{k=1}^{n}r\left(B_k,a_k\right)\leqslant\frac{\prod\limits_{k=1}^{n}r\left(B_k,a_k\right)}{n^{\frac{\gamma}{2}}\left[\prod\limits_{k=1}^{n}r\left(B_k,a_k\right)\right]^{\frac{\gamma}{n}}}=n^{-\frac{\gamma}{2}}\left(\prod_{k=1}^{n}r\left(B_k,a_k\right)\right)^{1-\frac{\gamma}{n}}.$$

Thus, Theorem 1 is proved. □

Remark 1. If $\gamma=n$, then by Theorem 1 the following inequality holds

$$r^{n}\left(B_0,0\right)\prod_{k=1}^{n}r\left(B_k,a_k\right)\leqslant n^{-\frac{n}{2}}.$$

For any n-radial system of points $A_n=\{a_k\}_{k=1}^{n}$, $|a_k|=1$, and for any pairwise non-overlapping domains $\{B_k\}_{k=1}^{n}$, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{1,n}$, the inequality

$$\prod_{k=1}^{n}r(B_k,a_k)\leqslant 2^{n}\prod_{k=1}^{n}\alpha_k$$

is valid (see, [9, Corollary 5.1.3]). Using this result we obtain the following statement.

Corollary 1. *Let $n\in\mathbb{N}$, $n\geqslant 2$, $\gamma\in(1,\,n]$. Then for any system of distinct points $\{a_k\}_{k=1}^{n}$ of the unit circle and any mutually non-overlapping domains B_k, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{0,n}$, $a_0=0$, the following inequality holds*

$$r^{\gamma}\left(B_0,0\right)\prod_{k=1}^{n}r\left(B_k,a_k\right)\leqslant 2^{(n-\gamma)}\cdot n^{-\frac{\gamma}{2}}\cdot\left(\prod_{k=1}^{n}\alpha_k\right)^{1-\frac{\gamma}{n}}.$$

In [8, Theorem 6.11] for any distinct points a_k on the circle $|a_k|=1$, $k=\overline{1,n}$ $(n\geqslant 2)$, and any pairwise non-overlapping domains $B_k\subset\overline{\mathbb{C}}$ such that $a_k\in B_k$, $k=\overline{1,n}$, the inequality

$$\prod_{k=1}^{n}r\left(B_k,a_k\right)\leqslant\left(\frac{4}{n}\right)^{n}$$

is proved. Thus, we have the next result.

Corollary 2. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (1, n]$. Then for any system of distinct points $\{a_k\}_{k=1}^{n}$ of the unit circle and any mutually non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0, n}$, $a_0 = 0$, the following inequality holds*

$$r^{\gamma}(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant n^{-\frac{\gamma}{2}} \left(\frac{4}{n}\right)^{n-\gamma}.$$

In view of Theorem 1 for any system of the distinct points $\{a_k\}_{k=1}^{n} \in \mathbb{C}\backslash\{0\}$ the following statement is true.

Corollary 3. *Let $n \in \mathbb{N}$, $n \geqslant 2$, $\gamma \in (1, n]$. Then, for any system of distinct points $\{a_k\}_{k=1}^{n} \in \mathbb{C}\backslash\{0\}$ and any mutually non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0, n}$, $a_0 = 0$, the inequality*

$$r^{\gamma}(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant n^{-\frac{\gamma}{2}} \left(\prod_{k=1}^{n} r(B_k, a_k)\right)^{1-\frac{\gamma}{n}} \left(\prod_{k=1}^{n} |a_k|\right)^{\frac{2\gamma}{n}}$$

holds.

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